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ALGORITHMS FOR MONOTONE VECTOR VARIATIONAL INEQUALITIES

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Abstract. In this paper, we propose three algorithms for monotone vector variational inequalities. Depending on monotonicity assumptions of the underlying objective mapping, our algorithms are either based on a projection method or a regularization technique. At the end of this paper, we apply the algorithms to a finite-dimensional affine vector variational inequality and generate an approximation of the solution set.

Keywords. Vector variational inequality; Affine vector variational inequality; Scalarization; Orthogonal projection; Regularization.

1. INTRODUCTION

The origin of variational inequalities goes back to the work of Fichera [9], who formulated a contact problem in elasticity, the so-called Signorini problem, as a variational inequality. One year later, in 1964, the first cornerstone for the theory of variational inequalities was posed by Stampacchia [28]. The latter works started an intensive study of the subject by numerous celebrated researchers. Some prolific applications of variational inequalities can be found, for example, in economics [23], structural mechanics [27], optimization [14], physics [24, 31], and in many other fields of pure and applied mathematics. The reader can also be referred to the book of Kinderlehrer and Stampacchia [25] and the books of Facchinei and Pang [7, 8].

Later, in 1980, Giannessi [11] extended the notion of variational inequalities to the one of finite-dimensional vector variational inequalities. Within the last 40 years, vector variational inequalities have turned out to be a powerful tool for studying numerous mathematical models in applied and industrial mathematics, for example, in multi-objective optimization and related fields which consist of the simultaneous investigation of contrary tasks; see, e.g., Ansari, Köbis and Yao [1], Elster, Hebestreit, Khan and Tammer [6], Giannessi and Mastroeni [12] and Göpfert, Tammer and Zălinescu [17]. Furthermore, a detailed introduction to some of the recent developments in the field of vector variational inequalities and related problems can be found in the survey papers by Giannessi, Mastroeni and Yang [13] and Hebestreit [18].

In recent years, hundreds of papers were devoted to various and very important aspects of vector variational inequalities and generalizations, such as, existence results, scalarization methods, inverse results, gap functions, image space analysis, stability and sensitivity analysis and many

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others; see, e.g., [1, 13, 18] and the Chapter 9 in [16]. To the best of our knowledge, there are only a handful of papers dealing with numerical methods for vector variational inequalities. For instance, in [15, Section 4], Goh and Yang studied the following affine (finite-dimensional) vector variational inequality: Find $x \in C$ such that

$$\begin{pmatrix} \langle F_1(x), y-x \rangle \\ \vdots \\ \langle F_k(x), y-x \rangle \end{pmatrix} \notin -\operatorname{int}\mathbb{R}^k_{\geq}, \quad \forall y \in C, \tag{1.1}$$

where $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$ and C are assumed to be of the following form:

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_k(x) \end{pmatrix} := \begin{pmatrix} Q^1 x + q^1 \\ \vdots \\ Q^k x + q^k \end{pmatrix},$$

$$C := \left\{ x \in \mathbb{R}^l \mid \langle a^j, x \rangle - b^j = 0,\ j \in I^1, \langle a^j, x \rangle - b^j \leq 0,\ j \in I^2 \right\}.$$

In the above, $k, l \in \mathbb{N}$, $\mathbb{R}^k_{\geq}$ is the positive orthant in $\mathbb{R}^k$, $Q^1, \ldots, Q^k \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ are symmetric and positive semi-definite matrices and $q^1, \ldots, q^k \in \mathbb{R}^l$. Further, $d \geq 2$, $a^1, \ldots, a^d \in \mathbb{R}^l$, $b^1, \ldots, b^d \in \mathbb{R}$ and I^1 and I^2 are non-empty and disjoint index sets with $I^1 \cup I^2 = \{1, \ldots, d\}$. Using the fact that F is conservative, that is, $F = \nabla f$ for some smooth and $\mathbb{R}^k_{\geq}$-convex function $f : \mathbb{R}^l \to \mathbb{R}^k$, Goh and Yang applied a linear scalarization method and investigated the associated optimization problem

$$\min_{x \in C} s^\top \nabla f(x), \tag{1.2}$$

where $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$. By varying s and by an application of the active set method for problem (1.2), Goh and Yang [15] derived the first algorithm for the computation of solutions of affine vector variational inequality (1.1). It should be noted that vector variational inequality (1.1) with $k = 2$, $Q^1 = 0$, $Q^2 = \Sigma$, $q^1 = -\mu$, $q^2 = 0$ and $C = \{x \in \mathbb{R}^l_{\geq} \mid \|x\|_1 = 1\}$ has been considered in [21, Section 4] for the investigation of the inverse problem of parameter identification in the bicriterial Markowitz portfolio problem.

Notice further that due to the fact that F is conservative with $F = \nabla f$, it is known that problem (1.1) is equivalent [3, Proposition 3.3] to the multi-objective optimization problem

$$\mathrm{WEff}\left(f[C], \mathbb{R}^k_{\geq}\right) := \left\{ x \in C \mid \left(f(x) - \operatorname{int}\mathbb{R}^k_{\geq}\right) \cap f[C] = \emptyset \right\}, \tag{1.3}$$

where $f[C] := \{f(x) \mid x \in C\}$. Thus, under mild conditions for f, powerful methods from the field of multi-objective optimization may be applied to calculate solutions of problem (1.1) or (1.3), respectively; see, for example, [17, Chapter 4] and [22, Part IV].

Recently, Chen presented a proximal-type method [2] for vector variational inequality (1.1), where $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$ is a $\mathbb{R}^k_{\geq}$-monotone, continuous and norm sequentially bounded mapping and C is a non-empty, closed and convex subset of $\mathbb{R}^l$. For this purpose, the author introduced a set-valued mapping $\mathrm{VN}_C : \mathbb{R}^l \rightrightarrows \mathrm{Mat}_{k\times l}(\mathbb{R})$, the weak normal mapping, by $\mathrm{VN}_C(x) := \{U \in \mathrm{Mat}_{k\times l}(\mathbb{R}) \mid \langle U, y-x \rangle \notin \operatorname{int}\mathbb{R}^k_{\geq},$ for all $y \in C\}$ for $x \in \mathbb{R}^l$; see [2, Definition 2.8]. Then, given some sequences $\{s_n\} \subseteq \{s \in \mathbb{R}^k_{\geq} \mid \|s\|_1 = 1\}$ and $\{\varepsilon_n\} \subseteq \mathbb{R}_{>}$ with $0 < \varepsilon_n \leq \varepsilon$ for some $\varepsilon > 0$, the proposed proximal-type method in [2, Section 3] reads as follows:

1. Let $x_0 \in C$.

2. Let $n = 1$ and take $x_n \in C$. If x_n is a solution of vector variational inequality (1.1), then stop. Else go to the next step.
3. Define $x_{n+1} \in \mathbb{R}^l$ as a solution of the following inclusion problem:

$$0 \in s_n^\top F(x_{n+1}) + s_n^\top \mathrm{VN}_C(x_{n+1}) + \varepsilon_n(x_{n+1} - x_n).$$

Go to Step 2 and update n.

