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CONVERGENCE THEOREMS OF COMMON SOLUTIONS FOR FIXED POINT, VARIATIONAL INEQUALITY AND EQUILIBRIUM PROBLEMS

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Dedicated to Professor Wataru Takahashi on the occasion of his 75th birthday

Abstract. The aim of this paper is to introduce an iterative process for a common solution of fixed point problem of a continuous pseudocontractive mapping, a variational inequality problem and an equilibrium problem provided their common solution exists. Moreover, a numerical example which supports our main convergence results is presented.

Keywords. Equilibrium problem; Fixed point; Monotone mapping; Strong convergence; Variational inequality problem.

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1. Introduction

Let C be a nonempty subset of a real Hilbert space H. A mapping $A : C \to H$ is said to be γ-inverse strongly monotone if there exists a positive real number γ such that

$$\langle x-y, Ax-Ay\rangle \geq \gamma||Ax-Ay||^2, \quad \forall x,y \in C.$$

If A is γ-inverse strongly monotone, then it is Lipschitz continuous with constant $\frac{1}{\gamma}$, i.e.,

$$||Ax-Ay|| \leq \frac{1}{\gamma}||x-y||, \quad \forall x,y \in C.$$

A is said to be strongly monotone if there exists $k > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq k||x-y||^2, \quad \forall x,y \in C.$$

A is said to be monotone if

$$\langle x-y, Ax-Ay\rangle \geq 0, \quad \forall x,y \in C.$$

Apart from being an important generalization of strongly monotone and γ-inverse strongly monotone mappings, interest in monotone mappings stems mainly from their firm connection with the important class of nonlinear pseudocontractive mappings. Recall that a mapping $T : C \to H$ is said to be pseudo-contractive if

$$\langle Tx-Ty, x-y\rangle \leq ||x-y||^2, \quad \forall x,y \in C.$$

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T is said to be strongly pseudocontractive if there exists $k \in (0,1)$ such that

$$\langle x-y, Tx-Ty\rangle \leq k||x-y||^2, \quad \forall x,y \in C.$$

T is said to be k-strict pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\langle x-y, Tx-Ty\rangle \leq ||x-y||^2 - k||(I-T)x-(I-T)y||^2, \quad \forall x,y \in C.$$

A mapping $T : C \to H$ is said to be L-Lipschitz if there exits $L \geq 0$ such that

$$||Tx-Ty| \leq L||x-y||, \quad \forall x,y \in C.$$

If $L = 1$ then A is said to be nonexpansive and if $L < 1$, then A is said to be contractive.

We observe that the class of pseudocontractive mappings includes the class of nonexpansive mappings, and T is pseudocontractive if and only if $A := I - T$ is monotone. Thus the fixed point set of T, $F(T) := \{x \in D(T) : Tx = x\}$, is the zero point sete of A, $N(A) := \{x \in D(A) : Ax = 0\}$.

Suppose that A is a monotone mapping from C to H. The variational inequality problem is formulated as finding

$$\text{a point } u \in C \text{ such that } \langle v-u, Au\rangle \geq 0, \text{ for all } v \in C. \tag{1.1}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$.

Variational inequality problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in C$ satisfying $0 \in Au$. It was initially studied in [7, 9] and ever since considerable research efforts have been devoted to iterative methods for approximating solutions of variational inequality and/or fixed points of T, when T is nonexpansive or its generalizations; see, for example, [5, 6, 20, 21] and the references contained therein.

In [3], Iiduka, Takahashi and Toyoda studied the projection algorithm given by

$$x_{n+1} = P_C(x_n - \alpha_n A x_n), \quad \forall n \geq 1, \tag{1.2}$$

with an initial point $x_1 \in C$, where P_C is the metric projection from H onto C and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that if A is γ-inverse strongly monotone, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some element of $VI(C,A)$. They also studied the following iterative scheme:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \alpha_n A x_n), \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 2\gamma]$. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{VI(C,A)}(x_0)$, where $P_{VI(C,A)}$ is the metric projection from H onto $VI(C,A)$ provided that A is γ-inverse strongly monotone.

Recently, the problem of finding a common element in the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem for a γ-inverse strongly monotone mapping has been considered by many authors; see, for example, [2, 7, 8, 10, 18, 19] and the references therein.

In 2005, Iiduka and Takahashi [4] considered a common element problem for a fixed point problem of nonexpansive mappings and a variational inequality problem via the following iterative algorithm:

$$x_0 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n) T P_C(x_n - \lambda_n A x_n), \quad n \geq 0,$$

where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is a γ-inverse strongly monotone mapping, $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\gamma)$. They proved that the sequence $\{x_n\}$ strongly converges to some point $z \in F(T) \cap VI(C,A)$.

