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ON DIAMETRICALLY MAXIMAL SETS, MAXIMAL PREMONOTONE OPERATORS AND PREMONOTONE BIFUNCTIONS

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Honoring Juan Enrique Martínez-Legaz in his 70th birthday

Abstract. First, we study diametrically maximal sets in the Euclidean space (those which are not properly contained in a set with the same diameter), establishing their main properties. Then, we use these sets for exhibiting an explicit family of maximal premonotone operators. We also establish some relevant properties of maximal premonotone operators, like their local boundedness, and finally we introduce the notion of premonotone bifunctions, presenting a canonical relation between premonotone operators and bifunctions, that extends the well known one, which holds in the monotone case.

Keywords. Diametrically maximal sets; Premonotone operators; Premonotone bifunctions.

1. Introduction

We recall that, given a Hilbert space H, a point-to-set operator $T : H \rightrightarrows H$ is said to be *monotone* whenever $\langle u-v, x-y\rangle \geq 0$ for all $x,y \in H$, all $u \in T(x)$ and all $v \in T(y)$. Monotone operators have been proved to be an essential concept in several pure and applied areas. A monotone operator is said to be *maximal* when its graph is not properly contained in the graph of another monotone operator. Maximal monotone operators enjoy several important properties. One of them, known as Minty's Theorem, states that the sum of a maximal monotone operator and the identity operator is onto, and its inverse is point-to-point (see [10]).

Several extensions of the class of monotone operators have been considered in the literature, like *submonotone* operators (see [12]), *hypo-monotone* operators (see [8, 11]), etc. Another of these extensions, which the one of interest in this paper, is the notion of *premonotone* operators, which was introduced in [7], where it was shown that, under adequate assumptions, they enjoy a surjectivity property akin to Minty's Theorem for maximal monotone operators. They are defined as follows.

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Given a function $\sigma : \mathbb{R}^n \to [0,\infty)$, a point-to-set operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be σ-*premonotone* if $\langle u-v, x-y\rangle \geq -\min\{\sigma(x),\sigma(y)\}\,\|x-y\|$ for all $x,y \in \mathbb{R}^n$, all $u \in T(x)$ and all $v \in T(y)$.

An adequate notion of *maximal premonotone* operators, which extend the concept of maximal monotone operators, was developed in [7]. Several relevant results on premonotone operators can be found in [1, 2, 9].

In Section 3, we will deepen the results on maximal premonotone operators given in [7], establishing several of their properties, like, e.g., the local boundedness. We will also define the notion of *premonotone hull* of a premonotone operator, extending again the similar notion for the monotone case.

An obstacle confronted in [7] refers to the difficulty in exhibiting explicit instances of maximal premonotone operators in dimension larger than 1. In fact, the example of this kind given in [7] consists of the sum of a monotone operator with a fixed ball. We overcome this obstacle through the introduction of *diametrically maximal sets*, meaning those sets $\in \mathbb{R}^n$, which are not properly contained in a set with the same *diameter*, defined as the maximal distance between points in the set. We study this class of sets in Section 2. We present sets of this kind which are not balls, prove that they have non-empty interior, and establish several other interesting properties.

Diametrically maximal sets are then used in Section 4 for constructing an explicit class on maximal premonotone operators in $\mathbb{R}^n$; namely, the sums of the form $T(x) = U(x) + C(x)$, where U is a strongly monotone operator and $C(x)$ is an inner Lipschitz semicontinuous operator such that $C(x)$ is a diametrically maximal set for all $x \in \mathbb{R}^n$.

The theory of monotone operators $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has been extended to functions $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, to be called *bifunctions* in the sequel. The prototypical bifunction associated to an operator T is given by

$$f(x,y) = \sup_{u \in T(x)} \langle u, y-x\rangle. \tag{1.1}$$

Bifunctions turned out to be the right notion for moving from variational inequalities, related to operators, to abstract equilibrium problems (see [3]). A bifunction is said to be monotone when $f(x,y) + f(y,x) \leq 0$ for all $x,y \in \mathbb{R}^n$. Note that, with this definition, f as in (1.1) in monotone if and only if T is monotone.

When $f(x,\cdot)$ is convex for all $x \in \mathbb{R}^n$, we can associate to a bifunction f the operator T defined as $T(x) = \partial f(x,\cdot)(x)$, i.e. the subdifferential of the bifunction in its second argument evaluated at its first argument. Maximality of this operator, called the *diagonal subdifferential*, was proved in [6]. Under adequate additional assumptions on the bifunction and the operator, this relation, together with one given by (1.1), are mutual inverses. We will call it the *canonical relation* between operators and bifunctions.

In Section 5, we give an appropriate definition of *premonotone bifunction*, extending the notion of monotone bifunctions, establish several properties of this class of bifunctions, and extend to premonotone operators and premonotone bifunctions the above mentioned canonical relation.

We close this section with a remark. Most of the materials in this paper can be extended without much difficulty to infinite dimensional spaces, at least to Hilbert ones. We have refrained to do so in order to avoid a few technicalities which might unnecessarily complicate some proofs.

We intend to continue this research project with a rather complete generalization of our current results on diametrically maximal sets, maximal premonotone operators and premonotone bifunctions to the infinite dimensional realm.

2. DIAMETRICALLY MAXIMAL SETS

Given a bounded set $C \subset \mathbb{R}^n$, its *diameter* $\operatorname{diam}(C)$ is defined as $\operatorname{diam}(C) = \sup\{\|a-b\| : a, b \in C\}$.

Definition 2.1. A bounded set $C \subset \mathbb{R}^n$ is said to be *diametrically maximal* (*diam-max* for short), if for all set $D \subset \mathbb{R}^n$ such that $C \subset D$ and $\operatorname{diam}(C) = \operatorname{diam}(D)$, it holds that $C = D$.

Diam-max sets are closely related to *constant width sets*, defined as those convex sets for which the distance between any pair of parallel supporting hyperplanes is the same (see [4]).

It follows immediately from the definition that closed balls are diam-max sets. We will devote most of this section to exhibit diam-max sets that are not balls. We continue with some elementary properties of max-diam sets.

Proposition 2.1. *Diametrically maximal sets are closed and convex.*

Proof. If C is diam-max and there exists $x \in \operatorname{cl}(C) \setminus C$, then the set $C \cup \{x\}$ has the same diameter as C, contradicting the maximality of C. Hence C is closed.

Assume now that C is diam-max and closed. Take any x, y in the convex hull of C. It is known that x, y are convex combinations of finite subsets of C, and then it follows easily that $\|x-y\| \leq \operatorname{diam}(C)$. Thus, C and its convex hull have the same diameter. It follows from the maximality of C that it coincides with its convex hull, and hence it is convex. □

Proposition 2.2. *Take $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, and a diam-max set $C \subset \mathbb{R}^n$. Then the set $\alpha C + a$ is diam-max.*

Proof. Elementary. □

Proposition 2.3. *Every bounded set $C \subset \mathbb{R}^n$ is contained in a diam-max set with the same diameter.*

Proof. It follows from a standard application of Zorn's Lemma. □

Given a bounded set C, any max-diam set containing it and having the same diameter will be said to be a *diam-max extension* of C.

It would be nice if every bounded set C had a unique diam-max extension (we'd call it the *diam-max hull* of C), but this is not true, as the following example shows.

Example 2.1. Consider a right triangle $T \subset \mathbb{R}^2$, with equal legs, so that T is also isosceles. Let a, b be the extreme points of the hypothenuse and c the vertex at the right angle. It is immediate that $\operatorname{diam}(T) = \|a-b\|$. Consider the circle S centered at $\frac{a+b}{2}$ with diameter $\|a-b\|$. It is rather immediate that c belongs to S (in fact, it belongs to its boundary), so that $T \subset S$. Since T and S have the same diameter, and circles are diam-max, S is a diam-max extension of T. We will exhibit another one. Consider a point $c' \in \mathbb{R}^2$ such that a, b, c' are the vertices of an equilateral triangle, say T'. Clearly, $T \subset T'$ and $c' \notin S$. It is also immediate that the diameter of an equilateral triangle is the common length of its edges, in this case $\|a-b\|$, so that $\operatorname{diam}(T') = \operatorname{diam}(T)$. By Proposition 2.3, T' has some diam-max extension, say S', which is obviously a diam-max extension of T because T and T' have the same diameter. Since $c' \in T' \subset S'$ and $c' \notin S$, it follows that S and S' are two different diam-max extensions of S'.