Chen showed that every sequence $\{x_n\}$, generated by the above method, converges to a solution of problem (1.1). Later, the results in [2] were extended by Chen, Pu and Wang [4] to an infinite-dimensional setting. In their results, they proposed a class of generalized proximal-type methods by the virtue of Bregman functions. To this end, they replaced Step 3 by

3*. Define $x_{n+1} \in X$ as a solution of the following inclusion problem:

$$0 \in s_n \circ F(x_{n+1}) + s_n \circ \mathrm{VN}_C(x_{n+1}) + \varepsilon_n(\nabla f(x_{n+1}) - \nabla f(x_n)),$$

Go to Step 2 and update n,

where X is a real Banach space and $f : X \to \mathbb{R}$ is a coercive Bregman function; see [4, Definition 2.9].

Inspired by the latter results, the algorithms in this paper are also based on a linear scalarization technique; see Proposition 2.1. To be more precise, we assume that C is a non-empty, closed and convex subset of the real Hilbert space X. Suppose further that K is a proper, closed, convex and solid cone in the real Banach space Y and let $F : X \to \mathrm{L}(X,Y)$ be a given mapping. For corresponding definitions, see the next section.

In this paper, we investigate the vector variational inequality of finding $x \in C$ such that

$$\langle F(x), y - x \rangle \notin -\operatorname{int} K, \quad \forall y \in C, \tag{1.4}$$

where $\langle F(x), y - x \rangle := F(x)(y - x)$. Now, let $\rho > 0$, $s \in K^* \setminus \{0\}$ and denote the composition of s and F by F_s. In Section 3 of this paper, we will show that every fixed-point of

$$\mathrm{Proj}\left(I - \rho^{-1} F_s\right) : C \to C,$$

is a solution of problem (1.4) (see Lemma 3.1). Thus, by applying Banach's fixed-point theorem, we propose a novel algorithm for the calculation of solutions of vector variational inequality (1.4); see Theorem 3.1. However, our methods require that F_s is strongly monotone and Lipschitz continuous. We therefore propose another method which is based on a regularization technique. By this, we only need to assume that F is K-monotone and v-hemicontinuous only. If, in addition, the regularizing mapping is appropriately chosen and X is a finite-dimensional real Euclidean space, the whole sequence of regularized solutions converges to a solution of problem (1.4); see Lemma 3.2. In order to further relax our monotonicity assumptions, we investigate the extragradient method, which requires the calculation of two projections per iteration (see Theorem 3.4). At the end of this paper, we apply our new algorithms to some finite-dimensional affine vector variational inequalities and generated an approximation of the solution set.

2. Notations and Preliminaries

In order to make this paper self contained, we briefly set forth below some important notations, definitions and results which we use here.

2.1. **Notations.** In the following, let X be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The Hilbert norm in X will be denoted by $\|\cdot\|_X$, that is, $\|\cdot\|_X := \sqrt{\langle\cdot,\cdot\rangle}$. If Y is another real Banach space, we denote by $\mathrm{L}(X,Y)$ the space of all linear and bounded operators from X to Y. With some abuse of the notation, we let $\langle A,x\rangle := A(x)$ for every $A\in\mathrm{L}(X,Y)$ and $x\in X$. Further, the duality pairing in Y will also be denoted by $\langle\cdot,\cdot\rangle$, that is, $\langle s,y\rangle := s(y)$ for every $s\in Y^*$ and $y\in Y$. In the case that X and Y are real finite-dimensional Euclidean spaces, say, $X=\mathbb{R}^l$ and $Y=\mathbb{R}^k$, where $k,l\in\mathbb{N}$ are positive integers, we use the identification $\mathrm{L}(\mathbb{R}^l,\mathbb{R}^k)\cong\mathrm{Mat}_{k\times l}(\mathbb{R})$. Further, $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^l$.

The Minkowski sum of two non-empty sets A and B in Y is defined by $A+B := \{a+b \mid a\in A,\ b\in B\}$ while the multiplication of a scalar $\lambda\in\mathbb{R}$ with A is defined by the rule $\lambda\cdot A := \{\lambda a \mid a\in A\}$. In particular, we let $-A := (-1)\cdot A$.

Now, let K be a non-empty subset of Y. We call K a cone if $\lambda\cdot K\subseteq K$ for every $\lambda\in\mathbb{R}_{\geq}$. The cone K is called convex if $K+K\subseteq K$, proper if $K\neq\{0\}$ and $K\neq Y$, closed if $\mathrm{cl}\,K=K$, pointed if $K\cap(-K)=\{0\}$ and solid if the topological interior $\mathrm{int}\,K$ is non-empty; see [17, Chapter 2] for more details. As usual, we denote the coordinates of any point $y\in\mathbb{R}^k$ with respect to the canonical basis $\{\mathrm{e}^1,\ldots,\mathrm{e}^k\}$ by $y_1,\ldots,y_k$, that is, $y=\sum_{j=1}^k y_j\mathrm{e}^j$. In $\mathbb{R}^k$, the non-negative orthant $\mathbb{R}^k_{\geq} := \{y=(y_1,\ldots,y_k)^\top \mid y_j\geq 0 \text{ for } j\in\{1,\ldots,k\}\}$ is a cone that enjoys all of the latter properties. We also use the notation $K^* := \{s\in Y^* \mid \langle s,y\rangle\geq 0 \text{ for every } y\in K\}$ for the dual cone of K. Notice that $(\mathbb{R}^k_{\geq})^*=\mathbb{R}^k_{\geq}$, that is, $\mathbb{R}^k_{\geq}$ is self-dual.

2.2. **Preliminaries.** The following basic definitions can be found, for example, in [31].

Definition 2.1. Let C be a non-empty subset of the real Hilbert space X and $A: X\to X$. Then A is said to be

(i) *monotone* if $\langle A(x)-A(y),x-y\rangle\geq 0$ for all $x,y\in X$,
(ii) *strictly monotone* if $\langle A(x)-A(y),x-y\rangle > 0$ for all $x,y\in X$ with $x\neq y$,
(iii) *strongly monotone* (with modulus c) if there exists $c>0$ such that $\langle A(x)-A(y),x-y\rangle\geq c\|x-y\|_X^2$ for all $x,y\in X$,
(iv) *pseudomonotone* on C if for all $x,y\in C$, $\langle A(x),y-x\rangle\geq 0$ implies $\langle A(y),y-x\rangle\geq 0$,
(v) *hemicontinuous* if the mapping $x\mapsto\langle A(x+ty),z\rangle$ is continuous at 0^+ for fixed $x,y,z\in X$,
(vi) *Lipschitz continuous* (with modulus L) if there exists $L>0$ such that $\|A(x)-A(y)\|_X\leq L\|x-y\|_X$ for all $x,y\in X$,
(vii) a *contraction* if the mapping is Lipschitz continuous with modulus $L\in(0,1)$.

Remark 2.1. Obviously, any strongly monotone operator is strictly monotone and every strictly monotone operator is monotone. Further, any monotone operator is pseudomonotone. It is also easy to see that any contraction is Lipschitz continuous and every (Lipschitz) continuous mapping is hemicontinuous.

Next we collect some useful properties of the orthogonal projection, which we use in the sequel.

Theorem 2.1 ([29, Section 1])**.** *Let C be a non-empty, closed and convex subset of the real Hilbert space X and let $x\in C$. Then, there exists exactly one element $p(x)\in C$ with*

$$\|x-p(x)\|_X = \inf_{y\in C}\|x-y\|_X. \tag{2.1}$$

The operator $\operatorname{Proj}: X \to C$ *with* $x \mapsto p(x)$ *is called (orthogonal) projection and has the following properties:*

(i) Proj *is non-expansive, that is,* $\|\operatorname{Proj}(x) - \operatorname{Proj}(y)\|_X \leq \|x-y\|_X$ *for every* $x, y \in X$.
(ii) *It holds* $\|\operatorname{Proj}(x)\|_X \leq \|x\|_X$ *for every* $x \in X$.
(iii) *Condition* (2.1) *is equivalent to* $\langle x - \operatorname{Proj}(x), \operatorname{Proj}(x) - y\rangle \geq 0$ *for every* $y \in C$. *The latter condition is called variational characterization.*

The following definitions can be found, for example, in [18, Section 4.1].

