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A REMARK ON VARIATIONAL INEQUALITIES IN SMALL BALLS

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Abstract. In this paper, we prove the following result. Let $(H,\langle\cdot,\cdot\rangle)$ be a real Hilbert space, B a ball in H centered at 0 and $\Phi : B \to H$ a $C^{1,1}$ function with $\Phi(0) \neq 0$ such that the function $x \to \langle\Phi(x), x-y\rangle$ is weakly lower semicontinuous in B for all $y \in B$. Then, for each $r > 0$ small enough, there exists a unique point $x^* \in H$ with $\|x^*\| = r$ such that $\max\{\langle\Phi(x^*), x^*-y\rangle, \langle\Phi(y), x^*-y\rangle\} < 0$ for all $y \in H \setminus \{x^*\}$ with $\|y\| \leq r$.

Keywords. Variational inequality; $C^{1,1}$ function; Saddle-point; Ball.

1. INTRODUCTION

In the sequel, $(H,\langle\cdot,\cdot\rangle)$ is a real Hilbert space. For each $r > 0$, set

$$B_r = \{x \in H : \|x\| \leq r\}$$

and

$$S_r = \{x \in H : \|x\| = r\}.$$

Let $\Phi : B_r \to H$ be a given function.

We are interested in the classical variational inequality associated to Φ, which consists of finding $x_0 \in B_r$ such that

$$\sup_{y\in B_r}\langle\Phi(x_0), x_0 - y\rangle \leq 0. \tag{1.1}$$

If H is finite-dimensional, the mere continuity of Φ is enough to guarantee the existence of solutions, in view of the classical result of Hartman and Stampacchia [3]. This is no longer true when H is infinite-dimensional. Actually, in that case, Frasca and Villani [2], for each $r > 0$, constructed a continuous affine operator $\Phi : H \to H$ such that, for each $x \in B_r$,

$$\sup_{y\in B_r}\langle\Phi(x), x - y\rangle > 0.$$

We also mention the related wonderful paper [7]. Another existence result was obtained by assuming the following condition:

(a) for each $y \in B_r$, the function

$$x \to \langle\Phi(x), x-y\rangle$$

is weakly lower semicontinuous in B_r.

Such a result is a direct consequence of the famous Ky Fan minimax inequality [1].

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In particular, condition (a) is satisfied when Φ is weakly continuous and monotone (i.e. $\langle \Phi(x) - \Phi(y), x-y \rangle \geq 0$ for all $x,y \in B_r$). Moreover, when Φ is continuous and monotone, (1.1) is equivalent to the following inequality

$$\sup_{y \in B_r} \langle \Phi(y), x_0 - y \rangle \leq 0 \tag{1.2}$$

(see [6]).

On the basis of the above remarks, a quite natural question is to find non-monotone functions Φ such that there is a solution of (1.1) which also satisfies (1.2).

The aim of the present note is just to give a first contribution along this direction, assuming, besides condition (a), that Φ is of class $C^{1,1}$, with $\Phi(0) \neq 0$ (Theorem 2.3).

2. Results

We first establish the following saddle-point result.

Theorem 2.1. *Let Y be a non-empty closed convex set in a Hausdorff real topological vector space, let $\rho > 0$ and let $J : B_\rho \times Y \to \mathbf{R}$ be a function satisfying the following conditions:*
(a_1) *for each $y \in Y$, the function $J(\cdot, y)$ is C^1, weakly lower semicontinuous and $J'_x(\cdot, y)$ is Lipschitzian with constant L (independent of y);*
(a_2) *$J(x, \cdot)$ is upper semicontinuous and concave for all $x \in B_\rho$ and $J(x_0, \cdot)$ is sup-compact for some $x_0 \in B_\rho$;*
(a_3) $\delta := \inf_{y \in Y} \|J'_x(0,y)\| > 0$.