Later on, we will explicitly construct a diam-max extension of an equilateral triangle. First, since we are particularly interested in diam-max sets, which are not balls, we will give necessary and sufficient conditions for a set to have a ball as a diam-max extension.

Given a set $C \subset \mathbb{R}^n$, a pair $(a,b) \in C \times C$ is said to be *antipodal* if $\|a-b\| = \operatorname{diam}(C)$. If (a,b) is antipodal, the point $\frac{a+b}{2}$ is said to be a *mid-diam* point of C. Clearly, a set can have many antipodal pairs and just one mid-diam pair; a circle in $\mathbb{R}^2$, for instance, has an infinite number of antipodal pairs, but its only mid-diam point is its center. Regular polygons in $\mathbb{R}^2$ with an even number m of edges have $\frac{m}{2}$ antipodal pairs and only one mid-diam point (again, its center), while regular polygons in $\mathbb{R}^2$ with an odd number m of edges have m antipodal pairs and m mid-diam points (which are the the vertices of a smaller regular polygon). Our characterization result for sets admitting a ball as a diam-max extension is as follows.

Proposition 2.4. *A bounded set $C \subset \mathbb{R}^n$ admits a ball as a diam-max extension if and only if it has a unique mid-diam point, and C is contained in the closed ball B centered at the mid-diam point with diameter equal to* $\operatorname{diam}(C)$ *(in this case, B is a diam-max extension of C).*

Proof. We prove the "if" statement. By definition, $\operatorname{diam}(B) = \operatorname{diam}(C)$. Since $C \subset B$ and closed balls are diam-max sets, B is a diam-max extension of C.

Now we deal with the "only if" statement. Assume that C has a ball B as a diam-max extension. By definition, $C \subset B$ and $\operatorname{diam}(B) = \operatorname{diam}(C)$. Take any antipodal pair (a,b) of C, and let $c = \frac{a+b}{2}$. Since $C \subset B$, we have that $a,b \in B$. Note that $\|a-b\| = \operatorname{diam}(C) = \operatorname{diam}(B)$. An elementary property of a ball ensures that if it contains two points a,b such that the distance between them is the diameter of the ball, then the mid point between them must be the center of the ball. So B is centered at $\frac{a+b}{2}$. Since (a,b) is an arbitrary antipodal pair of C, it follows that there exists a unique mid-diam point, and the ball B is the one specified in the statement of the proposition. □

It follows from Proposition 2.4 that no diam-max extension of a set C with more than one mid-diam point can be a ball. Maximal extensions of such sets (including, e.g., all regular polygons in $\mathbb{R}^2$ with an odd number of edges) provide examples of diam-max sets which are not balls. On the other hand, a regular polygon in $\mathbb{R}^2$ with an even number of edges, admits as a diam-max extension the circle centered at its center (its unique mid-diam point) and passing through all the vertices (this construction does not work for regular polygons with an odd number of edges, because the diameter of such circle is larger than the diameter of the polygon). We continue with two additional properties of diam-max sets.

Proposition 2.5. *Let C be a diam-max set with diameter s, and c a mid-diam point of C. Then the ball $B(c,r)$ centered at c with radius $r = \left(1 - \frac{\sqrt{3}}{2}\right) s$ is contained in C.*

Proof. Define $C' = C \cup B(c,r)$. We claim that $\operatorname{diam}(C') = s$. Take any $x,y \in C'$. If $x,y \in C$, then $\|x-y\| \le s$. If $x,y \in B(c,r)$, then $\|x-y\| \le \left(2-\sqrt{3}\right) s \le s$. Finally, assume that $x \in C, y \in B(c,r)$. Let (a,b) be an antipodal pair in CC with $c = \frac{a+b}{2}$. Then $2c = a+b$, and so $0 = \langle x-c, a-c\rangle + \langle x-c, b-c\rangle$, so that one of these two inner products is nonpositive. Without loss of generality, assume that it is the first one. It follows that

$$s^2 \geq \|x-a\|^2 \geq \|x-c\|^2 + \|c-a\|^2 = \|x-c\|^2 + \left(\frac{s}{2}\right)^2,$$

so that $\|x-c\| \leq \frac{\sqrt{3}}{2}s = s-r$. Hence, since $y \in B(c,r)$, we get

$$\|x-y\| \leq \|x-c\| + \|c-y\| \leq (s-r) + r = s.$$

Thus, $\|x-y\| \leq s$ for all $x, y \in C'$ and the claim holds, i.e., diam (C)= diam(C'). Since $C \subset C'$ and C is diam-max, we get that $C = C'$. So, $B(c,r) \subset C$. □

Corollary 2.1. *Max-diam sets have nonempty interiors.*

Proof. It follows from Proposition 2.5. □

Proposition 2.6. *Any point in the boundary of a max-diam set C is part of an antipodal pair.*

Proof. Let $s = \text{diam}(C)$ and take $a \in \text{bd}(C)$. It suffices to prove that $\|a-b\| = s$ for some $b \in C$. Otherwise $\max_{x\in C} \|a-x\| = t < s$. Define

$$r = \min\{s-t, \frac{s}{2}\}, C' = C \cup B(a,r).$$

We claim that $\text{diam}(C') = s$. Take any $x, y \in C'$. If $x, y \in C$, then $\|x-y\| \leq s$. If $x, y \in B(a,r)$, then $\|x-y\| \leq 2r \leq s$. If $x \in C, y \in B(a,r)$, then

$$\|x-y\| \leq \|x-a\| + \|a-y\| \leq t + r \leq s.$$

Hence the claim holds, i.e., diam(C)= diam(C'). Since $C \subset C'$ and C is diam-max, we get that $C = C'$, and so $B(a,r) \subset C$, contradicting the assumption that $a \in \text{bd}(C)$. □

We construct now a diam-max extension of an equilateral triangle $T \subset \mathbb{R}^2$. Let a^1, a^2, a^3 be its vertices and s the length of any of the edges. Let B_i $(1 \leq i \leq 3)$ be the closed ball centered at a^i with radius s. Note that all three vertices of T lie in the boundary of any of the B_i's. Let $S = \cap_{i=1}^3 B_i$. We claim that S is a diam-max extension of T. Being the intersection of three balls, S is convex, and it contains all vertices of T. Since T is convex, it follows that $T \subset S$. Now we must prove that $\text{diam}(S) = \text{diam}(T)$. The diameter of any triangle is the length of its largest edge, and hence $\text{diam}(T) = s$. Take now any $x \in S$. Since $x \in B_i$ for all i, we have that $\left\|x-a^i\right\| \leq s$ for all i. It follows from convexity of T and $\|\cdot\|$ that $\|x-y\| \leq s$ for all $y \in T$, but we need to prove that

$$\|x-y\| \leq s \tag{2.1}$$

also for all $y \in S \setminus T$.

Since the three balls B_i have simple analytical definitions, estimation of $\|x-y\|$ for $x, y \in S$ is a matter of simple algebra. We omit the details, but it turns out that (2.1) holds, and hence $\text{diam}(S) = \text{diam}(T) = s$. Now it remains to prove that S is a diam-max set. Take any $z \notin S$. So there exists i such that $z \notin B_i$ and hence $\left\|z-a^i\right\| > s$, implying that $\text{diam}(S \cup \{z\}) > s$. Thus, there exists no set S' containing S with $\text{diam}(S) = \text{diam}(S')$, which entails that S is a diam-max set. This set is known as *Reuleaux triangle*, see [4].