Definition 2.2. Let X be a real Hilbert space, K a proper, closed, convex and solid cone in the real Banach space Y and $F : X \to \mathrm{L}(X,Y)$. Then, F is said to be

(i) *K-monotone* if $\langle F(x) - F(y), x-y\rangle \in K$ for all $x, y \in X$,
(ii) *v-hemicontinuous* if the mapping $t \mapsto \langle F(x+ty), z\rangle$ is continuous at 0^+ for fixed elements $x, y, z \in X$.

The next proposition shows that vector variational inequality (1.4) can be completely characterized by scalar variational inequalities. Recall that F_s denotes the composition of $s \in Y^* \setminus \{0\}$ and $F : X \to \mathrm{L}(X,Y)$.

Proposition 2.1 ([5, Proposition 2.1])**.** *Let C be a non-empty, closed and convex subset of the real Hilbert space X. Further let K be a proper, closed, convex and solid cone in the Banach space Y and let $F : X \to \mathrm{L}(X,Y)$. Let $s \in Y^* \setminus \{0\}$ and consider the following variational inequality: Find $x = x(s) \in C$ such that*

$$\langle F_s(x), y-x\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Then, denoting the solution set of vector variational inequality (1.4) *and problem* (2.2) *by* $\operatorname{Sol}(\mathrm{VVI})$ *and* $\operatorname{Sol}(\mathrm{VI}_s)$*, respectively, we have*

$$\operatorname{Sol}(\mathrm{VVI}) = \bigcup_{s \in K^* \setminus \{0\}} \operatorname{Sol}(\mathrm{VI}_s).$$

Remark 2.2. In particular, any solution of problem (2.2) w.r.t. $s \in K^* \setminus \{0\}$ is a solution of vector variational inequality (1.4).

3. Main results

For convenience, we recall the general setting once again. Let C be a non-empty, closed and convex subset of the real Hilbert space X. Suppose further that K is a proper, closed, convex and solid cone in the real Banach space Y and let $F : X \to \mathrm{L}(X,Y)$. Then, the vector variational inequality, which will be studied in this paper, consists of finding $x \in C$ such that

$$\langle F(x), y-x\rangle \notin -\operatorname{int} K, \quad \forall y \in C. \tag{3.1}$$

Recall that we use the notation $\langle F(x), y-x\rangle = F(x)(y-x)$. In order to derive algorithms for the computation of solutions of problem (3.1), we will apply a linear scalarization technique; see Proposition 2.1. To this end, let $s \in K^* \setminus \{0\}$ and $\rho > 0$. Then, variational inequality (2.2) may be stated in the following way: Find $x \in C$ such that

$$\langle x, y-x\rangle \geq \langle x - \rho^{-1} F_s(x), y-x\rangle, \quad \forall y \in C,$$

where F_s is an operator from X to $X^* \cong X$. Consequently, by applying the orthogonal projection, problem (2.2) can be equivalently written as the following (parametric) fixed-point problem: Find $x \in C$ such that

$$\text{Proj}(x - \rho^{-1} F_s(x)) = x. \tag{3.2}$$

Thus, in order to find solutions of vector variational inequality (3.1), the necessary fixed-point problem (3.2) may be studied.

Our previous observations are summarized in the following lemma, where I denotes the identity operator from X to X.

Lemma 3.1. *Under the previously made assumptions, every fixed-point of* $\text{Proj}(I - \rho^{-1} F_s) : C \to C$ *is a solution of vector variational inequality* (3.1).

The next theorem is motivated by [8, Theorem 12.1.2] and based on Lemma 3.1. Under the assumption that F_s is strongly monotone and Lipschitz continuous, we are going to show that $\text{Proj}(I - \rho^{-1} F_s)$ is a contraction which makes it possible to apply Banach's fixed-point theorem to problem (3.2). Clearly, our method requires the evaluation of the orthogonal projection on the constraining set C. In general, such a projection method is conceptually simple and does not require the use of derivatives. Besides that, if the orthogonal projection on C is easily computable, the method becomes very cheap and fast.

Theorem 3.1 ([19, Theorem 6.1.1])**.** *Let C be a non-empty, closed and convex subset of the real Hilbert space X, let K be a proper, closed, convex and solid cone in the real Banach space Y, let $F : X \to \mathrm{L}(X,Y)$ and $\rho > 0$. Further let $x_0 \in C$ and $s \in K^* \setminus \{0\}$ and suppose that $F_s : X \to X$ is strongly monotone and Lipschitz continuous with modulus $c > 0$ and $L > 0$, respectively. If*

$$L^2 < 2c\rho, \tag{3.3}$$

and variational inequality (2.2) *w.r.t. s has a solution, then the iterative $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by*

$$x_{n+1} := x_{n+1}(\rho, s) := \text{Proj}(x_n - \rho^{-1} F_s(x_n)), \tag{3.4}$$

converges to a solution of vector variational inequality (3.1).

Proof. The proof of this theorem is based on Banach's fixed-point theorem and the linear scalarization method for vector variational inequalities (see [30, Theorem 1.A] and Proposition 2.1). We will therefore show that $\text{Proj}(I - \rho^{-1} F_s) : C \to C$ is a contraction. Let $x, y \in X$. Since Proj is non-expansive (see Theorem 2.1 (i)), we immediately have

$$\begin{aligned} &\| \text{Proj}(x - \rho^{-1} F_s(x)) - \text{Proj}(y - \rho^{-1} F_s(y)) \|_X \\ &\leq \|x - \rho^{-1} F_s(x) - y + \rho^{-1} F_s(y)\|_X. \end{aligned}$$

Further, by the strong monotonicity and Lipschitz continuity of F_s, we deduce

$$\begin{aligned} &\|x - \rho^{-1} F_s(x) - y + \rho^{-1} F_s(y)\|_X^2 \\ &= \|x - y\|_X^2 - 2\rho^{-1} \langle F_s(x) - F_s(y), x - y \rangle + \rho^{-2} \|F_s(x) - F_s(y)\|_X^2 \\ &\leq \left(1 - 2\rho^{-1} c + \rho^{-2} L^2\right) \|x - y\|_X^2. \end{aligned}$$

Finally, using relation (3.3), we have

$$1 - 2\rho^{-1} c + \rho^{-2} L^2 < 1,$$

which shows that $\mathrm{Proj}(I-\rho^{-1}F_s)$ is a contraction. Applying Banach's fixed-point theorem, the sequence $\{x_n\}$, given by (3.4), converges to the unique fixed-point of $\mathrm{Proj}(I-\rho^{-1}F_s)$. Invoking Lemma 3.1, the fixed-point is a solution of vector variational inequality (3.1) and the proof is complete. □

Remark 3.1. (i) Evidently, if for some $n \in \mathbb{N}_0$ it holds that $x_{n+1} = x_n$, then x_n is a solution of vector variational inequality (3.1) (see Lemma 3.1).