Then, for each $r \in \left]0, \min\left\{\rho, \frac{\delta}{2L}\right\}\right]$ and for each non-empty closed convex $T \subseteq Y$, there exist $x^* \in S_r$ and $y^* \in T$ such that

$$J(x^*, y) \leq J(x^*, y^*) \leq J(x, y^*)$$

for all $x \in B_r$, $y \in T$.

Proof. Fix $r \in \left]0, \min\left\{\rho, \frac{\delta}{2L}\right\}\right]$ and a non-empty closed convex $T \subseteq Y$. We also fix $y \in T$. Notice that the equation

$$J'_x(x,y) + Lx = 0$$

has no solution in $\mathrm{int}(B_r)$. Indeed, let $\tilde{x} \in B_\rho$ be such that

$$J'_x(\tilde{x}, y) + L\tilde{x} = 0.$$

In view of (a_1), we have

$$\|L\tilde{x} + J'_x(0,y)\| \leq \|L\tilde{x}\|.$$

In turn, using the Cauchy-Schwarz inequality, this readily implies that

$$\|\tilde{x}\| \geq \frac{\|J'_x(0,y)\|}{2L} \geq \frac{\delta}{2L} \geq r.$$

Using (a_1) again, we find that

$$x \to \frac{L}{2}\|x\|^2 + J(x,y)$$

is convex in B_ρ (see the proof of Corollary 2.7 of [5]). As a consequence, the set of its global minima B_r is non-empty and convex. But, by the remark above, this set is contained in S_r and hence it is a singleton. Thus, let $\hat{x} \in S_r$ be the unique global minimum of the restriction of the function

$$x \to \frac{L}{2}\|x\|^2 + J(x,y)$$

to B_r. So, we have

$$\frac{1}{2}\|\hat{x}\|^2 + J(\hat{x}, y) < \frac{1}{2}\|x\|^2 + J(x,y)$$

for all $x \in B_r \setminus \{\hat{x}\}$. Of course, this implies that

$$J(\hat{x}, y) < J(x, y)$$

for all $x \in B_r \setminus \{\hat{x}\}$. That is to say, $\hat{x}$ is the unique global minimum of $J(\cdot, y)_{|B_r}$. Hence, if we consider B_r with the weak topology, the restriction of J to $B_r \times T$ satisfies the assumptions of Theorem 1.2 of [4]. Consequently, we have

$$\sup_T \inf_{B_r} J = \inf_{B_r} \sup_T J.$$

Due the semicontinuity and compactness assumptions, this implies the existence of $x^* \in B_r$ and $y^* \in T$ such that

$$J(x^*, y) \leq J(x^*, y^*) \leq J(x, y^*)$$

for all $x \in B_r$, $y \in T$. Finally, observe that $x^* \in S_r$. Indeed, if $x^* \in$int(B_r), we would have

$$J'_x(x^*, y^*) = 0$$

and then

$$\delta \leq \|J'_x(0, y^*)\| \leq L\|x^*\| \leq \frac{\delta}{2},$$

an absurd. The proof is complete. □

Next, we give our main theorem.

Theorem 2.2. *Let $\rho > 0$ and let $\Phi : B_\rho \to H$ be a C^1 function whose derivative is Lipschitzian with constant γ. Moreover, assume that, for each $y \in B_\rho$, the function $x \to \langle \Phi(x), x - y\rangle$ is weakly lower semicontinuous. Set*

$$\theta := \sup_{x \in B_\rho} \|\Phi'(x)\|_{\mathscr{L}(H)},$$

$$M := 2(\theta + \rho\gamma)$$

and assume that

$$\sigma := \inf_{y \in B_\rho} \sup_{\|u\|=1} |\langle \Phi(0), u\rangle - \langle \Phi'(0)(u), y\rangle| > 0 .$$

Then, for each $r \in \left]0, \min\left\{\rho, \frac{\sigma}{2M}\right\}\right]$, there exists a unique $x^* \in S_r$ such that

$$\max\{\langle \Phi(x^*), x^* - y\rangle, \langle \Phi(y), x^* - y\rangle\} < 0$$

for all $y \in B_r \setminus \{x^*\}$.