This construction can be extended to all regular polygons in $\mathbb{R}^2$ with an odd number of vertices $a^1, \ldots, a^m$. If s is the diameter of the polygon (equal to the distance between any vertex and a vertex lying farthest away from it), and we define B_i as the ball centered at a^i with radius s, then the set $S = \cap_{i=1}^m B_i$ is a diam-max extension of the polygon. The argument for proving this statement is basically the same as the one used for the equilateral triangle. The difficulty lies in establishing that $\text{diam}(S) = s$, and the algebra is more involved than in the case of the triangle, but it works. In the same spirit, a diam-max extension of a tetrahedron in $\mathbb{R}^3$ is obtained as the intersection of the four balls centered at each one of the four vertices and passing through the remaining three. We leave as an exercise the construction of a diam-max extension S of a (nonequilateral) isosceles triangle in $\mathbb{R}^2$, such that the unequal edge is the smallest one, in which case there exist two mid-diam points, and hence S cannot be a ball (if the unequal edge is the largest one, its midpoint is the unique mid-diam point, and the circle centered at it passing through the extreme points of this edge is a diam-max extension of the triangle, like in the case of the right triangle in Example 1).

It is also easy to present diam-max extensions of balls in other norms, which happen to be Euclidean balls of appropriate radiuses. Let B be the ball centered at x with radius r in the elliptic norm given by $\|x\|^2 = x^t Ax$, with $A \in \mathbb{R}^{n\times n}$ symmetric and positive definite. A diam-max extension of B is the closed Euclidean ball with the same center and radius $r\lambda^{-1}$, where λ is the smallest eigenvalue of A. If B now is the ball centered at x with radius r in the p-norm ($1 \le p \le \infty$), a diam-max extension of B is the closed Euclidean ball with the same center and radius given by $rn^{1/2-1/p}$ when $2 < p \le \infty$ and by r when $1 \le p \le 2$. These facts can be established through elementary calculations. Note that in all these cases the unique mid-diam point is the center x of the ball.

All given examples of diam-max sets are intersections of a finite number of balls with the same radius; this could be a general property of diam-max sets, but it could be just a consequence of the fact that all of them are diam-max extensions of sets with a rather regular geometry. Diam-max sets will be used in Section 4 in order to exhibit a large class of examples of maximal premonotone operators.

3. Some properties of maximal premonotone operators

The notion of premonotone operators was introduced in [7], where it was shown that, under adequate assumptions, they enjoy a surjectivity property akin to Minty's Theorem for maximal monotone operators. The definition has already been presented in Section 1; we repeat it here for the sake of completeness.

Definition 3.1. Given a function $\sigma : \mathbb{R}^n \to \mathbb{R}_+$, a point-to-set operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be σ-*premonotone* if

$$\langle u - v, x - y\rangle \ge -\min\{\sigma(x), \sigma(y)\} \|x - y\| \tag{3.1}$$

for all $x, y \in \mathbb{R}^n$, all $u \in T(x)$ and all $v \in T(y)$.

It was observed in [7] that T is σ-premonotone if and only if

$$\langle u - v, x - y\rangle \ge -\sigma(y) \|x - y\| \tag{3.2}$$

for all $x, y \in \mathbb{R}^n$, all $u \in T(x)$ and all $v \in T(y)$.

Monotone operators are σ-premonotone with $\sigma(x) = 0$ for all $x \in \mathbb{R}^n$.

Mirroring the definition of the maximal monotonicity, an operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *maximal* σ-*premonotone* when it is σ-premonotone, and whenever $T(x) \subset T'(x)$ for a σ-premonotone operator $T' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, it holds that $T = T'$.

Note now that if T is σ-premonotone and $\sigma'(x) \ge \sigma(x)$ for all $x \in \mathbb{R}^n$, then T is also σ'-premonotone, but clearly it could be maximal σ-premonotone and not maximal σ'-premonotone.

This fact suggests the desirability of a notion of the maximal premonotonicity of an operator T, which does not depend on σ. This goal can be achieved in the following way. Given an operator $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$, define $\hat{\sigma}_T, \sigma_T : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ as

$$\hat{\sigma}_T(y) = \sup_{x\in\mathbb{R}^n, x\neq y} \left\{ \sup_{u\in T(x), v\in T(y)} \frac{\langle v-u, x-y\rangle}{\|x-y\|} \right\}, \tag{3.3}$$

$$\sigma_T(x) = \max\{0, \hat{\sigma}_T(x)\}. \tag{3.4}$$

It has been proved in [7] that T is σ-premonotone for some σ if and only if $\sigma_T(y) < +\infty$ for all $y\in\mathbb{R}^n$, in which case T is also σ_T-premonotone and $\sigma_T(y)\le\sigma(y)$ for all $y\in\mathbb{R}^n$. If $\sigma_T(y)=\infty$ for some $x\in\mathbb{R}^n$, then T is not σ-premonotone for any σ. In other words, σ_T, when finite everywhere, is the smallest function σ such that T is σ-premonotome.

This result leads to the following definitions $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *premonotone* when σ_T is finite everywhere, and it is *maximal premonotone* whenever it is maximal σ_T-premonotone, achieving thus the σ-independent definitions of the premonotonicity and the maximal premonotonicity.

Next, we present some elementary properties of premonotone operators. We mention first that, following an usual convention, we will identify an operator $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with its graph, i.e., with the set $\{(x,u): x\in\mathbb{R}^n, u\in T(x)\}\subset\mathbb{R}^n\times\mathbb{R}^n$. With this notation, the assertions $u\in T(x)$ and $(x,u)\in T$ are equivalent.

Given $A\subset\mathbb{R}^n$, we denote as $\mathrm{cl}(A)$ the closure of A, and as $\mathrm{co}(A)$ the convex hull of A. Now, given $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we consider the operators $\mathrm{cl}(T)$, $\mathrm{co}(T)$, defined as $\mathrm{cl}(T)(x)=\mathrm{cl}(T(x))$, and $\mathrm{co}(T)(x)=\mathrm{co}(T(x))$. We consider also the operator $\overline{T}$ whose graph is the closure of the graph of T. Finally, $\mathrm{dom}(T)=\{x\in\mathbb{R}^n : T(x)\neq\emptyset\}$.

Proposition 3.1. *Take $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Then,*

i) $\mathrm{dom}(\mathrm{co}(T))=\mathrm{dom}(\mathrm{cl}(T))=\mathrm{dom}(T)\subset\mathrm{dom}(\overline{T})$.
ii) *If $x\in\mathrm{dom}(T)$, then $T(x)\subset\mathrm{co}(T)(x), T(x)\subset\mathrm{cl}(T)(x)\subset\overline{T}(x)$.*
iii) *If $x\in\mathrm{dom}(T)$ and one among $\{T(x), \mathrm{cl}(T)(x), \mathrm{co}(T)(x), \overline{T}(x)\}$ is bounded, then all of them are bounded.*
iv) *If one among $\{T, \mathrm{cl}(T), \mathrm{co}(T), \overline{T}\}$ is σ-premonotone, then all of them are σ-premonotone.*
v) *If T is premonotone, then there exists a maximal premonotone operator T' such that $T\subset T', \sigma_T=\sigma_{T'}$.*

Proof. The proofs of items (i)-(iv) are elementary, using basic topological properties, continuity of the inner product and the definition of premonotonicity. Item (v) was proved in Proposition 5 of [7], with a standard application of Zorn's Lemma. □