(ii) The convergence rate of $\{x_n\}$ is linear. Indeed, denoting the limit point of $\{x_n\}$ by x, we have $\mathrm{Proj}(x-\rho^{-1}F_s(x)) = x$ and it follows

$$\begin{aligned}\|x_{n+1}-x\|_X &= \|\mathrm{Proj}(x_n-\rho^{-1}F_s(x_n)) - \mathrm{Proj}(x-\rho^{-1}F_s(x))\|_X \\ &\leq (1-2\rho^{-1}c+\rho^{-2}L^2)^{\frac{1}{2}}\|x_n-x\|_X.\end{aligned}$$

Thus,

$$\limsup_{n\to+\infty}\frac{\|x_{n+1}-x\|_X}{\|x_n-x\|_X} = (1-2\rho^{-1}c+\rho^{-2}L^2)^{\frac{1}{2}} < 1.$$

(iii) In case we have $C = X$, the iterate $\{x_n\}$ becomes

$$x_{n+1} = x_n - \rho^{-1}F_s(x_n)$$

for $n \in \mathbb{N}_0$.

(iv) If in addition to the assumptions of Theorem 3.1 the set C is bounded or F_s satisfies some certain coercivity condition, then variational inequality (2.2) has a unique solution (see Theorem 1.4 and Corollary 1.8 in [25]).

Under the assumptions of Theorem 3.1, the latter method can be summarized in the following algorithm.

Algorithm 1: Basic projection method

Result: Calculation of a solution of vector variational inequality (3.1).

1 **Input:** The set C, $x_0 \in C$, $s \in K^* \setminus \{0\}$, $F : X \to \mathrm{L}(X,Y)$ and $\rho > 0$.

2 $\mathrm{Sol(VVI)} \leftarrow \emptyset$.

3 $n \leftarrow 0$.

4 **if** $x_n = \mathrm{Proj}(x_n - \rho^{-1}F_s(x_n))$ **then**

5     $\mathrm{Sol(VVI)} \leftarrow \mathrm{Sol(VVI)} \cup \{x_n\}$ **stop**.

6 **else**

7     $x_{n+1} \leftarrow \mathrm{Proj}(x_n - \rho^{-1}F_s(x_n))$.

8     $n \leftarrow n+1$.

9   Go to line 4.

10 **end**

11 **Output:** An element of the solution set $\mathrm{Sol(VVI)}$.

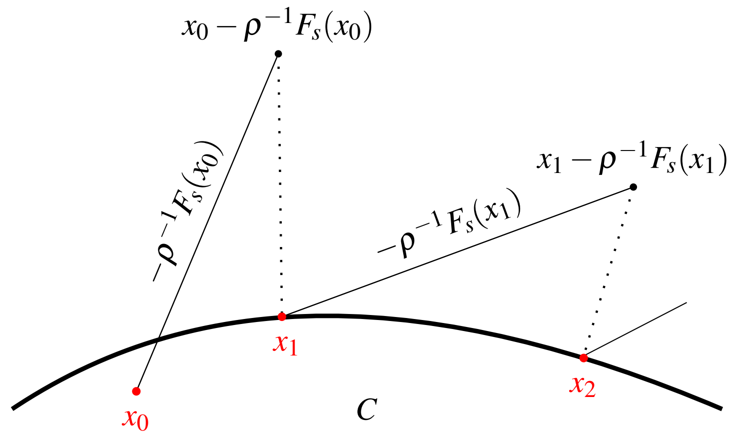


FIGURE 1. Illustration of the basic projection method

In general, the computation of the strongly monotonicity or Lipschitz continuity modul is not an easy task. However, by replacing the fixed step size ρ^{-1} with a variable step size ρ_n, no knowledge of c and L is needed; see also Exercise 12.8.2 in [8] and [10].

Theorem 3.2. *Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X, let K be a proper, closed, convex and solid cone in the real Banach space Y and let $F : X \to \mathrm{L}(X,Y)$. Further let $x_0 \in C$ and $s \in K^* \setminus \{0\}$ and suppose that $F_s : X \to X$ is strongly monotone and Lipschitz continuous. If $\{\rho_n\} \subseteq \mathbb{R}_{\geq}$ is a sequence with*

$$\lim_{n\to+\infty} \rho_n = 0 \qquad \text{and} \qquad \sum_{n=0}^{+\infty} \rho_n = +\infty,$$

and variational inequality (2.2) w.r.t. s has a solution, then the iterative $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by

$$x_{n+1} := x_{n+1}(\rho_n, s) := \mathrm{Proj}(x_n - \rho_n F_s(x_n)),$$

converges to a solution of vector variational inequality (3.1).

Proof. This proof follows similar to the one of Theorem 3.1 and is therefore omitted. □

We have shown that, under a strong monotonicity and Lipschitz continuity assumption for F_s, the sequence $\{x_n\}$ in (3.4) converges strongly to a solution of vector variational inequality (3.1). In order to weaken the assumptions in Theorem 3.1, we apply a regularization technique; see also [8, 20, 25]. We therefore assume that $F : X \to \mathrm{L}(X,Y)$ is K-monotone and v-hemicontinuous only which is a standard assumption. Roughly speaking, we will replace problem (2.2) by a family of well-behaving and regularized variational inequalities. Due to some nice features of the regularizing mapping R only, $F_s + \varepsilon_n R$ has significantly better properties than F_s and problem (3.6) attains a (unique) solution. Depending on R, we will show that the sequence $\{x_n\}$ of regularized solutions has an accumulation point or a limit point that solves vector variational inequality (3.1).

Theorem 3.3. *Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X and let K be a proper, closed, convex and solid cone in the real Banach space Y. Assume that $F : X \to \mathrm{L}(X,Y)$ is K-monotone and v-hemicontinuous, $R : X \to X$ is strongly* *monotone and hemicontinuous and $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ is a sequence of positive parameters with $\varepsilon_n \downarrow 0$. Suppose further that vector variational inequality (3.1) has a solution. If C is unbounded and there exists $\tilde{x} \in C$ such that*

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\langle R(x) - R(\tilde{x}), x - \tilde{x} \rangle}{\|x - \tilde{x}\|_X} = +\infty, \tag{3.5}$$

then we can find $s \in K^* \setminus \{0\}$ such that the iterative $\{x_n\}$, where for every $n \in \mathbb{N}$, the element $x_n := x(\varepsilon_n, s) \in C$ solves the regularized variational inequality

$$\langle F_s(x_n) + \varepsilon_n R(x_n), y - x_n \rangle \geq 0, \quad \forall y \in C, \tag{3.6}$$

has a subsequence that converges to a solution of vector variational inequality (3.1).