Proof. Consider the function $J : B_\rho \times B_\rho \to \mathbf{R}$ defined by

$$J(x, y) = \langle \Phi(x), x - y\rangle$$

for all $x, y \in B_\rho$. Of course, for each $y \in B_\rho$, the function $J(\cdot, y)$ is C^1 and one has

$$\langle J'_x(x, y), u\rangle = \langle \Phi'(x)(u), x - y\rangle + \langle \Phi(x), u\rangle$$

for all $x \in B_\rho, u \in H$. Fixing $x, v \in B_\rho$ and $u \in S_1$, we have

$$\begin{aligned}
&|\langle J'_x(x, y), u\rangle - \langle J'_x(v, y), u\rangle| \\
&= |\langle \Phi(x) - \Phi(v), u\rangle + \langle \Phi'(x)(u), x - y\rangle - \langle \Phi'(v)(u), v - y\rangle| \\
&\leq \|\Phi(x) - \Phi(v)\| + |\langle \Phi'(x)(u) - \Phi'(v)(u), v - y\rangle + \langle \Phi'(x)(u), x - v\rangle| \\
&\leq \theta\|x - v\| + 2\rho\|\Phi'(x) - \Phi'(v)\|_{\mathscr{L}(H)} + \theta\|x - v\| \\
&\leq 2(\theta + \rho\gamma)\|x - v\|.
\end{aligned}$$

Hence, the function $J'_x(\cdot, y)$ is Lipschitzian with constant M. At this point, we can apply Theorem 2.1 by taking $Y = B_\rho$ with the weak topology. Therefore, for each $r \in \left]0, \min\left\{\rho, \frac{\sigma}{2M}\right\}\right]$, there exist $x^* \in S_r$ and $y^* \in B_r$ such that

$$\begin{aligned} \langle \Phi(x^*), x^* - y \rangle &\leq \langle \Phi(x^*), x^* - y^* \rangle \\ &\leq \langle \Phi(x), x - y^* \rangle \end{aligned} \tag{2.1}$$

for all $x, y \in B_r$. Notice that $\Phi(x^*) \neq 0$. Indeed, if $\Phi(x^*) = 0$, we would have

$$\|\Phi(0)\| = \|\Phi(0) - \Phi(x^*)\| \leq \theta r$$

and hence

$$r \leq \frac{\|\Phi(0)\|}{2M} < \frac{\|\Phi(0)\|}{\theta} \leq r$$

since $\sigma \leq \|\Phi(0)\|$. Consequently, the infimum in B_r of the linear functional $y \to \langle \Phi(x^*), y \rangle$ is equal to $-\|\Phi(x^*)\| r$ and is attained only at the point $-r\frac{\Phi(x^*)}{\|\Phi(x^*)\|}$. But, from the first inequality in (2.1), it just follows that y^* is the global minimum in B_r of the functional $y \to \langle \Phi(x^*), y \rangle$, and hence

$$y^* = -r\frac{\Phi(x^*)}{\|\Phi(x^*)\|}.$$

Moreover, from (2.1) (taking $y = x^*$ and $x = y^*$), we have

$$\langle \Phi(x^*), x^* - y^* \rangle = 0.$$

Consequently, we have

$$\begin{aligned} \langle \Phi(x^*), x^* \rangle &= \langle \Phi(x^*), y^* \rangle \\ &= \left\langle \Phi(x^*), -r\frac{\Phi(x^*)}{\|\Phi(x^*)\|} \right\rangle \\ &= -\|\Phi(x^*)\| r. \end{aligned}$$

