J. Nonlinear Var. Anal. 3 (2019), No. 2, pp. 225-233
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.3.2019.2.10

ON A NEW ALGORITHM FOR SOLVING VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

FENGHUI WANG[1,*], HONGTRUONG PHAM[2]

[1]*Department of Mathematics, Luoyang Normal University, Luoyang 471022, China*
[2]*Department of Mathematics, Thainguyen University of Economics and Business Administration, Thainguyen 250000, Vietnam*

Abstract. The paper is concerned with the problem of finding a common solution of a variational inequality problem governed by Lipschitz continuous monotone mappings and a fixed point problem of nonexpansive mappings. To solve this problem, we introduce a new iterative algorithm which is based on Tseng's extragradient method. Moreover we prove the strong convergence of the algorithm to a solution of the above-stated problem without the hypothesis of the asymptotical regularization.
Keywords. Extragradient method; Fixed point; Iterative algorithms; Nonexpansive mapping; Variational inequality.
2010 Mathematics Subject Classification. 47H05, 47H09, 47J20, 49J40.

1. INTRODUCTION

Throughout the paper, $\mathscr{H}$ is a real Hilbert space and C is a nonempty closed convex subset of $\mathscr{H}$. A variational inequality problem (VIP) is formulated as a problem of finding a point $x^* \in C$ with the property:

$$\langle Ax^*, z - x^* \rangle \geq 0, \quad \forall z \in C, \tag{1.1}$$

where $A : C \to \mathscr{H}$ is a single-valued mapping. We denote the solution set of VIP (1.1) by $\mathrm{VI}(C,A)$. A fixed point problem (FPP) is to find a point $\hat{x}$ with the property:

$$\hat{x} \in C, \quad S\hat{x} = \hat{x}, \tag{1.2}$$

where $S : C \to C$ is a nonlinear mapping. The set of fixed points of S is denoted as $\mathrm{Fix}(S)$.

In this paper, we are interested in finding a common solution of VIP (1.1) and of FPP (1.2). Namely, we seek a point x^* such that

$$x^* \in \mathrm{Fix}(S) \cap \mathrm{VI}(C,A). \tag{1.3}$$

This problem was first introduced by Takahashi and Toyoda [11]. Since then, many algorithms have been built for approximating a solution of problem (1.3); see, e.g., [3, 4, 8, 9, 13, 14, 16] and the references therein.

[*]Corresponding author.
E-mail address: wfenghui@gmail.com (F. Wang).
Received March 19, 2019; Accepted June 1, 2019.

In the case where $A : C \to \mathscr{H}$ is inverse strongly monotone and $S : C \to C$ is nonexpansive, Takahashi and Toyoda [11] introduced an algorithm which generates a sequence $\{x_n\}$ by the iterative procedure:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n SP_C(x_n - \lambda_n Ax_n), \tag{1.4}$$

where P_C is the projection of C onto $\mathscr{H}$. The Halpern-type algorithm:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{1.5}$$

where $u \in C$ is fixed, was introduced by Iiduka and Takahashi [3]. In both algorithms (1.4) and (1.5), the sequence $\{\alpha_n\}$ is chosen from the interval $[0,1]$. Under certain assumptions, the sequence $\{x_n\}$ generated by algorithm (1.4) (resp., (1.5)) can be weakly (resp., strongly) convergent to a solution of problem (1.3).

In general, the above algorithm dose not work whenever A is only a k-Lipschitz-continuous and monotone mapping. In this situation, the following iterative method:

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = P_C(x_n - \lambda_n Ay_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sz_n, \end{cases} \tag{1.6}$$

where $\lambda_n \in (0, 1/k)$ and $\alpha_n \in (0,1)$, was proposed by Nadezhkina and Takahashi [7] for solving problem (1.3). It is worth noting that this algorithm is motivated by Korpelevich's extragradient method [5] for solving varational inequalities. For any initial guess $x_0 \in C$, Korpelevich's extragradient method generates an iterative sequence recursively by

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

where $\lambda \in (0, 1/k)$. Subsequently, Zeng and Yao [16] introduced a Halpern-type algorithm, which generates an iterative sequence recursively by

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = P_C(x_n - \lambda_n Ay_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Sz_n, \end{cases} \tag{1.7}$$

where $\lambda_n \in (0, 1/k)$ and $\alpha_n \in (0,1)$. Under some mild assumptions, the sequence (x_n) generated by algorithm (1.6) (resp., (1.7)) can be weakly (resp., strongly) convergent to a solution of problem (1.3).