Recently, Zegeye and Shahzad [25] investigated the problem of finding a common element in fixed point sets of a Lipschitz pseudocontractive mapping T and solution sets of a variational inequality problem of a γ-inverse strongly monotone mapping A by considering the following iterative algorithm:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = P_C[(1-\alpha_n)(\delta_n x_n + \theta_n T y_n + \gamma_n P_C[I - \gamma A]x_n)], \end{cases}$$

where $\{\delta_n\}, \{\theta_n\}, \{\gamma_n\}, \{\alpha_n\}, \{\beta_n\}$ are in $(0,1)$ satisfying certain conditions. Then, they proved that the sequence $\{x_n\}$ converges strongly to the minimum-norm point of $F(T) \cap VI(C,A)$.

In this paper, one of our concerns is the following:

Question 1. Is it possible to construct an iterative scheme which converges strongly to a common element in the fixed point set of a pseudocontractive mapping and the solution set of a variational inequality problem of a monotone mapping?

Let $f : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is to

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall\, y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(f)$. A number of real world problems can be investigated via the framework of the equilibrium problem; see, e.g., [1, 10].

For studying equilibrium problem (1.4), we assume that f satisfies the following conditions:
(A1) $f(x,x) = 0$ for all $x \in C$,
(A2) f is monotone, i.e, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$,
(A3) for each $x, y, z \in C$, $\lim_{t \to 0} f(tz + (1-t)x, y) \leq f(x,y)$,
(A4) for each $x \in C, y \to f(x,y)$ is convex and lower semicontinuous.

Recently, many authors have considered the problem of finding a common element in the fixed point set of a nonexpansive mapping, the solution set of an equilibrium problem and the solution set of a variational inequality problem of γ-inverse strongly monotone mappings; see, e.g. [8, 13, 15, 16, 20, 21] and the references therein. In [15], Tada and Takahashi investigated fixed points of nonexpansive mappings and solutions of equilibrium problem (1.4). They obtained the following result.

Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences given by

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ u_n \in C, \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0,\ \forall\ y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S u_n, n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset [a,b]$ for some $a,b \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(F)$, where $w = \lim_{n\to\infty} F(S) \cap EP(F) x_n$.

In connection with the strong convergence, Tada and Takahashi [15] also introduced the following iterative scheme for approximating the common element. Their algorithm is as follows.

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ u_n \in C, \text{ such that } f(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \ \forall \ y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n T u_n, \\ C_n = \{z \in H : ||w_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\} \subset [a,b]$ for some $a,b \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. They proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in EP(f) \cap F(T)$, where $z = P_{EP(f)\cap F(T)}(x_0)$.

For finding an element in $F(T) \cap VI(C,A) \cap EP(f)$, Kumam [8] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ u_n \in C, \text{ such that } f(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \ \forall \ y \in C, \\ w_n = \alpha_n x_n + (1-\alpha_n) T P_C(u_n - \lambda_n A u_n), \\ C_n = \{z \in H : ||w_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 0, \end{cases} \tag{1.6}$$

where $A : C \to H$ is a γ-inverse strongly monotone mapping and T is a nonexpansive mapping. They proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap VI(C,A) \cap EP(f)$.

We remark that the computation of x_{n+1} in Algorithms (1.5) and (1.6) are not simple in applications because of the involvement of computation of C_{n+1} from C_n for each $n \geq 1$.

This brings us to the second concern in this paper

Question 2. Can we construct an iterative scheme for a common element in the fixed point set of a pseudocontractive mapping, the solution set of a variational inequality problem for a monotone mapping and the solution set of an equilibrium problem?

It is our purpose in this paper to introduce an iterative scheme $\{x_n\}$ which converges strongly to a common element in the fixed point set of a continuous pseudoconcontractive mapping, the solution set of a variational inequality problem for a Lipschitz monotone mapping and the solution set of an equilibrium problem. In addition, a numerical example which supports our main convergence result is presented. Our scheme does not involve computation of C_n and Q_n to obtain x_{n+1} for each $n \geq 1$. Our theorems extend and unify most of the results that have been proved for this important class of nonlinear mappings.

2. PRELIMINARIES

Let C be a nonempty, closed and convex subset of a real Hilbert space H. We recall that for each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, satisfying

$$||x - P_C x|| \leq ||x - y||, \quad \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping and is characterized by the following property (see, e.g., [17])

$$||y - P_C x||^2 \leq ||x - y||^2 - ||x - P_C x||^2, \quad \forall x \in H, y \in C. \tag{2.1}$$

In the sequel, we shall make use of the following lammas.

Lemma 2.1. [24] *Let C be a nonempty closed and convex subset of a real Hilbert space H. If $A : C \to H$ is continuous monotone mapping, then $VI(C,A)$ is closed and convex.*

Lemma 2.2. [23] *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0,1]$ the following equality holds:*

$$||\alpha x + (1-\alpha)y||^2 = \alpha ||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$$

Lemma 2.3. [17] *Let H be a real Hilbert space. Then for any given $x, y \in H$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle.$$

Lemma 2.4. [17] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.5. [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [11] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.7. [22] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x := \{z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \leq 0, \forall y \in C\}.$$

Then the following hold:
$(C1)$ T_r is single-valued;
$(C2)$ $F(T_r) = F(T)$;

(C3) $F(T)$ *is closed and convex.*
(C4) $||T_r x - p||^2 + ||T_r x - x||^2 \leq ||x - p||^2$, $\forall p \in F(T_r), x \in H$.