Given a set $A\in\mathbb{R}^n\times\mathbb{R}^n$, a pair $(x,u)\in\mathbb{R}^n$ is it monotonically related to A if $\langle u-v x-y\rangle\ge 0$ for all $(y,v)\in A$. When A is a monotone operator, the set of pairs monotonically related to A is called the *monotone hull of A*. In the same vein, given a premonotone operator T, a pair $(x,u)\in\mathbb{R}^n\times\mathbb{R}^n$, was defined in [7] as being *premonotonically related to T* when

$$\langle u-v, y-x\rangle \le \min\{\sigma_T(x),\sigma_T(y)\}\,\|x-y\| \;\forall (y,v)\in T,$$

and the *premonotone hull of T*, denoted as $T^h\subset\mathbb{R}^n\times\mathbb{R}^n$, consists of all the pairs $(x,u)\in\mathbb{R}^n$, which are premonotonically related to T, i.e.,

$$T^h(x)=\{u\in\mathbb{R}^n : \langle u-v, y-x\rangle \le \min\{\sigma_T(x),\sigma_T(y)\}\,\|x-y\| \;\forall (y,v)\in T\}. \tag{3.5}$$

At this point, it is important to emphasize that substituting $\sigma_T(x)$ for $\min\{\sigma_T(x),\sigma_T(y)\}$ in (3.5) makes a difference (contrary to what was wrongly stated in [7], after Definition 24). At a variance with the definition of the premonotonicity in (3.1), (3.2), which is symmetric in x,y, exchanging x for y in (4.10) defines a different set, namely, $T^h(y)$. One could consider also another operator, say T^c, defined as

$$T^c(x) = \{u \in \mathbb{R}^n : \langle u-v, y-x\rangle \le \sigma_T(y)\,\|x-y\| \;\forall (y,v)\in T\}. \tag{3.6}$$

The following example shows that T^h and T^c may differ. Define $T:\mathbb{R}\to\mathbb{R}$ as $T(x)=\sin(x)$. It is easy to check that $\sigma_T(x) = 1+|\sin(x)|$. Taking into account that $\sin(x)$ is periodical, the condition in (4.10) is equivalent to $|u-\sin(y)| \le 1+|\sin(y)|$ for all $y\in\mathbb{R}$, which leads to $T^c(x) = [-1,1]$ for all $x\in\mathbb{R}$, while the condition in (3.6) demands additionally $|u-\sin(y)| \le 1+|\sin(x)|$ for all $y\in\mathbb{R}$, which implies $1+|u|\le 1+|\sin(x)|$, so that $T^h(x) = [-|\sin(x)|, |\sin(x)|]$. It is easy to check that $\sigma_{T^h}(x) = \sigma_T(x) = 1+|\sin(x)|$, and that T^h is maximal premonotone. T^c is also maximal premonotone, but $\sigma_{T^c}(x) = 2$ for all $x\in\mathbb{R}$, so that $\sigma_{T^c}\ne\sigma_T$.

We recall that a *premonotone extension of* T is an operator T' such that $T\subset T'$, $\sigma_T=\sigma_{T'}$. Hence, in the example above T^c is not a premonotone extension of T. We observe that if T is a monotone operator, in which case $\sigma_T\equiv 0$, the definitions of T^c and T^h coincide, and reduce to the well known monotone hull.

From now on, we will restrict our attention to T^h, some of whose properties, which extend those which hold for the monotone hull, we prove next.

Proposition 3.2. *Assume that $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is premonotone. Then,*

i) $T\subset T^h$.

ii) $(z,w)\in T^h$ *if and only if* $\sigma_{T'}=\sigma_T$, *with* $T'=T\cup\{(z,w)\}$

iii) T *is maximal premonotone if and only if* $T=T^h$.

iv) T^h *is the union of all premonotone extensions of* T.

v) $T^h(x)$ *is closed and convex for all* $x\in dom(T)$.

vi) *If* T,U *are premonotone and* $T\subset U$, *then* $\sigma_T(x)\le\sigma_U(x)$ *for all* $x\in\mathbb{R}^n$.

Proof. i) Follows from the definitions of the premonotonicity and T^h.

ii) We prove first the "only if" statement. Since $T\subset T'$, we have that $\sigma_T(x)\le\sigma_{T'}(x)$ for all $x\in\mathbb{R}^n$, so that it suffices to check that $\sigma_T(x)\ge\sigma_{T'}(x)$. By (3.3),

$$\hat{\sigma}_{T'}(y) = \sup_{x\in\mathbb{R}^n, x\ne y}\left\{\sup_{u\in T'(x), v\in T'(y)} \frac{\langle v-u, x-y\rangle}{\|x-y\|}\right\}. \tag{3.7}$$

If $y\ne z$, it suffices to look at the expression

$$\frac{\langle v-u, x-y\rangle}{\|x-y\|}, \tag{3.8}$$

with $(x,u)=(z,w)$, because any other pair $(x,u)\in T'$ belongs also to T, in which case the expression (3.8) is no larger that $\sigma_T(y)$, by definition of σ_T. Since $(z,w)\in T^h$, by (3.6)

$$\frac{\langle v-w, z-y\rangle}{\|z-y\|} \le \sigma_T(y).$$

We conclude that $\sigma_{T'}(y) \le \sigma_T(y)$. If $y = z$, by the same token, it suffices to look at (3.8) with $(y, v) = (z, w)$, in which case we get

$$\frac{\langle w - u, x - z \rangle}{\|x - z\|}. \tag{3.9}$$

Looking again at (3.6), and noting the presence of $\min\{\sigma_T(x), \sigma_T(y)\}$ in its right hand side, we realize that the expression in (3.9) is no larger than $\sigma_T(z)$, so that $\sigma_{T'}(z) \le \sigma_T(z)$. We conclude that $\sigma_T = \sigma_{T'}$. We prove now the "if" statement. If $\sigma_T = \sigma_{T'}$, then, for all $(x, u) \in T$,

$$\frac{\langle w - u, x - z \rangle}{\|x - z\|} \le \min\{\sigma_{T'}(z), \sigma_{T'}(x)\} = \min\{\sigma_T(z), \sigma_T(x)\}, \tag{3.10}$$

for all $(x, u) \in T$, using the definition of $\sigma_{T'}$ in the inequality. In view of (3.10) and the definition of T^h, we conclude that $(z, w) \in T^h$.

iii) Assume first that T is maximal premonotone. Take any $(x, w) \in T^h$ and let $T' = T \cup \{(z, w)\}$. Clearly $T \subset T'$ and by item (ii), $\sigma_T = \sigma_{T'}$. By maximality of T, $T = T'$, so that $(z, w) \in T$, implying that $T^h \subset T$, and hence $T^h = T$ by item (i). Assume now that $T = T^h$, and take any T' such that $T \subset T, \sigma_T = \sigma_{T'}$. If $(x, u) \in T'$, then the fact that $\sigma_T = \sigma_{T'}$ implies easily that $(x, u) \in T^h = T$, so that $(x, u) \in T$ and hence $T' = T$, entailing maximality of T.

iv) Take any premonotone extension T' of T and any $(z, w) \in T'$. By item (iii), $(z, w) \in T^h$, so that $T' \subset T^h$. Now take any $(z, w) \in T^h$. Then, by item (iii) again, $T' = T \cup \{(z, w)\}$ is a premonotone extension of T. So any point in T^h belongs to some premonotone extension of T.

v) Follows easily from bilinearity and continuity of $\langle \cdot, \cdot \rangle$.

vi) Follows from the definitions of σ_T and T^h.

□

We continue with two additional properties of T^h. We recall that the recession cone A^∞ of a convex set $A \subset \mathbb{R}^n$ is defined as $A^\infty = \{d \in \mathbb{R}^n : x + t \in A, \ \forall t \ge 0, x \in A\}$, and the normal cone $N_A(x)$ of A at a point $x \in \mathrm{cl}(\mathrm{co}(A))$ is defined as $N_A(x) = \{d \in \mathbb{R}^n : \langle d, y - x \rangle \le 0 \ \forall y \in A\}$. It is easy to check that the normal cones of A and of $\mathrm{cl}(\mathrm{co}(A))$ coincide.