We note that one sufficient condition for the convergence of algorithm (1.7) is $\|x_{n+1} - x_n\| \to 0$ (see [16, Theorem 3.1]). However, from a practical point of view, such condition is often difficult to verify. In this paper, we propose a new algorithm for solving problem (1.3) in the case where the governed mapping is only Lipschitz-continuous monotone. The potential advantage of this algorithm is that we can prove its strong convergence without assuming $\|x_{n+1} - x_n\| \to 0$. Our algorithm is mainly based on Tseng's splitting method [12] for solving variational inequalities. For any initial guess $x_0 \in C$, Tseng's splitting method generates an iterative sequence recursively by

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(y_n - \lambda(Ay_n - Ax_n)), \end{cases}$$

where $\lambda \in (0, 1/k)$. The paper is organized as follows. In the next section, some useful lemmas are given. In Section 3, we propose our algorithm and prove its strong convergence to a solution of problem (1.3).

2. Preliminary and notation

We shall use the following notation:

- $x_n \to x$: strong convergence of (x_n) to x;
- $x_n \rightharpoonup x$: weak convergence of (x_n) to x;
- I the identity mapping, and $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$.

A mapping $S : C \to C$ is said to be k-Lipschitz continuous if there exists a constant $k > 0$ so that

$$\|Sx - Sy\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $k = 1$, then we say S is a nonexpansive mapping. A mapping $A : C \to \mathscr{H}$ is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

k-inverse strongly monotone if there exists a constant $k > 0$ so that

$$\langle Ax - Ay, x - y\rangle \geq k\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

We use P_C to denote the projection from $\mathscr{H}$ onto C, namely, for $x \in \mathscr{H}$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

It is well known that $P_C x$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x\rangle \leq 0, \quad \forall z \in C. \tag{2.1}$$

The lemma below is referred to as the demiclosedness principle for nonexpansive mappings (see [2]).

Lemma 2.1 (Demiclosedness principle)**.** *Let $T : C \to \mathscr{H}$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in$* $\mathrm{Fix}(T)$.

A multi-valued mapping $T : \mathscr{H} \to 2^{\mathscr{H}}$ is said to be monotone if $\langle u - v, x - y\rangle \geq 0$, for any $u \in Tx, v \in Ty$; maximal monotone if its graph

$$\mathscr{G}(T) = \{(x, y) : x \in \mathscr{D}(A), u \in Ax\}$$

is not properly contained in the graph of any other monotone operator. The normal cone to C at $x \in C$ is a multi-valued mapping defined by

$$N_C x = \{w \in \mathscr{H} : \langle x - u, w\rangle \geq 0,\ u \in C\}.$$

It is known that N_C is maximal monotone, and its resolvent is P_C, that is, $P_C = (I + \lambda N_C)^{-1}$ (see [1, p. 334]). Define a mapping $T : \mathscr{H} \to 2^{\mathscr{H}}$ by

$$Tx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{2.2}$$

Lemma 2.2. *Let T be defined as (2.2) and A a single-valued monotone mapping. Then*

(i) $\mathscr{G}(T)$ is sequentially weakly-strongly closed, that is, if $u_n \in Tx_n, x_n \rightharpoonup x$, and $u_n \to u$, then $u \in Tx$;
(ii) $\mathrm{VI}(C, A) = T^{-1}(0) = \{x \in \mathscr{H}, 0 \in Tx\}$;

(iii) for $\lambda > 0$, $y = P_C(x - \lambda Ax)$ iff

$$\frac{x-y}{\lambda} + Ay - Ax \in Ty;$$

in particular, $x = P_C(x - \lambda Ax)$ iff $x \in \mathrm{VI}(C,A)$.