Proof. Since vector variational inequality (3.1) attains a solution, we deduce from Proposition 2.1 that there exists $s \in K^* \setminus \{0\}$ such that variational inequality (2.2) w.r.t. s has a non-empty solution set $\mathrm{Sol}(\mathrm{VI}_s)$. Consider the regularized variational inequality (3.6), that is, given s and $\varepsilon_n > 0$, we are looking for $x_n = x(\varepsilon_n, s) \in C$ such that

$$\langle F_s(x_n) + \varepsilon_n R(x_n), y - x_n \rangle \geq 0, \quad \forall y \in C.$$

We shall show that all of the regularized problems attain a (unique) solution. Indeed, since F is K-monotone and v-hemicontinuous, F_s is monotone and hemicontinuous (see the proof of Theorem 3.9 in [20]). Further, due to the monotonicity of F_s, we immediately have

$$\langle S_n(x) - S_n(\tilde{x}), x - \tilde{x} \rangle \geq \langle R(x) - R(\tilde{x}), x - \tilde{x} \rangle,$$

for every $x \in C$, where $S_n := F_s + \varepsilon_n R$ and $\tilde{x} \in C$ is given by relation (3.5). Thus, S_n also satisfies coercivity condition (3.5). Applying [25, Corollary 1.8] and using the fact that S_n is strictly monotone and hemicontinuous, every regularized problem (3.6) attains a unique solution. Thus, the sequence $\{x_n\}$ of regularized solutions is well-defined. In order to show that $\{x_n\}$ produces a sequence that converges to a solution of vector variational inequality (3.1), let $x^* \in \mathrm{Sol}(\mathrm{VI}_s)$ be arbitrarily chosen, that is, invoking Minty's lemma [25, Lemma 1.5], it holds that $x^* \in C$ satisfies

$$\langle F_s(y), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{3.7}$$

Consequently, inserting x_n in problem (3.7) and x^* in the regularized problem (3.6), respectively, we conclude

$$\langle R(x_n), x^* - x_n \rangle \geq 0. \tag{3.8}$$

By the strong monotonicity of R, we can find $c > 0$ with

$$\langle R(x^*) - R(x_n), x^* - x_n \rangle \geq c\|x^* - x_n\|_X^2.$$

Thus, combining the previous inequalities, we finally have

$$c\|x^* - x_n\|_X^2 \leq \langle R(x^*), x^* - x_n \rangle \leq \|R(x^*)\|_X \|x^* - x_n\|_X.$$

The previous line shows that $\{x_n\}$ is bounded. Since X is finite-dimensional, there exists a convergent subsequence $\{x_{n_j}\}$ with $x_{n_j} \to x$ and $x \in C$. In what follows, we will show that x

solves vector variational inequality (3.1). Indeed, invoking [25, Lemma 1.5] once again, we have

$$\langle F_s(y) + \varepsilon_{n_j} R(y), y - x_{n_j} \rangle \geq 0, \quad \forall y \in C.$$

Then, passing in the above variational inequality to the limit and using the fact that $F_s(y) + \varepsilon_{n_j} R(y) \to F_s(y)$ for every $y \in C$, we deduce that $x \in C$ satisfies

$$\langle F_s(y), y - x \rangle \geq 0, \quad \forall y \in C.$$

Hence, x is a solution of variational inequality (2.2) w.r.t. $s \in K^* \setminus \{0\}$ and therefore one of vector variational inequality (3.1); see [25, Lemma 1.5] and Proposition 2.1. The proof is complete. □

Remark 3.2. Theorem 3.3 remains correct if C is bounded. In this case, we can drop condition (3.5) and it is enough to assume that R is strictly monotone and hemicontinuous only.

If the regularizing operator $R : X \to X$ in Theorem 3.3 is the identity operator, we even have that the whole sequence of regularized solutions converges to a solution of problem (3.1).

Lemma 3.2. *Let the assumptions of Theorem 3.3 hold. If R is the identity operator, then there exists $s \in K^* \setminus \{0\}$ such that the iterative $\{x_n\}$, where for every $n \in \mathbb{N}$, $x_n := x(\varepsilon_n, s) \in C$ solves the regularized variational inequality*

$$\langle F_s(x_n) + \varepsilon_n x_n, y - x_n \rangle \geq 0, \quad \forall y \in C, \tag{3.9}$$

converges to a solution of vector variational inequality (3.1).

Proof. Following the proof of Theorem 3.3, every problem (3.9) attains a (unique) solution and $\{x_n\}$ is well-defined. Since X is a real Hilbert space, it is strictly convex and we can find a unique least-norm solution $x^* \in C$ of problem (2.2); see [31]. By inequality (3.8), we have $\langle x_n, x^* - x_n \rangle \geq 0$, which implies

$$\|x_n\|_X \leq \|x^*\|_X. \tag{3.10}$$

Thus, $\{x_n\}$ is bounded and we can find a convergent subsequence $\{x_{n_j}\}$ with $x_{n_j} \to x$ and $x \in C$. Again, following the arguments in the proof of Theorem 3.3, we have $x \in \mathrm{Sol}\,(\mathrm{VI}_s)$. But passing in inequality (3.10) to the limit gives $x = x^*$. We have therefore that all subsequences of $\{x_n\}$ have the same limit point. Thus, the whole sequence $\{x_n\}$ converges to the least-norm solution x^*, which is, in view of Proposition 2.1, a solution of vector variational inequality (3.1). The proof is complete. □

Until now, the proposed algorithms have executed one projection per iteration at most. In 1976, Korpelevich [26] introduced the so-called extragradient method which requires the calculation of two orthogonal projections per iteration. The idea behind this method is to rewrite fixed-point problem (3.2) in the following way:

$$x = \mathrm{Proj}(x - \rho F_s(y)), \qquad x = y.$$

This seems artificial but it allows us to introduce a new iteration and to decide on different update rules for the artificial variable y. By introducing an extra projection, the strongly monotonicity of F_s can be weakened to a pseudomonotonicity assumption only. See [7, 8, 31] for some extragradient based methods for variational inequalities, quasi-variational inequalities and optimization problems.

The following theorem is motivated by [8, Theorem 12.1.11]. Notice that we assume that X is a finite-dimensional Hilbert space once again.

Theorem 3.4 ([19, Theorem 6.1.11])**.** *Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X, let K be a proper, closed, convex and solid cone in the real Banach space Y, let $F : X \to \mathrm{L}(X,Y)$ and $\rho > 0$. Further let $x_0 \in C$ and $s \in K^* \setminus \{0\}$ and assume that variational inequality (2.2) w.r.t. s attains a solution. Suppose that $F_s : X \to X$ is Lipschitz continuous with modulus $L > 0$ and pseudomonotone on C. If*

$$0 < \rho < \frac{1}{L}, \tag{3.11}$$

then the iterative $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by

$$\begin{aligned} y_n &:= y_n(\rho, s) := \mathrm{Proj}(x_n - \rho F_s(x_n)), \\ x_{n+1} &:= x_{n+1}(\rho, s) := \mathrm{Proj}(x_n - \rho F_s(y_n)), \end{aligned} \tag{3.12}$$

converges to a solution of vector variational inequality (3.1).

Proof. Using the pseudomonotonicity of F_s and the variational characterization of the projection operator (see Theorem 2.1 (iii)), one can show that, for every $\tilde{x} \in \mathrm{Sol}(\mathrm{VI}_s)$ and every $n \in \mathbb{N}_0$,

$$\|x_{n+1} - \tilde{x}\|_X^2 \leq \|x_n - \tilde{x}\|_X^2 - (1 - \rho^2 L^2)\|y_n - x_n\|_X^2, \tag{3.13}$$

with $\{x_n\}$ and $\{y_n\}$ generated by formula (3.12) (see Lemma 12.1.10 in [8]). According to (3.13), we further have, for every $\tilde{x} \in \mathrm{Sol}(\mathrm{VI}_s)$ and $N \in \mathbb{N}$,

$$(1 - \rho^2 L^2) \sum_{n=0}^{N} \|x_n - y_n\|_X^2 \leq \sum_{n=0}^{N} \|x_n - \tilde{x}\|_X^2 - \|x_{n+1} - \tilde{x}\|_X^2 \leq \|x_0 - \tilde{x}\|_X^2,$$

which implies the convergence of the series $\sum_{n=0}^{+\infty} \|x_n - y_n\|_X^2$. We thus get