Proposition 3.3. *If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is premonotone, then $(T^h(x))^\infty = N_{H(T)}(x)$ for all $x \in \mathrm{cl}(H(T))$, with $H(T) = \mathrm{co}(\mathrm{dom}(H(T))$.*

Proof. Take $(x, u) \in T^h$ and $d \in N_{H(T)}(x)$. Then

$$\langle u - v, y - x \rangle \le \min\{\sigma_T(x), \sigma_T(y)\} \|x - y\| \tag{3.11}$$

for all $(y, v) \in T$ and

$$\langle d, y - x \rangle \le 0 \tag{3.12}$$

for all $y \in \mathrm{dom}(T)$. Multiplying (3.12) by $t \ge 0$ and adding (3.11), we obtain

$$\langle u + td - v, y - x \rangle \le \min\{\sigma(x), \sigma(y)\} \|x - y\|$$

for all $(y, v) \in T$ and all $t \ge 0$. It follows that $u + td \in T^h(x)$ for all $t \ge 0$, and hence $d \in (T^h(x))^\infty$.

Conversely, take $d \in (T(x))^\infty$, $u \in T^h(x)$. Then,

$$\langle u + td - v, y - x \rangle \le \min\{\sigma_T(x), \sigma_T(y)\} \|x - u\|$$

for all $(y,v) \in T$ and all $t > 0$. So,

$$\frac{\langle u-v, y-x\rangle}{t} + \langle d, y-x\rangle \leq \frac{\min\{\sigma_T(x), \sigma_T(y)\}\,\|y-x\|}{t} \tag{3.13}$$

for all $(y,v) \in T$ and all $t > 0$. Taking limits with $t \to \infty$ in (3.13), we obtain $\langle d, y-x\rangle \leq 0$ for all $y \in \mathrm{dom}(T)$. Thus $\langle d, y-x\rangle \leq 0$ for all $y \in \mathrm{cl}(H(T))$. Therefore, $d \in N_{H(T)}(x)$. □

For our next result, we will use the following lemma, proved in [5].

Lemma 3.1. *Let $D = \mathrm{co}\{x^0, x^1, \cdots, x^n\}$ be an n-dimensional simplex of $\mathbb{R}^n$. Take a closed and convex set $V \subset int(D)$. Then, for all $x \in V$ and all $c \in \mathbb{R}^n$, the linear programming problem*

$$\min_u \sum_{i=0}^{n} u_i \text{ s.t. } \sum_{i=0}^{n} u_i(x^i - x) = c,\ \ u \geq 0, \tag{3.14}$$

is feasible and has a unique optimal solution, say $u(c,x)$, which is continuous as a function of c,x on $\mathbb{R}^n \times V$.

Proof. See Lemma 2.2 in [5] □

We establish next local boundedness of T^h under some additional assumptions.

Proposition 3.4. *Consider a premonotone $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Let $D(T) = \mathrm{int}(\mathrm{cl}(\mathrm{co}(\mathrm{dom}(T))))$. Then, for all $\bar{x} \in D(T)$ there exists a compact set K and a neighborhood V of $\bar{x}$ such that $\emptyset \neq T^h(x) \subset K$ for all $x \in V$.*

Proof. Since D is open, for each $\bar{x} \in D$, there exists $\{\bar{t}_i\}_{i=1}^n \subset (0,1)$ and $\{(x^i, u^i)\}_{i=0}^n \subset T$ such that the vectors $x^1 - x^0, x^2 - x^0, \cdots, x^n - x^0$ are linearly independent, $\bar{x} = \sum_{i=0}^n \bar{t}_i x^i$ and $\sum_{i=0}^n \bar{t}_i = 1$. Taking $\varepsilon > 0$ such that $\varepsilon < \bar{t}_i$ for all $i \in \{0, 1, \cdots, n\}$, we define

$$V = \left\{ x = \sum_{i=0}^{n} t_i x^i : \sum_{i=0}^{n} t_i = 1,\ \{t_i\}_{i=0}^n \subset [\varepsilon, 1) \right\}.$$

By construction, V is a nonempty, convex and compact neighborhood of $\bar{x}$ contained in D. Given $c \in \mathbb{R}^n$ and $x \in D$, we define

$$\alpha(c,x) = \sup_u \{\langle c, u\rangle : u \in T^h(x)\}, \tag{3.15}$$

$$\beta(c,x) = \sup_u \{\langle c, u\rangle : A^t u \leq b\}, \tag{3.16}$$

where $A \in \mathbb{R}^{n\times(n+1)}$ is the matrix with columns $x^i - x$ $(0 \leq i \leq n)$, and $b \in \mathbb{R}^{n+1}$ is defined as

$$b_i = \min\{\sigma_T(x), \sigma_T(x^i)\}\,\left\|x^i - x\right\| - \langle u_i, x - x_i\rangle.$$

By convention, we take $\alpha(c,x) = -\infty$ if $T^h(x) = \emptyset$. From (3.15), (3.16), we get

$$-\infty \leq \alpha(c,x) \leq \beta(c,x) \tag{3.17}$$

Let P_e be the linear programming problem defined by the right hand side of (3.16). Its dual is the problem D_e given by

$$\widetilde{\beta}(c,x) = \min_y \langle y, b\rangle \text{ s.t. } \sum_{i=0}^{n} y_i(x^i - x) = c, y \geq 0.$$

Problem D_e satisfies the assumptions of Lemma 3.1. Hence, D_e is feasible, there is no duality gap between P_e and D_e, and

$$\beta(c,x) = \widetilde{\beta}(c,x) \leq \rho(c,x) := \sum_{i=0}^{n} u_i(c,x)\langle (x^i)^*, x^i - x\rangle, \tag{3.18}$$

with $u(c,x)$ as in the statement of Lemma 3.1.

Let $M = \sup_{x,c}\{\rho(c,x) : x \in V,\ \|c\| \leq 1\}, K = B(0,M)$, with ρ as defined in (3.18).

Since V is compact and the function u is continuous on $\mathbb{R}^n \times V$ by Lemma 3.1, M is finite and K is a compact convex set.

Then, for all c with $\|c\| \leq 1$ and all $x \in V$, we obtain, using (3.17) and ρ as in (3.18),

$$\alpha(c,x) \leq \beta(c,x) \leq \rho(c,x) \leq M. \tag{3.19}$$

Define the quantities

$$A(x) = \sup_u\{\|u\| : u \in T^h(x)\},$$

$$B(x) = \sup_u\{\|u\| : \langle u, x^i - x\rangle \leq b_i, \quad (0 \leq i \leq n)\}.$$

Then, we obtain

$$A(x) = \sup_{c,u}\{\langle c,u\rangle \ : \ u \in T^h(x),\ \|c\| \leq 1\} = \sup_c\{\alpha(c,x) : \|c\| \leq 1\},$$

$$B(x) = \sup_c\{\beta(c,x) : \|c\| \leq 1\}.$$

In view of (3.19), we have $A(x) \leq B(x) \leq M$ for all $x \in V$, implying

$$T^h(x) \subset \{u \ : \ \langle u, x^i - x\rangle \leq b_i, i = 0,1,\cdots,n\} \subset K. \tag{3.20}$$

□

Corollary 3.1. *For each $x \in \operatorname{int}(\operatorname{cl}(\operatorname{co}(\operatorname{dom}(T)))$, $T^h(x)$ is a nonempty, convex and compact set.*

Proof. Follows immediately from Proposition 3.4. □

4. A FAMILY OF MAXIMAL PREMONOTONE OPERATORS

In [7], it was shown that, given a maximal monotone and continuous operator $U : \mathbb{R}^n \to \mathbb{R}^n$ and a closed ball $B(0,r) \subset \mathbb{R}^n$, the operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as $T(x) = U(x) + B(0,r)$ is maximal premonotone with $\sigma_T(y) = 2r$ for all $y \in \mathbb{R}^n$. No other examples of maximal premonotone operators were available at that time, and the difficulty for producing them lied in the computation of σ_T for operators other than those above.

We will next present a significantly larger class of maximal premonotone operators. We will replace $B(0,r)$ in the example above by a rather general diam-max $C(x) \subset \mathbb{R}^n$, letting this set change with x in an appropriate way.