Proof. (i) It is known (see [1, Pro. 20.33]) that if T is maximal monotone, then its graph $\mathscr{G}(T)$ is sequentially weakly-strongly closed. Since A is single-valued, by [10, Theorem 3], T is maximal monotone and thus the first assertion follows.

(ii) To see this, we note that $0 \in Tx = (A + N_C)x$ if and only if $-Ax \in N_C x$. By the definition of the normal cone, this is equivalent to $\langle Ax, x - z\rangle \leq 0, \forall z \in C$.

(iii) Since P_C is the resolvent of N_C, it follows

$$\begin{aligned} y = P_C(x - \lambda Ax) &\Leftrightarrow y = (I + \lambda N_C)^{-1}(x - \lambda Ax) \\ &\Leftrightarrow y + \lambda N_C y \ni x - \lambda Ax \\ &\Leftrightarrow \lambda(A + N_C)y \ni x - y + \lambda(Ay - Ax). \end{aligned}$$

Particularly, we have $x = P_C(x - \lambda Ax)$ if and only if $0 \in Tx$. By part (ii), we get the desired assertion. □

We end this section by some useful lemmas.

Lemma 2.3. [6] *Let $\{s_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_k}\}$ so that*

$$s_{n_k} \leq s_{n_k+1} \text{ for all } k \geq 0.$$

For every $n > n_0$ define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$

$$\max(s_{\tau(n)}, s_n) \leq s_{\tau(n)+1}. \tag{2.3}$$

Lemma 2.4. [15] *Let $\{s_n\}$ be a nonnegative real sequence satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \varepsilon_n,$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\varepsilon_n\}$ is a real sequence. Then $s_n \to 0$ as $n \to \infty$ provided that

(i) $\sum \alpha_n = \infty, \lim_n \alpha_n = 0$;
(ii) $\overline{\lim}_n \varepsilon_n \leq 0$ *or* $\sum \alpha_n |\varepsilon_n| < \infty$.

Lemma 2.5. *Let $x, y \in \mathscr{H}$ and let $t, s \geq 0$. Then*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(ii) $\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x - y\|^2$.

3. A STRONG CONVERGENCE THEOREM

We now introduce our iterative algorithm. Take an initial guess $x_0 \in C$; choose $\{\alpha_n\} \subseteq (0,1), \{\beta_n\} \subseteq (0,1), \{\gamma_n\} \subseteq (0,1)$ and $\{\lambda_n\} \subseteq (0,1/k)$; and define a sequence $\{x_n\}$ by the iterative procedure:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_C(y_n - \lambda_n (A y_n - A x_n)), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S z_n, \end{cases} \tag{3.1}$$

where $u \in C$ is fixed, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are positive real sequence such that $\alpha_n + \beta_n + \gamma_n = 1$. To state the convergence of the algorithm, we need the following lemma.

Lemma 3.1. *Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be defined by (3.1). Then, for any solution z to problem (1.3), there holds the inequality:*

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - (1 - (k\lambda_n)^2)\|y_n - x_n\|^2. \tag{3.2}$$

Proof. Since P_C is nonexpansive, we have

$$\begin{aligned} \|z_n - z\|^2 &= \|P_C(y_n - \lambda_n(Ay_n - Ax_n)) - z\|^2 \\ &\le \|y_n - z - \lambda_n(Ay_n - Ax_n)\|^2 \\ &= \|y_n - z\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\ &\quad - 2\lambda_n \langle y_n - z, Ay_n - Ax_n \rangle \\ &= \|x_n - z\|^2 + \|y_n - x_n\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\ &\quad + 2\langle x_n - z, y_n - x_n \rangle - 2\lambda_n \langle y_n - z, Ay_n - Ax_n \rangle \\ &= \|x_n - z\|^2 - \|y_n - x_n\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\ &\quad + 2\langle y_n - z, y_n - x_n \rangle - 2\lambda_n \langle y_n - z, Ay_n - Ax_n \rangle. \end{aligned}$$