Lemma 2.8. [16] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows:*

$$F_r x := \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall\, y \in C\}$$

for all $x \in E$. Then the following hold:
(1) F_r is single-valued;
(2) $F(F_r) = EP(f)$;
(3) $EP(f)$ is closed and convex;
(4) $||F_r x - p||^2 + ||F_r x - x||^2 \leq ||x - p||^2$, $\forall p \in F(F_r), x \in H$.

3. Main results

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). In what follows, $T_{r_n}, F_{r_n} : H \to C$ are defined as follows.

For $x \in H$ and $\{r_n\} \subset (0,\infty)$ satisfying $\inf_{n\to\infty} r_n > 0$, define

$$T_{r_n} x := \{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \forall\, y \in C\}$$

and

$$F_{r_n} x := \{z \in C : f(z,y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \forall\, y \in C\}.$$

For the rest of this paper, P_C is the metric projection from H onto C and $\{\alpha_n\} \subset (0,c] \subset (0,1)$ for all $n \geq 0$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Now, we prove our main convergence theorem.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping and let $A : C \to H$ be a L-Lipschitz monotone mapping with Lipschitz constant L. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Assume that $\mathscr{F} = F(T) \cap VI(C,A) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)\left[\beta_n y_n + (1 - \beta_n) u_n\right], \end{cases} \tag{3.1}$$

where $y_n = F_{r_n} T_{r_n} x_n$, $u_n = P_C[x_n - \gamma_n A z_n]$, $\{\gamma_n\} \subset [a,b] \subset (0, \frac{1}{L})$ and $\{\beta_n\} \subset [e,1) \subset (0,1)$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathscr{F}$ which is nearest to u.

Proof. Let $p \in \mathscr{F}$ and $v_n = T_{r_n} x_n$. Then, we get that $y_n = F_{r_n} v_n$. From Lemmas 2.7 and 2.8 we have

$$\begin{aligned} ||y_n - v||^2 &= ||F_{r_n} v_n - F_{r_n} p||^2 \\ &\leq ||v_n - p||^2 - ||v_n - y_n||^2 \\ &\leq ||x_n - p||^2 - ||v_n - x_n||^2 - ||v_n - y_n||^2 \leq ||x_n - p||^2. \end{aligned} \tag{3.2}$$

From (2.1) and (3.1) we get

$$\begin{aligned} ||u_n - p||^2 &\leq ||x_n - \gamma_n A z_n - p||^2 - ||x_n - \gamma_n A z_n - u_n||^2 \\ &= ||x_n - p||^2 - ||x_n - u_n||^2 + 2\gamma_n \langle A z_n, p - u_n \rangle \\ &= ||x_n - p||^2 - ||x_n - u_n||^2 + 2\gamma_n \big(\langle A z_n - A p, p - z_n \rangle \\ &\quad + \langle A p, p - z_n \rangle + \langle A z_n, z_n - u_n \rangle \big) \\ &\leq ||x_n - p||^2 - ||x_n - u_n||^2 + 2\gamma_n \langle A z_n, z_n - u_n \rangle \\ &= ||x_n - p||^2 - ||x_n - z_n||^2 - ||z_n - u_n||^2 \\ &\quad + 2\langle \gamma_n A z_n - x_n + z_n, z_n - u_n \rangle. \end{aligned} \tag{3.3}$$

Using the Lipschitz property of A, we obtain

$$\begin{aligned} \langle \gamma_n A z_n - x_n + z_n, z_n - u_n \rangle &= \langle x_n - \gamma_n A x_n - z_n, u_n - z_n \rangle + \langle \gamma_n A x_n - \gamma_n A z_n, u_n - z_n \rangle \\ &\leq \langle \gamma_n A x_n - \gamma_n A z_n, u_n - z_n \rangle \\ &\leq \gamma_n L ||x_n - z_n|| ||u_n - z_n||. \end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4), we obtain

$$\begin{aligned} ||u_n - p||^2 &\leq ||x_n - p||^2 - ||x_n - z_n||^2 - ||z_n - u_n||^2 \\ &\quad + 2\gamma_n L ||x_n - z_n|| ||u_n - z_n|| \\ &\leq ||x_n - p||^2 - ||x_n - z_n||^2 - ||z_n - u_n||^2 \\ &\quad + \gamma_n L (||x_n - z_n||^2 + ||z_n - u_n||^2) \\ &\leq ||x_n - p||^2 + (\gamma_n L - 1) \big[||x_n - z_n||^2 + ||z_n - u_n||^2 \\ &\leq ||x_n - p||^2. \end{aligned} \tag{3.5}$$