We will say that a point-to-set operator $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *inner Lipschitz semicontinuous with constant* β if for all $x, y \in \mathbb{R}^n$ and all $u \in T(x)$ there exists $v \in T(y)$ such that $\|u - v\| \leq \beta\, \|x - y\|$. We recall also that an operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *strongly monotone with constant* γ if, for all $x, y \in \mathbb{R}^n$, all $u \in T(x)$ and all $v \in T(y)$, it holds that $\langle u - v, x - y\rangle \geq \gamma\|x - y\|^2$. We will next compute σ_T for an operator T of the form $T(x) = U(x) + C(x)$, with a strongly monotone and continuous U and an inner Lipschitz semicontinuous C.

Proposition 4.1. *Let $U:\mathbb{R}^n\to\mathbb{R}^n$ be continuous and strongly monotone with constant γ, $C:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ inner Lipschitz semicontinuous with constant β, and assume that $C(x)$ is compact for all $x\in\mathbb{R}^n$ and that $\gamma\geq\beta$. Define $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ as $T(x)=U(x)+C(x)$ for all $x\in\mathbb{R}^n$. Then $\sigma_T(y)=\operatorname{diam}(C(y))$ for all $y\in\mathbb{R}^n$.*

Proof. We prove first that

$$\sigma_T(y)\leq\operatorname{diam}(C(y)). \tag{4.1}$$

Fix $y\in\mathbb{R}^n$ and take any $x\in\mathbb{R}^n, x\neq y$, any $u\in T(x)$ and any $v\in T(y)$. Look at the definition of T and write $u=U(x)+u', v=U(y)+v'$, with $u'\in C(x), v'\in C(y)$. Then

$$\begin{aligned}\frac{\langle v-u,x-y\rangle}{\|x-y\|}&=\frac{\langle v'-u',x-y\rangle}{\|x-y\|}-\frac{\langle U(x)-U(y),x-y\rangle}{\|x-y\|}\\&\leq\frac{\|v'-u'\|\,\|x-y\|}{\|x-y\|}-\gamma\frac{\|x-y\|^2}{\|x-y\|}\\&=\left\|v'-u'\right\|-\gamma\|x-y\|,\end{aligned}\tag{4.2}$$

using Cauchy-Schwartz's inequality and the strong monotonicity of U in the inequality. Use now the inner Lipschitz semicontinuity of C for finding $v''\in C(y)$ such that $\|u'-v''\|\leq\beta\,\|x-y\|$. In view of (4.2), we have

$$\begin{aligned}\frac{\langle v-u,x-y\rangle}{\|x-y\|}&\leq\left\|v'-v''\right\|+\left\|v''-u'\right\|-\gamma\|x-y\|\\&\leq\left\|v'-v''\right\|+\beta\,\|x-y\|-\gamma\|x-y\|\\&\leq\operatorname{diam}(C(y))-(\gamma-\beta)\,\|x-y\|\\&\leq\operatorname{diam}(C(y)),\end{aligned}\tag{4.3}$$

using the assumption that $\gamma\geq\beta$ in the last inequality. Taking supremum over $x\neq y, u\in T(x)$ and $v\in T(y)$ in (4.3), and recalling the definition of σ_T, we obtain the inequality in (4.1).

Next, we prove the converse inequality, namely,

$$\sigma_T(y)\geq\operatorname{diam}(C(y)). \tag{4.4}$$

Fix $y\in\mathbb{R}^n$ and use compactness of $C(y)$ for finding $w,w'\in C(y)$ such that $\|w-w'\|=\operatorname{diam}(C(y))$. Take any $\varepsilon>0$, and consider the point $x=y+\varepsilon(w-w')$. Invoke the inner Lipschitz semicontinuity of C for finding $u'\in C(x)$ such that

$$\left\|u'-w'\right\|\leq\beta\,\|x-y\|=\beta\varepsilon\left\|w-w'\right\|. \tag{4.5}$$

Take now $u=U(x)+u'$, $v=U(y)+w$, so that $u\in T(x), v\in T(y)$. Define

$$z=\frac{w-w'}{\|w-w'\|}=\frac{x-y}{\|x-y\|}.$$

Then,

$$\begin{aligned}\sigma_T(y) &\geq \frac{\langle v-u, x-y\rangle}{\|x-y\|}\\ &= \langle v-u, z\rangle\\ &= \langle w-u', z\rangle - \langle U(y+\varepsilon(w-w'))-U(y), z\rangle\\ &= \langle w-w', z\rangle + \langle w'-u', z\rangle + \langle U(y+\varepsilon(w-w'))-U(y), z\rangle\\ &\geq \|w-w'\| - \|w'-u'\| - \|U(y+\varepsilon(w-w'))-U(y)\|\\ &\geq (1-\beta\varepsilon)\|w-w'\| - \|U(y+\varepsilon(w-w'))-U(y)\|,\end{aligned} \tag{4.6}$$

using the definition of T in the second equality, the definition of z in the first inequality and (4.5), together with the fact that $\|z\| = 1$, in the second inequality. Since the inequality in (4.6) holds for all $\varepsilon > 0$, we take limit with $\varepsilon \to 0$ in the rightmost expression of (4.6) and get, using the continuity of U, that

$$\sigma_T(y) \geq \|w-w'\| = \operatorname{diam}(C(y)),$$

establishing (4.4) and completing the proof. □

Using the notion of diam-max sets, it is easy to give conditions on operators as considered in Proposition 4.2, which ensure the maximal premonotonicity. We proceed to do this.

Proposition 4.2. *An operator of the form $T = U + C$ satisfying the assumptions of Proposition 4.1 is maximal premonotone if and only if $C(x)$ is a diametrically maximal set for all $x \in \mathbb{R}^n$.*

Proof. We prove first the "only if" statement. Consider an operator $T' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $T(x) \subset T'(x)$ for all $x \in \mathbb{R}^n$ and $\sigma_T = \sigma_{T'}$. Call $C'(x) = T'(x) - U(x)$ (recall that $U(x)$ is a singleton), so that $C(x) \subset C'(x)$ for all $x \in \mathbb{R}^n$. Suppose that there exits $y \in \mathbb{R}^n$ and $w' \in C'(y) \setminus C(y)$. Since $C(y)$ is diam-max, it follows that $\operatorname{diam} C'(y)) > \operatorname{diam}(C(y))$, and in particular there exists $w \in C(y))$ such that

$$\|w'-w\| > \operatorname{diam}(C(y)). \tag{4.7}$$

Now we take, as in the proof of Proposition 4.1, $\varepsilon > 0$, $x = y + \varepsilon(w-w')$, and $u' \in C(x) \subset C'(x)$ such that $\|u'-w\| \leq \beta\|x-y\|$ (note that we need only inner Lipschitz semicontinuity of C, and not of C', which in principle may fail to enjoy this property). Define now $u = u' + U(x), v = w + U(y)$, so that $u \in T(x), v \in T'(y)$. Hence

$$\sigma'_T(y) \geq \frac{\langle v-u, x-y\rangle}{\|x-y\|},$$

and the same calculation as in (4.6) leads to

$$\sigma_{T'}(y) \geq (1-\beta\varepsilon)\|w-w'\| - \|U(y+\varepsilon(w-w'))-U(y)\|$$

for all $\varepsilon > 0$, and henceforth

$$\sigma_{T'}(y) \geq \|w-w'\| > \operatorname{diam}(C(y)) = \sigma_T(y),$$

using (4.7) and Proposition 4.1. We have contradicted the assumption that $\sigma_T = \sigma_{T'}$. It follows that for all $y \in \mathbb{R}^n, C'(y) \setminus C(y) = \emptyset$, so that $C = C'$ and hence $T = T'$, establishing maximal premonotonicity of T.