Having in mind that $y_n = P_C(x_n - \lambda_n A x_n)$, we deduce from (2.1) that

$$\langle y_n - z, y_n - x_n \rangle \le -\lambda_n \langle y_n - z, Ax_n \rangle.$$

Since A is k-Lipschitz continuous, it follows that

$$\begin{aligned} \|z_n - z\|^2 &\le \|x_n - z\|^2 - \|y_n - x_n\|^2 \\ &\qquad + \lambda_n^2 \|Ay_n - Ax_n\|^2 - 2\lambda_n \langle y_n - z, Ay_n \rangle \\ &\le \|x_n - z\|^2 - (1 - k^2\lambda_n^2)\|y_n - x_n\|^2 \\ &\qquad - 2\lambda_n \langle y_n - z, Ay_n \rangle. \end{aligned}$$

On the other hand, we observe that

$$\langle y_n - z, Ay_n \rangle = \langle y_n - z, Ay_n - Az \rangle + \langle y_n - z, Az \rangle.$$

Since $z \in \mathrm{VI}(C, A)$ and A is monotone, this implies $\langle y_n - z, Ay_n \rangle \ge 0$. Hence the desired inequality (3.2) at once follows. □

Below is the convergence of algorithm (3.1).

Theorem 3.1. *Let $A : C \to \mathscr{H}$ be a $k(>0)$-Lipschitz continuous monotone mapping and $S : C \to C$ a nonexpansive mapping. Suppose that*

(i) $0 < \underline{\lim}_n \beta_n \leq \overline{\lim}_n \beta_n < 1$;

(ii) $\lim_n \alpha_n = 0, \sum_n \alpha_n = \infty$;

(iii) $0 < \underline{\lim}_n \lambda_n \leq \overline{\lim}_n \lambda_n < 1/k$.

If the solution set of problem (1.3), denoted by Ω, is nonempty, then the sequence (x_n) generated by (3.1) converges strongly to $P_\Omega(u)$.

Proof. Let $z := P_\Omega(u)$. We first show that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. By Lemma 3.1, one has

$$\|z_n - z\| \leq \|x_n - z\|.$$

Thus

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n u + \beta_n x_n + \gamma_n S z_n - z\| \\ &\leq \alpha_n \|u - z\| + \beta_n \|x_n - z\| + \gamma_n \|z_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\|,\end{aligned}$$

where the last inequality follows from the fact that $\alpha_n + \beta_n + \gamma_n = 1$. By induction, we have

$$\|x_n - z\| \leq \max\{\|u - z\|, \|x_0 - z\|\}.$$

Then $\{x_n\}$ is bounded and so is $\{z_n\}$. Since P_C is nonexpansive, one has

$$\begin{aligned}\|y_n - z\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(z - \lambda_n A z)\| \\ &\leq \|x_n - z - \lambda_n (A x_n - A z)\| \\ &\leq \|x_n - z\| + \lambda_n k \|x_n - z\| \\ &\leq 2\|x_n - z\|.\end{aligned}$$

Therefore $\{y_n\}$ is bounded.

We next show the following inequality:

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\quad - \sigma(\|S z_n - x_n\|^2 + \|y_n - x_n\|^2),\end{aligned} \tag{3.3}$$

where σ is a positive number. Indeed, it follows from Lemmas 2.5 and 3.1 that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n (u - z) + \beta_n (x_n - z) + \gamma_n (S z_n - z)\|^2 \\ &\leq \|\beta_n (x_n - z) + \gamma_n (S z_n - z)\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &= \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|S z_n - z\|^2 \\ &\quad - \gamma_n \beta_n \|S z_n - x_n\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|z_n - z\|^2 \\ &\quad - \gamma_n \beta_n \|S z_n - x_n\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 - \gamma_n (1 - \alpha_n)(1 - (k\lambda_n)^2)\|y_n - x_n\|^2 \\ &\quad - \gamma_n \beta_n \|S z_n - x_n\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle.\end{aligned}$$