Using (3.1), (3.2), (3.5), Lemma 2.2 and the fact that $L\gamma_n < 1$, we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n u + (1-\alpha_n)[\beta_n y_n + (1-\beta_n) u_n] - p||^2 \\ &= ||\alpha_n (u - p) + (1-\alpha_n)([\beta_n y_n + (1-\beta_n) u_n] - p)||^2 \\ &\leq \alpha_n ||u - p||^2 + (1-\alpha_n)\beta_n ||y_n - p||^2 \\ &\quad + (1-\alpha_n)(1-\beta_n) ||u_n - p||^2 \\ &\leq \alpha_n ||u - p||^2 + (1-\alpha_n)\beta_n ||x_n - p||^2 \\ &\quad + (1-\alpha_n)(1-\beta_n) ||x_n - p||^2 \\ &\leq \alpha_n ||u - p||^2 + (1-\alpha_n) ||x_n - p||^2. \end{aligned} \tag{3.6}$$

Therefore, by induction, we get that

$$||x_{n+1} - p||^2 \leq \max\{||u - p||^2, ||x_0 - p||^2\}, \quad \forall n \geq 0,$$

which implies that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded.

Let $x^* = \Pi_{\mathscr{F}} u$. From (3.1), (3.2), (3.5), Lemma 2.2 and 2.3, and the fact that $L\gamma_n < 1$, we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||\alpha_n u + (1-\alpha_n)[\beta_n y_n + (1-\beta_n)u_n] - x^*||^2 \\ &= ||\alpha_n (u - x^*) + (1-\alpha_n)[\beta_n y_n + (1-\beta_n)u_n - x^*]||^2 \\ &\leq (1-\alpha_n)||\beta_n (y_n - x^*) + (1-\beta_n)(u_n - x^*)||^2 \\ &\quad +2\langle x_{n+1} - x^*, \alpha_n (u - x^*)\rangle \\ &\leq (1-\alpha_n)[\beta_n ||y_n - x^*||^2 + (1-\beta_n)||u_n - x^*||^2] \\ &\quad +2\alpha_n \langle x_{n+1} - x^*, u - x^*\rangle \\ &\leq (1-\alpha_n)\beta_n \big(||x_n - x^*||^2 - ||v_n - x_n||^2 - ||v_n - y_n||^2\big) \\ &\quad +(1-\alpha_n)(1-\beta_n)\Big(||x_n - x^*||^2 + (L\gamma_n - 1)\big[||x_n - z_n||^2 + ||z_n - u_n||^2\big]\Big) \\ &\quad +2\alpha_n \langle x_{n+1} - x^*, u - x^*\rangle . \end{aligned}$$

Hence

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq (1-\alpha_n)||x_n - x^*||^2 - (1-\alpha_n)\beta_n (||v_n - x_n||^2 + ||v_n - y_n||^2) \\ &\quad +(1-\alpha_n)(1-\beta_n)(L\gamma_n - 1)\big[||x_n - z_n||^2 + ||z_n - u_n||^2\big] \\ &\quad +2\alpha_n \langle x_{n+1} - x^*, u - x^*\rangle \\ &\leq (1-\alpha_n)||x_n - x^*||^2 + 2\alpha_n ||x_{n+1} - x_n||||u - x^*|| \\ &\quad +2\alpha_n \langle x_n - x^*, u - x^*\rangle . \end{aligned} \tag{3.7}$$

Next, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - x^*||\}$ is decreasing for all $n \geq n_0$. Then, we get that, $\{||x_n - x^*||\}$ is convergent. Thus, from (3.7) and the fact that $\gamma_n < b < 1$ for all $n \geq 0$ and $\alpha_n \to 0$ as $n \to \infty$, we have that

$$v_n - x_n \to 0, v_n - y_n \to 0, x_n - z_n \to 0, z_n - u_n \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Moreover, from (3.1), we also have

$$\begin{aligned} ||x_{n+1} - x_n||^2 &= ||\alpha_n u + (1-\alpha_n)[\beta_n y_n + (1-\beta_n)u_n] - x_n||^2 \\ &\leq \alpha_n ||x_n - u||^2 + (1-\alpha_n)[\beta_n ||y_n - x_n||^2 + (1-\beta_n)||u_n - x_n||^2]. \end{aligned}$$

This together with (3.8) implies that

$$||x_{n+1} - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Furthermore, since $\{x_n\}$ is bounded in H, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$ and

$$\limsup_{n\to\infty} \langle x_n - x^*, u - x^*\rangle = \lim_{j\to\infty} \langle x_{n_j} - x^*, u - x^*\rangle .$$

This implies from (3.8) that $y_{n_j} \rightharpoonup z$, $u_{n_j} \rightharpoonup z$, $v_{n_j} \rightharpoonup z$ and $z_{n_j} \rightharpoonup z$ as $j \to \infty$.