Therefore, x^* is the global minimum in B_r of the functional $y \to \langle \Phi(x^*), y \rangle$ and hence $x^* = y^*$. Thus, (2.1) actually reads

$$\langle \Phi(x^*), x^* - y \rangle \leq 0 \leq \langle \Phi(x), x - x^* \rangle \tag{2.2}$$

for all $x, y \in B_r$. Finally, we fix $u \in B_r \setminus \{x^*\}$. By what seen above, the inequality

$$\langle \Phi(x^*), x^* - u \rangle < 0$$

is clear. Moreover, from the proof of Theorem 2.1, we know that, for each $y \in B_r$, the function $J(\cdot, y)_{|B_r}$ has a unique global minimum. But, the second inequality in (2.2) says that x^* is a global minimum of the function $J(\cdot, x^*)_{|B_r}$ and hence the inequality

$$\langle \Phi(u), x^* - u \rangle < 0$$

follows. Finally, to show the uniqueness of x^*, we argue by contradiction. Suppose that there is another $x_0 \in S_r$ with $x_0 \neq x^*$ such that

$$\max\{\langle \Phi(x_0), x_0 - y \rangle, \langle \Phi(y), x_0 - y \rangle\} < 0$$

for all $y \in B_r \setminus \{x_0\}$. So, we would have at the same time

$$\langle \Phi(x_0), x_0 - x^* \rangle < 0$$

and

$$\langle \Phi(x_0), x^* - x_0 \rangle < 0,$$

which is an absurd. The proof is complete. □

From Theorem 2.2, we obtain the following characterization.

Theorem 2.3. *Let $\rho > 0$ and let $\Phi : B_\rho \to H$ be a C^1 function with Lipschitzian derivative such that, for each $y \in B_\rho$, the function $x \to \langle \Phi(x), x-y\rangle$ is weakly lower semicontinuous.*

Then, the following assertions are equivalent:
(i) *for each $r > 0$ small enough, there exists a unique $x^* \in S_r$ such that*

$$\max\{\langle \Phi(x^*), x^* - y\rangle, \langle \Phi(y), x^* - y\rangle\} < 0$$

for all $y \in B_r \setminus \{x^*\}$;
(ii) $\Phi(0) \neq 0$.

Proof. The implication $(i) \to (ii)$ is clear. So, we assume that (ii) holds. Observe that the function

$$y \to \sup_{\|u\|=1} |\langle \Phi(0), u\rangle - \langle \Phi'(0)(u), y\rangle|$$

is continuous in H and takes the value $\|\Phi(0)\| > 0$ at 0. Consequently, for a suitable $r^* \in]0, \rho]$, we have

$$\inf_{y \in B_{r^*}} \sup_{\|u\|=1} |\langle \Phi(0), u\rangle - \langle \Phi'(0)(u), y\rangle| > 0.$$

At this point, we can apply Theorem 2.2 to the restriction of Φ to B_{r^*}, and (i) follows. □

Finally, it is also worth noticing the following further corollary of Theorem 2.2.

Theorem 2.4. *Let $\rho > 0$ and let $\Psi : B_\rho \to H$ be a C^1 function whose derivative vanishes at 0 and is Lipschitzian with constant γ_1. Moreover, assume that, for each $y \in B_\rho$, the function $x \to \langle \Psi(x), x-y\rangle$ is weakly lower semicontinuous. Set*

$$\theta_1 := \sup_{x \in B_\rho} \|\Psi'(x)\|_{\mathscr{L}(H)},$$

$$M_1 := 2(\theta_1 + \rho\gamma_1)$$

and let $w \in H$ satisfy

$$\|w - \Psi(0)\| \geq 2M_1\rho. \tag{2.3}$$

Then, for each $r \in]0, \rho]$, there exists a unique $x^* \in S_r$ such that

$$\max\{\langle \Psi(x^*) - w, x^* - y\rangle, \langle \Psi(y) - w, x^* - y\rangle\} < 0$$

for all $y \in B_r \setminus \{x^*\}$.

Proof. Set

$$\Phi := \Psi - w$$

and apply Theorem 2.2 to Φ. Since $\Phi' = \Psi'$, we have $M = M_1$. Since $\Phi'(0) = 0$, we have $\sigma = \|\Phi(0)\|$. Using (2.3), we have

$$\rho \leq \frac{\sigma}{2M}$$

and the conclusion follows. □

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