Now we prove the "if" statement. Suppose that T is maximal premonotone but there exists $y \in \mathbb{R}^n$ such that $C(y)$ is not a diam-max set, i.e., there exists $w \notin C(y)$ such that $\operatorname{diam}(C(y)) =$

$\text{diam}(C'(y))$ with $C'(y) = C(y) \cup \{w\}$. If we define now $C'(x) = C(x)$ for all $x \neq y$, and then $T'(x) = U(x) + C'(x)$, the same computations as in the proof of Proposition 4.1 lead to

$$\sigma_{T'}(y) = \text{diam}(C'(y)) = \text{diam}(C(y)) = \sigma_T(y) \tag{4.8}$$

for all $y \in \mathbb{R}^n$ (again, only inner Lipschitz semicontinuity of C, not of C', is needed). Since $T'(x) = T(x)$ for all $x \neq y$ and $T'(y) = T(y) \cup \{w\}$, we have that T is strictly contained in T', which, together with (4.8), contradicts the maximal premonotonicity of T. Thus, $C(x)$ is a diam-max set for all $x \in \mathbb{R}^n$. This completes the proof. □

We observe that if the operator C is constant, i.e. $C(x) = \widehat{C}$ for all $x \in \mathbb{R}^n$, where $\widehat{C}$ is a compact set, then we can proceed through the proofs of Propositions 4.1 and 4.2 with $\beta = 0$, in which case we can afford to have also $\gamma = 0$, i.e. U needs not be strongly monotone. We consider this case in the following corollary.

Corollary 4.1. *Consider a maximal monotone and continuous operator $U : \mathbb{R}^n \to \mathbb{R}^n$ and a compact set $\widehat{C} \subset \mathbb{R}^n$ Define $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as $T(x) = U(x) + \widehat{C}$. The T is premonotone with $\sigma_T(x) = diam(\widehat{C})$ for all $x \in \mathbb{R}^n$. T is maximal premonotone if and only if $\widehat{C}$ is a diametrically maximal set.*

Proof. Follows from the observation above and Propositions 4.1, 4.2. □

If the set $\widehat{C}$ is taken as a ball, which is certainly a diam-max set, we recover the maximal premonotone operators considered in [7].

We remark that, since U is point-to-point, $\text{diam}(C(x)) = \text{diam}(T(x))$ for all $x \in \mathbb{R}^n$. Also, in view of Proposition 2.2, $C(x)$ is a diam-max set if and only if $T(x)$ is max-diam. So we can also rephrase Propositions 4.1 and 4.2 in terms of of $T(x)$ rather than $C(x)$, as we do in the following corollary.

Corollary 4.2. *Consider a maximal monotone and continuous operator $U : \mathbb{R}^n \to \mathbb{R}^n$, which is also strongly monotone with constant γ, and a compact valued operator $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ which is inner Lipschitz semicontinuous with constant $\beta \leq \gamma$. Define $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as $T(x) = U(x) + C(x)$. The T is premonotone with $\sigma_T(x) = diam(T(x))$ for all $x \in \mathbb{R}^n$. T is maximal premonotone if and only if $T(x)$ is a diametrically maximal set for all $x \in \mathbb{R}^n$.*

Proof. Follows from the observation above and Propositions 4.1, 4.2. □

Corollary 4.2 suggests that we could obtain a class of maximal premonotone operators without involving a splitting of the form $T = U + C$, imposing the requested conditions directly on T. There are, however, obstacles in this path. We could indeed demand inner Lipschitz semicontinuity of T, but it does not make sense to ask T to be strongly monotone, or even monotone, because in such a case T is premonotone with $\sigma_T \equiv 0$, and the maximal premonotonicity reduces to the maximal monotonicity. If we go through the proof of Propositions 4.1 and 4.2, assuming just that T is compact valued and inner Lipschitz semicontinuous, we end up with the following estimate

$$\text{diam}(T(y)) \leq \sigma_T(y) \leq \text{diam}(T(y)) + \beta, \tag{4.9}$$

which unfortunately leads to no precise maximal premonotonicity result. In order to get the equality between $\sigma_T(y)$ and $\text{diam}(C(y))$, we need, in the estimation of the inner product $\langle v -$

$u, x-y\rangle$ with x far from y, the constant of strong monotonicity γ of U, in order to annihilate the term β in (4.9), so that getting rid of U turns out to be tricky.

An interesting conjecture involving maximal promonotone operators is the following:

Conjecture 4.1. If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is premonotone, then its premonotone hull T^h contains a maximal monotone operator.

An equivalent formulation of Conjecture 4.2 is the following.

Conjecture 4.2. If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal premonotone operator, then it contains a maximal monotone operator.

The equivalence between the two formulations of the conjecture follows from items (iii) and (iv) of Proposition 3.2.

We examine now this conjecture for operators satisfying the assumptions of Propositions 4.1 and 4.2. Assuming that $T = U + C$ as in Propositions 4.1 and 4.2, note that we cannot take $V = U$, because we are not requesting that $0 \in C(x)$ for all x, so that it is not true that $V \subset T$. If there exists some $x^* \in \cap_{x\in\mathbb{R}^n} C(x)$ (e.g., if C is constant), then the operator V given by $V(x) = U(x) + x^*$ is contained in T and does the job. The conjecture also holds in the following case.

Proposition 4.3. *Take $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of the form $T = U + C$, satisfying the assumptions of Propositions 4.1, 4.2. Assume that $C(x)$ is a closed ball for all x, say with center $c(x)$ and radius $r(x)$, so that T is maximal premonotone. Define $V(x) = U(x) + c(x)$. Then $V \subset T$ and V is maximal monotone.*

Proof. Since V is clearly contained in T, it suffices to check its maximal monotonicity. We claim by assuming inner Lipschitz semicontinuity of C with constant β that

$$\|c(x) - c(y)\| + |r(x) - r(y)| \le \beta \|x - y\| . \tag{4.10}$$

for all $x, y \in \mathbb{R}^n$. We establish (4.10). Without loss of generality, assume that $r(x) \ge r(y)$ and $c(x) \ne c(y)$. Define $z = \frac{c(x)-c(y)}{\|c(x)-c(y)\|}, u = c(x) + r(x)z$, so that u lies in the line passing through $c(x)$ and $c(y)$. Note that $u \notin C(y)$ because $r(x) \ge r(y)$. Let v be the projection of u onto $C(y)$. Clearly v is the intersection of the above mentioned line with the boundary of $C(y)$, namely, $c(y) + r(y)z$. By inner Lipschitz semicontinuity of C,

$$\begin{aligned}
\beta \|x - y\| &\ge \|u - v\| \\
&= \|c(x) - c(y) + (r(x) - r(y))z\| \\
&= \left(1 + \frac{r(x) - r(y)}{\|c(x) - c(y)\|}\right) \|c(x) - c(y)\| \\
&= \|c(x) - c(y)\| + |r(x) - r(y)|
\end{aligned}$$

using the fact that $r(x) \ge r(y)$. Thus, (4.10) holds, which allows us to prove monotonicity of V, under the assumptions on U and C given in Propositions 4.1:

$$\begin{aligned}
\langle V(x) - V(y), x - y\rangle &= \langle U(x) - U(y), x - y\rangle + \langle c(x) - c(y), x - y\rangle \\
&\ge \gamma \|x - y\|^2 - \|c(x) - c(y)\| \, \|x - y\| \\
&\ge (\gamma - \beta) \|x - y\|^2 \ge 0,
\end{aligned}$$

using (4.10) in the second inequality. Hence, V is monotone. Regarding its maximality, observe that (4.10) implies hte continuity of $c(\cdot)$, and consequently of V, because U is continuous by assumption. Since point-to-point, continuous and monotone operators are know to be maximal, we have that V is maximal monotone. □

We have not been able yet to establish the conjecture for cases in which the $C(x)$'s are diam-max sets but not balls. It would suffice to prove that every inner Lipschitz semicontinuous operator C with closed and convex values has a Lipschitz continuous selection, with the same Lipschitz constant as C (meaning a Lipschitz continuous $c:\mathbb{R}^n\to\mathbb{R}^n$ with $c(x)\in C(x)$ for all $x\in\mathbb{R}^n$). It is easy to check that such an operator has a continuous selection (take, e.g., $c(x)$ as the minimum norm element of $C(x)$), but finding a Lipschitz continuous one seems to be harder.