Now, we show that $z \in F(T)$. From the definition of v_{n_j} we have

$$\langle y - v_{n_j}, Tv_{n_j}\rangle - \frac{1}{r_{n_j}}\langle y - v_{n_j}, (r_{n_j}+1)v_{n_j} - x_{n_j}\rangle \leq 0 \quad \forall y \in C. \tag{3.10}$$

Put $z_t = tv + (1-t)z$ for all $t \in (0,1]$ and $v \in C$. Consequently, we get $z_t \in C$. From (3.10) and pseudo-contractivity of T, it follows that

$$\begin{aligned}\langle v_{n_j} - z_t, Tz_t\rangle &\geq \langle v_{n_j} - z_t, Tz_t\rangle + \langle z_t - v_{n_j}, Tv_{n_j}\rangle - \frac{1}{r_{n_j}}\langle z_t - v_{n_j}, (1+r_{n_j})v_{n_j} - x_{n_j}\rangle \\ &= -\langle z_t - v_{n_j}, Tz_t - Tv_{n_j}\rangle - \frac{1}{r_{n_j}}\langle z_t - v_{n_j}, v_{n_j} - x_{n_j}\rangle \\ &\quad -\langle z_t - v_{n_j}, v_{n_j}\rangle \\ &\geq -||z_t - v_{n_j}||^2 - \frac{1}{r_{n_j}}\langle z_t - v_{n_j}, v_{n_j} - x_{n_j}\rangle - \langle z_t - v_{n_j}, v_{n_j}\rangle \\ &= \langle v_{n_j} - z_t, z_t\rangle - \langle z_t - v_{n_j}, \frac{v_{n_j} - x_{n_i}}{r_{n_j}}\rangle.\end{aligned}$$

Since $v_n - x_n \to 0$ as $n \to \infty$, we obtain that $\frac{v_{n_j} - x_{n_j}}{r_{n_j}} \to 0$ as $j \to \infty$. Thus, as $j \to \infty$, it follows that $\langle z - z_t, Tz_t\rangle \geq \langle z - z_t, z_t\rangle$. Hence $-\langle v - z, Tz_t\rangle \geq -\langle v - z, z_t\rangle$, $\forall v \in C$. Letting $t \to 0$ and using the fact that T is continuous, we obtain that $-\langle v - z, Tz\rangle \geq -\langle v - z, z\rangle$ $\forall v \in C$, which implies that $z = Tz$.

Next, we show that $z \in EP(f)$. From (A2), we note that

$$\frac{1}{r_n}\langle v - y_n, y_n - x_n\rangle \geq -f(y_n, v) = f(v, y_n) \ \forall\, v \in C, \tag{3.11}$$

which implies that $f(v,z) \leq 0$, $\forall\, v \in C$. Put $z_t = tv + (1-t)z$ for all $t \in (0,1]$ and $v \in C$. Consequently, we get that $z_t \in C$ and $f(z_t, z) \leq 0$. Therefore, from (A1), we obtain that

$$0 = f(z_t, z_t) \leq tf(z_t, v) + (1-t)f(z_t, z) \leq tf(z_t, v).$$

Thus, $f(z_t, v) \geq 0$, $\forall v \in C$. Furthermore, as $t \to 0$, we have from (A3) that $f(z, v) \geq 0$, for all $v \in C$. This implies that $z \in EP(f)$.

Next, we show that $z \in VI(C,A)$. Since A is Lipschitz continuous, we have

$$||Au_{n_j} - Az_{n_j}|| \to 0 \text{ as } j \to \infty.$$

Let

$$Bx = \begin{cases} Ax + N_C x, & x \in C \\ \emptyset, & x \notin C, \end{cases} \tag{3.12}$$

where $N_C(x)$ is the normal cone to C at $x \in C$ given by

$$N_C(x) = \{w \in H : \langle x - u, w\rangle \geq 0 \text{ for all } u \in C\}.$$

Then, B is maximal monotone and $0 \in Bx$ if and only if $x \in VI(C,A)$ (see, [14]). Let $(v,w) \in G(B)$. Then, we have $w \in Bv = Av + N_C v$ and hence $w - Av \in N_C v$. Thus, we get $\langle v - u, w - Av\rangle \geq 0$, for all $u \in C$. On the other hand, since $u_{n_j} = P_C(x_{n_j} - \gamma_{n_j} Az_{n_j})$ and $v \in C$, we have $\langle x_{n_j} - \gamma_{n_j} Az_{n_j} - u_{n_j}, u_{n_j} - v\rangle \geq 0$. Hence, $\langle v - u_{n_j}, (u_{n_j} - x_{n_j})/\gamma_{n_j} + Az_{n_j}\rangle \geq 0$. Thus, as $u_{n_j} \in C$, the above inequality implies that

$$\begin{aligned}\langle v - u_{n_j}, w\rangle &\geq \langle v - u_{n_j}, Av\rangle \\ &\geq \langle v - u_{n_j}, Av\rangle - \langle v - u_{n_j}, (u_{n_j} - x_{n_j})/\gamma_{n_j} + Az_{n_j}\rangle \\ &= \langle v - u_{n_j}, Av - Au_{n_j}\rangle + \langle v - u_{n_j}, Au_{n_j} - Az_{n_j}\rangle - \langle v - u_{n_j}, (u_{n_j} - x_{n_j})/\gamma_{n_j}\rangle \\ &\geq \langle v - u_{n_j}, Au_{n_j} - Az_{n_j}\rangle - \langle v - u_{n_j}, (u_{n_j} - x_{n_j})/\gamma_{n_j}\rangle.\end{aligned}$$