$$\lim_{n \to +\infty} \|x_n - y_n\|_X = 0. \tag{3.14}$$

From relation (3.11) and (3.13), we have that $\{x_n\}$ is bounded. Now let $\{x_{n_j}\}$ be an appropriate subsequence with $x_{n_j} \to x$ and $x \in C$. By (3.14), we also have $y_{n_j} \to x$. Finally, passing in (3.12) to the limit yields

$$x = \lim_{j \to +\infty} y_{n_j} = \lim_{j \to +\infty} \mathrm{Proj}(x_{n_j} - \rho F_s(x_{n_j})) = \mathrm{Proj}(x - \rho F_s(x)),$$

where the last equation follows from the continuity of $\mathrm{Proj}(I - \rho F_s)$ (see also Theorem 2.1 (i)). Thus, $x \in \mathrm{Sol}(\mathrm{VI}_s)$. In order to complete the proof, we have to show that the whole sequence $\{x_n\}$ converges to x. To this end, it is enough to apply (3.13) with $\tilde{x}$ replaced with x. Following the previous arguments, the sequence $\{\|x_n - x\|_X\}$ is monotonically decreasing and therefore convergent. Since

$$\lim_{n \to +\infty} x_n = \lim_{j \to +\infty} x_{n_j} = x,$$

the whole sequence converges to a solution of problem (2.2) and consequently to one of vector variational inequality (3.1). The proof is complete. □

Remark 3.3. (i) Under some local error bound assumptions for the solution set $\mathrm{Sol}(\mathrm{VI}_s)$, one can show that the sequence $\{x_n\}$, generated by (3.12), converges to a solution $x \in C$ of vector variational inequality (3.1) at least R-linearly; see Section 12.6 in [8]. Here, R-linearity is understood in the sense that it holds

$$0 < \limsup_{n \to +\infty} \|x_n - x\|_X^{\frac{1}{n}} < 1.$$

(ii) If $C = X$, then the iterates $\{x_n\}$ and $\{y_n\}$, given by (3.12), become $y_n = x_n - \rho F_s(x_n)$ and $x_{n+1} = x_n - \rho F_s(y_n)$ for $n \in \mathbb{N}_0$.

Under the assumptions of Theorem 3.4, we have the following new algorithm for vector variational inequality (3.1).

Algorithm 2: Extragradient method

Result: Calculation of a solution of vector variational inequality (3.1).
1 **Input:** The set C, $x_0 \in C$, $s \in K^* \setminus \{0\}$, $F : X \to \mathrm{L}(X,Y)$ and $\rho > 0$.
2 $\mathrm{Sol}(\mathrm{VVI}) \leftarrow \emptyset$.
3 $n \leftarrow 0$.
4 **if** $x_n = \mathrm{Proj}(x_n - \rho F_s(x_n))$ **then**
5 $\quad \mathrm{Sol}(\mathrm{VVI}) \leftarrow \mathrm{Sol}(\mathrm{VVI}) \cup \{x_n\}$ **stop**.
6 **else**
7 $\quad y_n \leftarrow \mathrm{Proj}(x_n - \rho F_s(x_n))$.
8 $\quad x_{n+1} \leftarrow \mathrm{Proj}(x_n - \rho F_s(y_n))$.
9 $\quad n \leftarrow n+1$.
10 $\quad$ Go to line 4.
11 **end**
12 **Output:** An element of the solution set $\mathrm{Sol}(\mathrm{VVI})$.

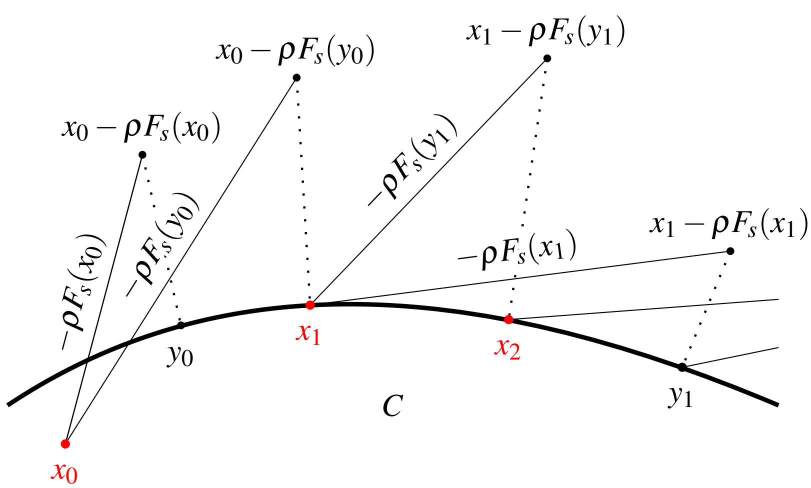


FIGURE 2. Illustration of the extragradient method

4. Applications

In what follows, we will apply Algorithm 1 and 2 in Section 3 to some finite-dimensional test problems.

4.1. A motivational example.

The following example has been intensively studied in [18] and will serve as a prototype example:

Given a non-empty, closed and convex set $C \subseteq \mathbb{R}^2$ and two different points $d^1, d^2 \in \mathbb{R}^2$, we are looking for all points $x \in C$ such that the Euclidean distance between $x - d^1$ and $x - d^2$ is minimal simultaneously. Here, minimization is understood in the sense that it is impossible to decrease the distance to d^1 (or d^2, respectively) without increasing the distance to d^2 (or d^1, respectively) at the same time. Using some geometric arguments, it can be easily shown that an element $x \in C$ is optimal (in the above sense) if for every $y \in C$ the angles between $x - d^1$ and $x - y$ as well as between $x - d^2$ and $x - y$ are not bigger than 90° at the same time, that is, either $\measuredangle(x - d^1, x - y) \leq 0$ or $\measuredangle(x - d^2, x - y) \leq 0$ for every $y \in C$. This is equivalent to saying that the element $x \in C$ is a solution of the following vector variational inequality: Find $x \in C$ such that

$$\begin{pmatrix} \langle x - d^1, y - x \rangle \\ \langle x - d^2, y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^2_{\geq}, \quad \forall y \in C.$$

Further, introducing a mapping $F : \mathbb{R}^2 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ by

$$F_1(x) := Ix - d^1 \quad \text{and} \quad F_2(x) := Ix - d^2,$$

where $I \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ denotes the unit matrix, the above problem can be written in form of the following affine vector variational inequality: Find $x \in C$ such that

$$\begin{pmatrix} \langle F_1(x), y - x \rangle \\ \langle F_2(x), y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^2_{\geq}, \quad \forall y \in C.$$

Denoting the solution set of the above problem by S, we have the following three cases, depending on the position of d^1 and d^2:

(i) $S = [d^1, d^2]$ if $d^1, d^2 \in C$.
(ii) $S = [d^1, d^2] \cap C$ if $d^j \in C$ for some $j \in \{1, 2\}$.
(iii) $S = \emptyset$ if $d^1, d^2 \notin C$.