By our hypothesis on $\{\lambda_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n\}$, we may assume without loss of generality that there is $\sigma > 0$ so that

$$\gamma_n \min(\beta_n, (1-\alpha_n)(1-(k\lambda_n)^2)) \geq \sigma, \quad \forall n.$$

Hence, inequality (3.3) immediately follows. If we let $s_n = \|x_n - z\|^2$ and

$$c_n = \sigma(\|Sz_n - x_n\|^2 + \|y_n - x_n\|^2),$$

then inequality (3.3) has the form:

$$s_{n+1} \leq (1-\alpha_n)^2 s_n + 2\alpha_n \langle u - z, x_{n+1} - z\rangle - c_n. \tag{3.4}$$

Finally, we prove $s_n \to 0$ by considering two possible cases on $\{s_n\}$.

CASE 1. $\{s_n\}$ is eventually decreasing (i.e., there exists $N \geq 0$ such that $\{s_n\}$ is decreasing for $n \geq N$). In this case, $\{s_n\}$ must be convergent. In view of (3.4), one has

$$0 \leq c_n \leq M\alpha_n + (s_n - s_{n+1}),$$

where $M > 0$ is a sufficient large number. Letting $n \to \infty$ in the last inequality yields $c_n \to 0$, this implies that $\{\|Sz_n - x_n\|\}$ and $\{\|y_n - x_n\|\}$ both converge to 0. Hence

$$\begin{aligned}
\|z_n - x_n\| &= \|P_C(y_n - \lambda_n(Ay_n - Ax_n)) - P_C x_n\| \\
&\leq \|y_n - x_n - \lambda_n(Ay_n - Ax_n)\| \\
&\leq (1 + k\lambda_n)\|y_n - x_n\| \\
&\leq 2\|y_n - x_n\| \to 0,
\end{aligned}$$

where we use the fact $x_n \in C$, the nonexpansive property of P_C and the Lipschitz continuity of A. By the nonexpansiveness of S, we have

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - Sz_n\| + \|Sz_n - Sx_n\| \\
&\leq \|x_n - Sz_n\| + \|z_n - x_n\| \to 0.
\end{aligned}$$

Using the demiclosedness principle yields $\omega_w(x_n) \subseteq \mathrm{Fix}(S)$.

We next show $\omega_w(x_n) \subseteq \mathrm{VI}(C,A)$. Having in mind that $\|y_n - x_n\| \to 0$, we have $\omega_w(x_n) = \omega_w(y_n)$. Let

$$v_n = \frac{x_n - y_n}{\lambda_n} - (Ax_n - Ay_n).$$

It then follows from Lemma 2.2 that $v_n \in (A + N_C)(y_n) =: T(y_n)$. By the Lipschitz continuity, one has $\|Ax_n - Ay_n\| \to 0$. Hence $v_n \to 0$. Using Lemma 2.2 yields $\omega_w(x_n) \subseteq T^{-1}(0) = \mathrm{VI}(C,A)$. Altogether, we have $\omega_w(x_n) \subseteq \Omega$. Consequently,

$$\overline{\lim_{n\to\infty}} \langle u - z, x_{n+1} - z\rangle = \max_{x' \in \omega_w(x_n)} \langle u - z, x' - z\rangle \leq 0,$$

where the inequality follows from (2.1). In view of (3.4), one has

$$s_{n+1} \leq (1-\alpha_n)^2 s_n + 2\alpha_n \langle u - z, x_{n+1} - z\rangle. \tag{3.5}$$

We therefore apply Lemma 2.4 to (3.5) to conclude $s_n \to 0$.