Therefore, we obtain that $\langle v-z,w\rangle \geq 0$. Then, the maximality of B gives that $z\in B^{-1}(0)$. Therefore, $z\in VI(C,A)$. Using Lemma 2.4, we immediately obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n-x^*,u-x^*\rangle &= \lim_{j\to\infty}\langle x_{n_j}-x^*,u-x^*\rangle\\ &= \langle z-x^*,u-x^*\rangle \leq 0. \end{aligned}\tag{3.13}$$

It follows from (3.7), (3.13) and Lemma 2.5 that $x_n\to x^*=\Pi_{\mathscr{F}}u$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$||x_{n_i}-x^*||^2<||x_{n_i+1}-x^*||^2,$$

for all $i\in\mathbb{N}$. Then, by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$, and

$$||x_{m_k}-x^*||\leq||x_{m_k+1}-x^*|| \text{ and } ||x_k-x^*||\leq||x_{m_k+1}-x^*||,\tag{3.14}$$

for all $k\in\mathbb{N}$. Now, from (3.7) and the facts that $\gamma_n<\frac{1}{L}$ for all $n\geq 0$ and $\alpha_n\to 0$ as $n\to\infty$, we get that $v_{m_k}-x_{m_k}\to 0, v_{m_k}-y_{m_k}\to 0$, $x_{m_k}-z_{m_k}\to 0$, and $z_{m_k}-u_{m_k}\to 0$ as $k\to\infty$. Thus, following the method in Case 1, we obtain

$$\limsup_{k\to\infty}\langle x_{m_k}-x^*,u-x^*\rangle\leq 0.\tag{3.15}$$

Now, from (3.7), we have that

$$\begin{aligned}||x_{m_k+1}-x^*||^2 &\leq (1-\alpha_{m_k})||x_{m_k}-x^*||^2+2\alpha_{m_k}\langle x_{m_k}-x^*,u-x^*\rangle\\ &\quad +2\alpha_{m_k}||x_{m_k+1}-x_{m_k}||||u-x^*||.\end{aligned}\tag{3.16}$$

Hence, (3.14) and (3.16) imply that

$$\begin{aligned}\alpha_{m_k}||x_{m_k}-x^*||^2 &\leq ||x_{mk}-x^*||^2-||x_{m_k+1}-x^*||^2+2\alpha_{m_k}\langle x_{m_k}-x^*,u-x^*\rangle\\ &\quad +2\alpha_{m_k}||x_{m_k+1}-x_{m_k}||||u-x^*||\\ &\leq 2\alpha_{m_k}\langle x_{m_k}-x^*,u-x^*\rangle+2\alpha_{m_k}||x_{m_k+1}-x_{m_k}||||u-x^*||,\end{aligned}\tag{3.17}$$

which implies that

$$||x_{m_k}-x^*||^2 \leq 2\langle x_{m_k}-x^*,u-x^*\rangle+2||x_{m_k+1}-x_{m_k}||||u-x^*||.$$

Thus, using (3.9) and (3.15) we get that $||x_{m_k}-x^*||\to 0$ as $k\to\infty$. This together with (3.16) implies that $||x_{m_k+1}-x^*||\to 0$ as $k\to\infty$. But $||x_k-x^*||\leq||x_{m_k+1}-x^*||$ for all $k\in\mathbb{N}$ gives that $x_k\to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point $x^*=\Pi_{\mathscr{F}}u$. The proof is complete. □

If, in Theorem 3.1, we assume that $A=0$, then we obtain the following corollary.

Corollary 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T:C\to H$ be a continuous pseudo-contractive mapping. Let $f:C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4). Assume that $\mathscr{F}=F(T)\cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0,u\in C$ by*

$$x_{n+1}=\alpha_n u+(1-\alpha_n)\left[\beta_n y_n+(1-\beta_n)x_n\right],$$

where $y_n = F_{r_n} T_{r_n} x_n$ *and* $\{\beta_n\} \subset [e,1) \subset (0,1)$. *Then,* $\{x_n\}$ *converges strongly to a point* x^* *in* $\mathscr{F}$ *which is nearest to u.*