4.2. Preliminary results and observations.

Motivated by the latter example, we are going to study the following (finite-dimensional) affine vector variational inequality: Let $k, l \in \mathbb{N}$ and denote by C a non-empty, closed and convex subset of $\mathbb{R}^l$. Further, let $q^1, \ldots, q^k \in \mathbb{R}^l$ and $Q^1, \ldots, Q^k \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ and define the affine mapping $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$ by

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_k(x) \end{pmatrix} := \begin{pmatrix} Q^1 x + q^1 \\ \vdots \\ Q^k x + q^k \end{pmatrix}. \tag{4.1}$$

Then, the affine vector variational inequality, which will be studied in this section, consists of finding $x \in C$ such that

$$\begin{pmatrix} \langle F_1(x), y - x \rangle \\ \vdots \\ \langle F_k(x), y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^k_{\geq}, \quad \forall y \in C. \tag{4.2}$$

For further use, given a vector $s = (s_1, \dots, s_k)^\top \in \mathbb{R}^k_\geq \setminus \{0\}$, we define the mapping $F_s : \mathbb{R}^l \to \mathbb{R}^l$ by

$$F_s(x) := \sum_{j=1}^{k} s_j F_j(x) = \sum_{j=1}^{k} s_j (Q^j x + q^j). \tag{4.3}$$

Notice that we have $F_{e^j} = F_j$, where e^j denotes the jth unit vector in $\mathbb{R}^k_\geq$. Now let $A \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ be a symmetric and positive semi-definite matrix, that is, $A = A^\top$ and $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^l$. It is easy to see that $\langle x, y\rangle_A := \langle Ax, y\rangle$ for $x, y \in \mathbb{R}^l$ defines an inner product in $\mathbb{R}^l$ and consequently, $x \mapsto \|x\|_A := \sqrt{\langle x, x\rangle_A}$ induces a norm in $\mathbb{R}^l$. One can further show that it holds

$$\lambda_{\min}(A)\|x\|_2^2 \leq \|x\|_A^2 \leq \lambda_{\max}(A)\|x\|_2^2,$$

for every $x \in \mathbb{R}^l$, where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denotes the smallest and biggest positive eigenvalue of A, respectively; see [7].

In order to apply the algorithms of Section 3, we have to ensure that the data of problem (4.2) satisfy all the required assumptions of Theorem 3.1 and 3.4, respectively. For this purpose, we introduce the following assumption, where 0 denotes the null matrix in $\mathrm{Mat}_{l\times l}(\mathbb{R})$.

(A_Q) For $j \in \{1, \dots, k\}$, the matrices $Q^j \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ are symmetric and positive semi-definite and it holds $Q^j \neq 0$.

We now collect some useful properties of F and F_s, given by (4.1) and (4.3), respectively.

Lemma 4.1. *Assume that assumption* (A_Q) *holds. Then*

(i) *F is $\mathbb{R}^k_\geq$-monotone and v-hemicontinuous,*
(ii) *for every vector $s \in \mathbb{R}^k_\geq \setminus \{0\}$, F_s is strongly monotone and Lipschitz continuous.*

Proof. Fix $j \in \{1, \dots, k\}$ and let $x, y \in \mathbb{R}^l$. We immediately have

$$\langle F_j(x) - F_j(y), x - y\rangle = \|x - y\|_{Q^j}^2 \geq \lambda_{\min}(Q^j)\|x - y\|_2^2$$

and

$$\|F_j(x) - F_j(y)\|_2 = \|Q^j(x - y)\|_2 \leq \|Q^j\|_2\|x - y\|_2,$$

where $\|Q^j\|_2$ denotes the spectral norm of Q^j, that is,

$$\|Q^j\|_2 = \sqrt{\lambda_{\max}(Q^j Q^j)}.$$

Thus, the mapping $F_j : \mathbb{R}^l \to \mathbb{R}^l$ is strongly monotone and Lipschitz continuous. Consequently, it is easily seen that F_s is strongly monotone and Lipschitz continuous with modulus

$$c(s, Q^1, \dots, Q^k) := \sum_{j=1}^{k} s_j \lambda_{\min}(Q^j) \qquad \text{and} \qquad L(s, Q^1, \dots, Q^k) := \sum_{j=1}^{k} s_j \|Q^j\|_2, \tag{4.4}$$

respectively. Since all components of F are monotone, F is $\mathbb{R}^k_\geq$-monotone. Finally. the v-hemicontinuity trivially holds since F is continuous. The proof is complete. □

The following lemma shows that all corresponding scalar variational inequalities of problem (4.2) attain a (unique) solution.

Lemma 4.2. *Assume that assumption* (A_Q) *holds. Then, affine vector variational inequality* (4.2) *attains a solution. Further, for every vector* $s=(s_1,\dots,s_k)^\top \in \mathbb{R}^k_{\geq}\setminus\{0\}$, *the corresponding variational inequality w.r.t. s has a unique solution. In other words, there exists a unique vector* $x=x(s)\in C$ *such that*

$$\langle \sum_{j=1}^{k} s_j(Q^j x+q^j), y-x\rangle \geq 0, \quad \forall y\in C. \tag{4.5}$$

Proof. Let us consider the case where C is bounded first. Then, invoking Brouwer's fixed-point theorem, see [30, Proposition 2.6], the operator $\mathrm{Proj}(I-F_s): C\to C$ attains a fixed-point which is a solution of variational inequality (4.5) (see Lemma 3.1). Since F_s is strictly monotone (see Lemma 4.1 (ii)), problem (4.5) has a unique solution. Further, applying Proposition 2.1, the fixed-point also solves affine vector variational inequality (4.2). However, if C is unbounded, we shall show that F_s is coercive in the sense of Corollary 1.8 in [25]. Indeed, let $\tilde{x}\in C$ be arbitrarily chosen. Since F_s is strongly monotone with modulus c, we conclude

$$\lim_{\substack{\|x\|_2\to+\infty\\ x\in C}} \frac{\langle F_s(x)-F_s(\tilde{x}), x-\tilde{x}\rangle}{\|x-\tilde{x}\|_2} \geq \lim_{\substack{\|x\|_2\to+\infty\\ x\in C}} c\|x-\tilde{x}\|_2 = +\infty,$$

where $c:=c(s,Q^1,\dots,Q^k)$ is given by relation (4.4). Thus, applying [25, Corollary 1.8] and Proposition 2.1, the proof is complete. □

Remark 4.1. (i) The proof of Theorem 4.2 shows that assumption (A_Q) can be dropped if C is bounded in addition.
(ii) Let us consider affine vector variational inequality (4.2) where the constraining set C is the whole space $\mathbb{R}^l$. Applying Proposition 2.1, we immediately have that the solution set of problem (4.2) is given by

$$\mathrm{Sol}\,(\mathrm{AVVI}) = \bigcup_{(s_1,\dots,s_k)^\top\in\mathbb{R}^k_{\geq}\setminus\{0\}} \left\{ -\left(\sum_{j=1}^{k} s_j Q^j\right)^{-1} \sum_{j=1}^{k} s_j q^j \right\}. \tag{4.6}$$

Apparently, even though ever (positive) linear combination of $Q^1,\dots,Q^k$ is non-singular, the calculation of $(s_1Q^1+\ldots+s_kQ^k)^{-1}$ in terms of ${Q^1}^{-1},\dots,{Q^k}^{-1}$ is not an easy task in general. One can therefore not expect to find a closed form for formula (4.6).

4.3. **Numerical illustration and approximation of the solution set.** In what follows, we assume that assumption (A_Q) holds. Thus, in view of Lemma 4.1, the mapping F_s is strongly monotone, Lipschitz continuous and pseudomonotone in particular. Therefore, we are in position to apply Algorithm 1 and 2 to some test problems. For this purpose, we introduce the fundamental data set

$$\mathscr{D} := \{k,l,r,q^1,\dots,q^k,Q^1,\dots,Q^k,s,x_0\},$$

which we use for the numerical illustration of our methods. The parameters of $\mathscr{D}$ can be described in the following way:

(i) dimension of $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively: $k,l\in\mathbb{N}$,
(ii) constraining set: $C:=\{x\in\mathbb{R}^l \mid \|x\|_2\leq r\}$ is a closed ball with center 0 and radius $r>0$, where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^l$,

(iii) affine mapping $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$: $F = (F_1,\dots,F_k)^\top$ with $F_j(x) = Q^j x + q^j$, $q^j \in \mathbb{R}^l$ and $Q^j \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ for $j \in \{1,\dots,k\}$,
(iv) scalarizing vector: $s \in \mathbb{R}^k_{\geq}$,
(v) initial value: $x_0 \in C$.