CASE 2. (s_n) is not eventually decreasing. Hence, we can find a subsequence $\{s_{n_k}\}$ so that $s_{n_k} \leq s_{n_k+1}$ for all $k \geq 0$. In this case, we may define an integer sequence $\{\tau(n)\}$ as in Lemma 2.3. Since $s_{\tau(n)} \leq$

$s_{\tau(n)+1}$ for all $n > n_0$, it follows again from (3.4) that $c_{\tau(n)} \le M\alpha_{\tau(n)} \to 0$, so that $\|S_{\tau(n)}z_{\tau(n)} - x_{\tau(n)}\| \to 0$ and $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$ as $n \to \infty$. In a similar way to Case 1, we deduce that $\omega_w(x_{\tau(n)}) \subseteq \Omega$. Therefore

$$\overline{\lim_{n\to\infty}} \langle u - z, x_{\tau(n)} - z\rangle \le 0. \tag{3.6}$$

On the other hand, we note that

$$\begin{aligned}\|x_{\tau(n)} - x_{\tau(n)+1}\| &= \|\alpha_{\tau(n)}(u - x_{\tau(n)}) + \gamma_{\tau(n)}(S_{\tau(n)}z_{\tau(n)} - x_{\tau(n)})\| \\ &\le \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + \gamma_{\tau(n)}\|S_{\tau(n)}z_{\tau(n)} - x_{\tau(n)}\| \\ &\le M(\alpha_{\tau(n)} + \|S_{\tau(n)}z_{\tau(n)} - x_{\tau(n)}\|) \to 0,\end{aligned}$$

which together with (3.6) yields

$$\overline{\lim_{n\to\infty}} \langle u - z, x_{\tau(n)+1} - z\rangle \le 0. \tag{3.7}$$

Since $s_{\tau(n)} \le s_{\tau(n)+1}$ (see (2.3)), it follows from (3.4) that

$$s_{\tau(n)} \le 2\langle u - z, x_{\tau(n)+1} - z\rangle$$

for all $n > n_0$. This together with (3.7) yields

$$\overline{\lim_{n\to\infty}} s_{\tau(n)} \le 0.$$

Hence $s_{\tau(n)} \to 0$. Moreover we have

$$\begin{aligned}\sqrt{s_{\tau(n)+1}} &= \|(x_{\tau(n)} - z) - (x_{\tau(n)} - x_{\tau(n)+1})\| \\ &\le \sqrt{s_{\tau(n)}} + \|x_{\tau(n)} - x_{\tau(n)+1}\| \to 0.\end{aligned}$$

Consequently, $s_n \to 0$ follows from (2.3) immediately. □

Remark 3.1. In a similar way to [16], we can apply our algorithms for finding a common fixed point of Lipschitz pseudocontractive and nonexpansive mappings, and for finding a common zero for two monotone mappings.

Acknowledgments

The authors would like to thank the referee for useful suggestions. This work was sponsored by Program for Science & Technology Innovation Talents in Universities of Henan Province with Grant no. 15HASTIT013.

References

[1] H.H. Bauschke, P.L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, Springer-Verlag, 2011.

[2] K. Goebel, W.A. Kirk, Topics on Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[3] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly-monotone mappings, Nonlinear Anal. 61 (2005), 341–350.

[4] H. Iiduka, W. Takahashi, Strong convergence theorem by a hybrid method for nonlinear mappings of nonexpansive and monotone type and applications, Adv. Nonlinear Var. Inequal. 9 (2006), 1–10.

[5] G.M. Korpelevich, An extragradient method for finding saddle points and for other problems, Ekonomika i Matematicheskie Metody, 12 (1976), 747–756.

[6] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008), 899–912.

[7] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006), 191–201.

[8] N. Nadezhkina, W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, SIAM J. Optim. 16 (2006), 1230–1241.

[9] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003), 372–379.

[10] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75–88.

[11] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003), 417–428.

[12] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. 38 (1998), 431–446.

[13] F. Wang, H.K. Xu, Strongly convergent iterative algorithms for solving a class of variational inequalities, J. Nonlinear Convex Anal. 11 (2010), 407–421.

[14] F. Wang, H.K. Xu, Weak and strong convergence theorems for variational inequality and fixed point problems with Tseng's extragradient method, Taiwanese J. Math. 16 (2012), 1125–1136.

[15] H.K. Xu, Iterative algorithms for nonlinear operators. J. London Math. Soc. 66 (2002), 240–256.

[16] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese J. Math. 10 (2006), 1293–1303.