If, in Theorem 3.1, we assume that $T = I$, the identity mapping on C, then we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let* $A : C \to H$ *be a L-Lipschitz monotone mapping with Lipschitz constant L. Let* $f : C \times C \to \mathbb{R}$ *be a disfunction satisfying (A1)-(A4). Assume that* $\mathscr{F} = VI(C,A) \cap EP(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0, u \in C$ *by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big[\beta_n y_n + (1-\beta_n) u_n\big], \end{cases}$$

where $y_n = F_{r_n} x_n$, $u_n = P_C[x_n - \gamma_n A z_n]$, $\{\gamma_n\} \subset [a,b] \subset (0, \frac{1}{L})$ *and* $\{\beta_n\} \subset [e,1) \subset (0,1)$. *Then,* $\{x_n\}$ *converges strongly to a point* x^* *in* $\mathscr{F}$ *which is nearest to u.*

If, in Theorem 3.1, we assume that $f = 0$, then we obtain the following corollary.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let* $T : C \to H$ *be a continuous pseudocontractive mapping and* $A : C \to H$ *be a L-Lipschitz monotone mapping with Lipschitz constant L. Assume that* $\mathscr{F} = F(T) \cap VI(C,A)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0, u \in C$ *by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big[\beta_n y_n + (1-\beta_n) u_n\big], \end{cases}$$

where $y_n = T_{r_n} x_n$, $u_n = P_C[x_n - \gamma_n A z_n]$, $\{\gamma_n\} \subset [a,b] \subset (0, \frac{1}{L})$ *and* $\{\beta_n\} \subset [e,1) \subset (0,1)$. *Then,* $\{x_n\}$ *converges strongly to a point* x^* *in* $\mathscr{F}$ *which is nearest to u.*

If, in Theorem 3.1, we assume that $f = A = 0$, then we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let* $T : C \to H$ *be a continuous pseudocontractive mapping. Assume that* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0, u \in C$ *by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\big[\beta_n T_{r_n} x_n + (1-\beta_n) x_n\big],$$

where $\{\beta_n\} \subset [e,1) \subset (0,1)$. *Then,* $\{x_n\}$ *converges strongly to a point* x^* *in* $F(T)$ *which is nearest to u.*

From Theorem 3.1, we can also obtain the following result on the common minimum norm solution for the fixed point problem of a continuous pseudocontractive mapping, the variational inequality problem for Lipschitz monotone mappings and the equilibrium problem.

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let* $T : C \to H$ *be a continuous pseudocontractive mapping and* $A : C \to H$ *be a L-Lipschitz monotone mapping with Lipschitz constant L. Let* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1)-(A4). Assume that* $\mathscr{F} = F(T) \cap VI(C,A) \cap EP(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0 \in C$ *by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = P_C\big[(1-\alpha_n)\big(\beta_n y_n + (1-\beta_n) u_n\big)\big], \end{cases} \tag{3.18}$$

where $y_n = F_{r_n} T_{r_n} x_n$, $u_n = P_C[x_n - \gamma_n A z_n]$, $\{\gamma_n\} \subset [a,b] \subset (0, \frac{1}{L})$ *and* $\{\beta_n\} \subset [e,1) \subset (0,1)$. *Then,* $\{x_n\}$ *converges strongly to a minimum norm point* x^* *of* $\mathscr{F}$.

Remark 3.1. Theorem 3.1 extends Theorem 3.1 and 4.1 of Tada and Takahashi [15] and Theorem 3 of Kumam [8] to a more general class of continuous pseudocontractive and monotone mappings. Our scheme does not involve computation of C_n and Q_n to obtain x_{n+1} for each $n \geq 1$. Corollary 3.3 extends Theorem 3.1 of Nadezhkina and Takahashi [12] and Theorem 3.1 of Zegeye and Shahzad [25] to a general class of continuous pseudocontractive mapping and Lipschitz monotone mappings. Our results provide affirmative answers to the questions raised in Section 1.

4. The numerical example

In this section, we give an example of a continuous pseudocontractive mapping T, a Lipschitz monotone mapping A and a bifunction f satisfying (A1)-(A4) with all the conditions of Theorem 3.1 and a numerical experiment result to support the conclusion of the theorem.

Example 4.1. Let $H = \mathbb{R}$ with the Euclidean norm. Let $C = [-1, 10]$ and let $T : C \to \mathbb{R}$ be a mapping defined by

$$Tx := \begin{cases} -4x - \frac{3}{2}, & x \in [-1, -\frac{1}{2}), \\ x, & x \in [-\frac{1}{2}, 10]. \end{cases}$$

Then, we see that $(I - T)$ is continuous and monotone and hence T is a continuous pseudocontractive mapping on C. In addition, if $x \in [-1, -\frac{1}{2})$, and $z \in [-1, \frac{1}{2})$, we have that

$$\langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C,$$

is equivalent to

$$\big[(1 + r)z - x + (4zr + \frac{3}{2}r)\big]y \geq \big[(1 + r)z - x + (4rz + \frac{3}{2}r)\big]z, \quad \forall y \in C.$$