In the next figure, we compare the convergence rate of the basic projection and the extragradient method; see the Algorithms 1 and 2. We took $l = k = 100$, and uniformly randomly generated the following data: $r \in [1,100]$, $q^j \in [0,1000]^l$ and $Q^j \in [-1000,1000]^l$ satisfying assumption (A_Q) for $j \in \{1,\dots,k\}$, $s \in [0,1]^k \setminus \{0\}$ and $x_0 \in [0,1]^l$ with $\|x_0\| \leq r$. The corresponding parameters ρ_{BPM} and ρ_{EGM} were chosen to be

$$\rho_{\mathrm{BPM}} := \frac{L^2}{2c} + 1 \quad \text{and} \quad \rho_{\mathrm{EGM}} := \frac{1}{2L},$$

respectively, where $c := c(s,Q^1,\dots,Q^k)$ and $L := L(s,Q^1,\dots,Q^k)$ are given in (4.4). It should be noted that the above parameters satisfy condition (3.3) and (3.11), respectively.

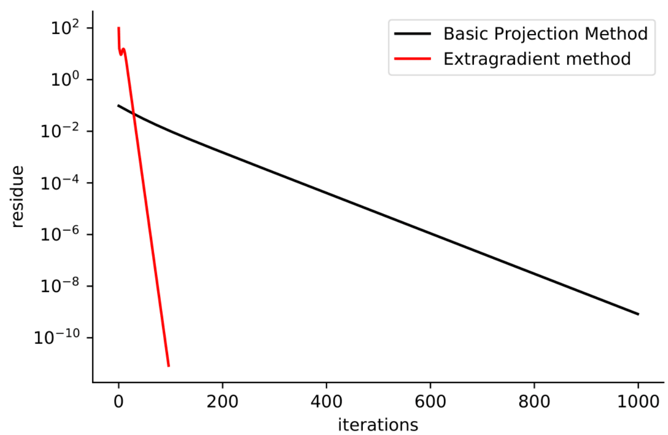


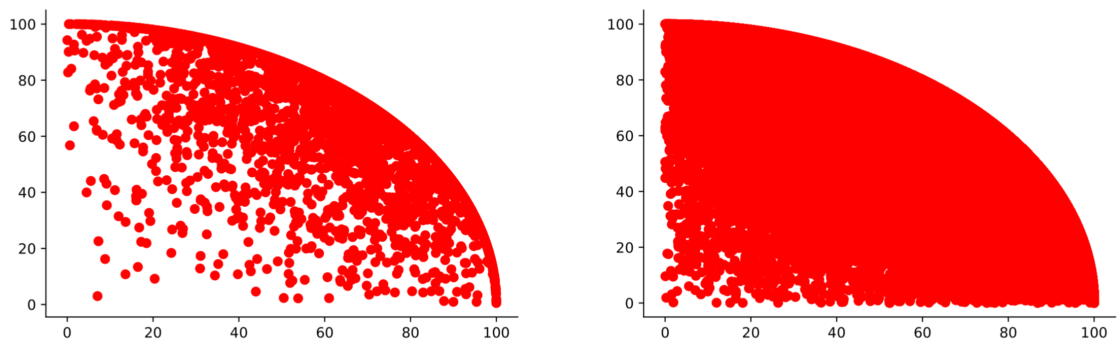
FIGURE 3. Comparison of the basic projection and the extragradient method

Clearly, the fundamental idea of our algorithms consists of the numerical generation of solutions of corresponding variational inequalities; see Proposition 2.1. Thus, by applying Algorithm 1 or 2, we find one solution of affine vector variational inequality (4.2) only. However, by varying the vector $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$, it is possible to approximate the solution set Sol(AVVI). Let $\kappa \in \mathbb{N}$ and denote by S_κ a finite subset of $\mathbb{R}^k_{\geq} \setminus \{0\}$ with cardinality κ. We then have

$$\bigcup_{s\in S_\kappa} \{x(s)\} \subseteq \mathrm{Sol}(\mathrm{AVVI}), \tag{4.7}$$

where $x(s)$ denotes the unique solution of problem (4.5) w.r.t. the vector s (see Proposition 2.1). The following figures show $U_\kappa := \bigcup_{s\in S_\kappa}\{x(s)\}$ for $\kappa \in \{10000, 500000\}$, where we let $k = 4$, $l = 2$, $r = 100$, $Q^j = I$ and $q^j = -d^j$ for $j \in \{1,\dots,4\}$ with $d^1 = (0,0)^\top$, $d^2 = (100,0)^\top$, $d^3 = (0,100)^\top$ and $d^4 = (100,100)^\top$. It should be noted that, using the above data, it holds

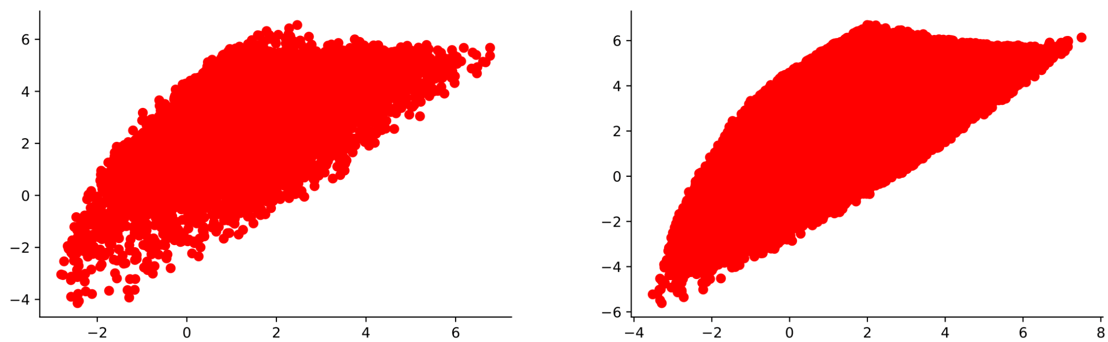
$$\mathrm{Sol}(\mathrm{AVVI}) = \left\{x \in \mathbb{R}^4 \mid x \in \mathbb{R}^4_{\geq} \text{ and } \|x\|_2 \leq 100\right\}.$$



(A) Illustration of U_κ for $\kappa = 10\,000$

(B) Illustration of U_κ for $\kappa = 500\,000$

The next figures show U_κ for $\kappa \in \{10\,000, 500\,000\}$, where we let $k = 4$, $l = 2$, $r = 100$, $q^j = -d^j$ for $j \in \{1,\ldots,4\}$, where $d^1 = (0,0)^\top$, $d^2 = (100,0)^\top$, $d^3 = (0,100)^\top$ and $d^4 = (100,100)^\top$. Here, the symmetric and positive semi-definite matrices Q^j are uniformly randomly generated from $[-1000,1000]^4$ for $j \in \{1,\ldots,4\}$.



(C) Illustration of U_κ for $\kappa = 10\,000$

(D) Illustration of U_κ for $\kappa = 500\,000$

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