But this holds, if $z = \frac{x - \frac{3r}{2}}{1 + 5r}$. If $x \in [-\frac{1}{2}, 10]$, we get from $z \in [-\frac{1}{2}, 10]$ that

$$\langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C,$$

is equivalent to $(y - z)z - \frac{1}{r}(y - z)\big[(1 + r)z - x\big] \leq 0$, $\forall y \in C$, which is further equivalent to $(z - x)y \geq (z - x)z$, $\forall y \in C$. But this holds, if $z = x$. Therefore, we get that

$$T_{r_n} x := \begin{cases} \frac{x - \frac{3r}{2}}{1 + 5r}, & x \in [-1, -\frac{1}{2}), \\ x, & x \in [-\frac{1}{2}, 10]. \end{cases}$$

Let $A : C \to \mathbb{R}$ be a mapping defined by

$$Ax := \begin{cases} 0, & x \in [-1, 1], \\ (x - 1)^2, & x \in (1, 10]. \end{cases}$$

Then, we easily see that A is monotone. Now, we show that A is Lipschitz.
Case 1: If $x, y \in [-1, 1]$, then

$$|Ax - Ay| = |0 - 0| \leq |x - y|.$$

Case 2: If $x \in [-1,1]$ and $y \in (1,10]$, then

$$\begin{aligned} |Ax - Ay| &= |0 - (y-1)^2| = (y-1)^2 \leq |y-x|^2 = |x-y|^2 \\ &\leq |x+y||x-y| \leq 11|x-y|. \end{aligned}$$

Case 3: If $x, y \in (1,10]$, then

$$\begin{aligned} |Ax - Ay| &= |(x-1)^2 - (y-1)^2| \\ &\leq |x^2 - y^2| + 2|x-y| \\ &\leq |x+y||x-y| + 2|x-y| \\ &\leq 20|x-y| + 2|x-y| = 22|x-y|. \end{aligned}$$

From Cases 1, 2 and 3, we obtain that A is Lipschitz with Lipschitz constant $L = 22$.
Let $f : C \times C \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} 0, & x \in [-1,0), \\ 2xy - 2x^2, & x \in (0,10]. \end{cases}$$

Then, we observe that $f(x,x) = 0$, $f(x,y) + f(y,x) \leq 0$, $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x,y,z \in C$ and for each $x \in C$, $y \to f(x,y)$ is convex and lower semicontinuous. Furthermore, if $x \in [-1,0)$, the inequality

$$F_r x = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}, \tag{4.1}$$

shows that we may take $F_r(x) = x$ and if $x \in [0,10]$, we obtain from (4.1) that

$$2r(zy - z^2) + (y-z)(z-x) \geq 0, \quad \forall y \in C,$$

which implies that $F_r(x) = z = \frac{x}{2r+1}$. Hence,

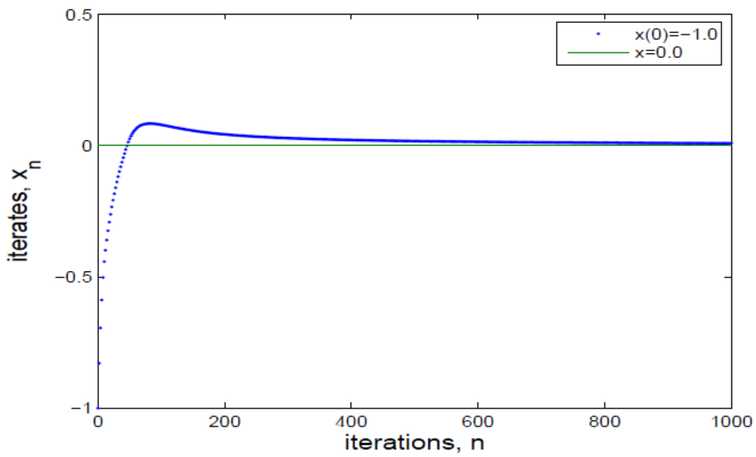
$$F_r(x) := \begin{cases} x, & x \in [-1,0), \\ \frac{x}{2r+1}, & x \in [0,10]. \end{cases}$$

It is also clear that $F(T) \cap VI(C,A) \cap EF(f) = [-\frac{1}{2}, 10] \cap [-1,1] \cap [-1,0] = [-\frac{1}{2}, 0]$. If $\alpha_n = \frac{1}{n+100}$, $\gamma_n = \frac{1}{n+100} + 0.01$, $\beta_n = \frac{1}{2n+100} + 0.05$, $r_n = 10$, $\forall n \geq 1$, and $u = 4.0$, then the conditions of Theorem 3.1 are satisfied and iterative scheme (3.1) is reduced to

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\left[\beta_n y_n + (1-\beta_n) u_n\right], \end{cases} \tag{4.2}$$

where $y_n = F_{r_n} T_{r_n} x_n$ and $u_n = P_C[x_n - \gamma_n A z_n]$. Thus, for $x_0 = -1.0$, the sequence generated in iterative scheme (4.2) converges strongly to $0 = P_{\mathscr{F}}(u)$. See the following table and Figure

n	0	100	200	500	1000	2000	3000	4000
x_n	-1.0000	0.00426	0.0211	0.0084	0.0042	0.0021	0.0014	0.00001



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