5. PREMONOTONE BIFUNCTIONS

The theory of monotone operators $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ has been extended to functions $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, to be called *bifunctions* in the sequel. As mentioned in Section 1, the prototypical bifunction associated to an operator T is given by $f(x,y)=\sup_{u\in T(x)}\langle u,y-x\rangle$.

In order to obtain significant results, the bifunctions are assumed to enjoy a couple of properties of the prototypical example. Also, it is useful to consider bifunctions with a domain smaller that $\mathbb{R}^n$. Given $K\subset\mathbb{R}^n$, we will consider bifunctions $f:K\times\mathbb{R}^n\to\mathbb{R}$ satisfying

B1. For each $x\in K$: $f(x,x)=0$,

B2. For each $x\in K$: $f(x,\cdot):K\to\mathbb{R}$ is a convex function.

A bifunction f is said to be *monotone* when $f(x,y)+f(y,x)\le 0$ for all $x,y\in K$. Note that in the prototypical example, the monotonicity of T is equivalent to the monotonicity of f. Next, we will define, in a similar fashion, *σ-premonotone bifunctions*.

Definition 5.1. Given a function $\sigma:K\to[0,+\infty)$, a bifunction $f:K\times K\to\mathbb{R}$ is σ-premonotone if, for each $y\in K$,

$$\sup_{x\in K\setminus\{y\}}\left\{\frac{f(x,y)+f(y,x)}{\|x-y\|}\right\}\le\sigma(y). \tag{5.1}$$

Now we consider an additional condition on the bifunctions.

B3. There exists $\rho:K\to[0,+\infty)$: $f(x,y)\le\rho(y)\,\|x-y\|$ for all $x,y\in K$.

Proposition 5.1. *Let $f:K\times\mathbb{R}^n\to\mathbb{R}$ be a bifunction satisfying assumptions* B1 *and* B2*. If f is σ-premonotone bifunction, then f satisfies* B3.

Proof. Note that $f(x,y)+\langle v,y-x\rangle\le f(x,y)+f(y,x)\le\sigma(y)\,\|y-x\|$ for all subgradient $v\in\partial f(y,\cdot)(y)$. Then,

$$f(x,y)\le(\sigma(y)+\inf\{\|v\|:v\in\partial f(y,\cdot)(y)\})\,\|y-x\|\,.$$

So, f satisfies B3, with $\rho(y):=\sigma(y)+\inf\{\|v\|\in\partial f(y,\cdot)(y)\}$. □

Next, we present a relation between operators and bifunctions similar to the one in the above mentioned prototypical example, that was already been announced in Section 1. This relation between operators and bifunctions will be called the *canonical relation*. In order to formally introduce it, we need to consider some assumptions on the operator T, which will shed some light into the canonical relation between operators and bifunctions.

A1. T is locally bounded on $D(T)\cap\text{dom}(T)$.
A2. $T(x)$ is closed and convex for all $x\in\text{dom}(T)$.
A3. There exists $\rho:\mathbb{R}^n\to[0,+\infty)$ such that $\langle u,y-x\rangle\le\rho(y)\|x-y\|$ for all $x\in D(T)\cap\text{dom}(T)$ all $u\in T(x)$ and all $y\in\mathbb{R}^n$.

For each operator $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ satisfying A3, define the bifunction $f_T:\text{int}(\text{dom}(T))\times\mathbb{R}^n\to\mathbb{R}$ as

$$f_T(x,y)=\sup_{u\in T(x)}\langle u,y-x\rangle\ \forall x\in\text{int}(\text{dom}(T))\text{ and }\forall y\in\mathbb{R}^n. \tag{5.2}$$

For each bifunction f satisfying assumption B2, define the operator $T_f:K\rightrightarrows\mathbb{R}^n$ as

$$T_f(x)=(\partial f(x,\cdot))(x)\ \forall x\in K \tag{5.3}$$

Proposition 5.2. *For all operator $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ satisfying* A3 *and all bifunction $f:K\times\mathbb{R}^n\to\mathbb{R}$ satisfying* B2*, the bifunction f_T and the operator T_f are well defined. Moreover f_T satisfies* B1, B2*, and if f satisfies* B3 *then T_f satisfies* A1-A3.

Proof. We only need to prove that if f satisfies B2 and B3, then T_f satisfies A1, because the proof of the remaining statements is elementary. Assume that T_f is not locally bounded. In such case, there exists $x\in\text{int}(K)$ and a sequence $\{x^k\}\subset K$ such that $\lim_{k\to\infty}x^k=x$ and a sequence $\{u^k\}$ such that $u^k\in T_f(x^k)$ for all k and $\lim_{k\to\infty}\left\|u^k\right\|=+\infty$. Take $\delta>0$ such that the closed ball $\bar{B}(x,\delta)\subset K$. Define $\bar{u}^k:=\frac{u^k}{\|u^k\|}$ and let $\bar{u}$ be a cluster point of the bounded sequence $\{\bar{u}^k\}$. Take $y=x+\delta\bar{u}$. Note that $y\in\bar{B}(x,\delta)\subset K$. In view of Assumption B3, $\langle u^k,y-x^k\rangle\le\rho(y)\left\|y-x^k\right\|$, i.e.,

$$\langle u^k,x+\delta\bar{u}-x^k\rangle\le\rho(x+\delta\bar{u})\left\|x-x^k+\delta\bar{u}\right\| \tag{5.4}$$

Dividing both sides of (5.4) by $\left\|u^k\right\|$, we get

$$\langle\bar{u}^k,x+\delta\bar{u}-x^k\rangle\le\frac{\rho(x+\delta\bar{u})\left\|x-x^k+\delta\bar{u}\right\|}{\left\|u^k\right\|}. \tag{5.5}$$

Taking limits in (5.5) with $k\to\infty$ along a subsequence of $\{\bar{u}^k\}$ converging to $\bar{u}$, and using the facts that $\lim_{k\to\infty}\left\|u^k\right\|=+\infty$ and $\lim_{k\to+\infty}x^k=x$, we get $\delta\le0$, in contradiction with the choice of δ. □

Corollary 5.1. *Consider $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ and $f:K\times\mathbb{R}^n\to\mathbb{R}$.*

i) *If T is monotone, then f_T is monotone and satisfies* B1-B3.
ii) *If T is σ-premonotone, then f_T is σ-premonotone and satisfies* B1-B3.
iii) *If f is monotone and satisfies B2, then T_f is monotone and satisfies* A1-A3.
iv) *If f is σ-premonotone and satisfies B2, then T_f is σ-premonotone and satisfies* A1-A3.

Proof. Follows from Proposition 5.2. □

We will denote by Γ the set of operators $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$, which satisfies A1–A3, and Θ the set of bifunctions $f:K\times\mathbb{R}^n\to\mathbb{R}$, which satisfies B1–B3, for some $K\subset\mathbb{R}^n$.

Now, consider the map F acting on those T, which satisfies A3 defined as $F(T)=f_T$, and the mapping G acting on the set of bifunctions, which satisfies B2 defined as $G(f)=T_f$. We will prove that under adequate assumptions G and F are inverses of each other, for which the following lemma is needed.

Lemma 5.1. *For each operator $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfying A3, $G(F(T))$ belongs to Γ. Moreover,* $\text{co}(\text{cl}(T(x))) = G(F(T))(x)$ *for all* $x \in \text{int}(\text{dom}(T))$.

Proof. Take $K = \text{int}(\text{dom}(T))$. Recall that $F(T) = f_T : K \times \mathbb{R}^n \to \mathbb{R}$ is defined as $f_T(x,y) = \sup_{u\in T(x)}\langle u, y-x\rangle$. Note that $G(F(T))(x)$ is closed and convex for all $x \in K$, because $G(F(T))(x) = \partial f_T(x,\cdot)(x)$. Since $T(x) \subset \partial f_T(x,\cdot)(x)$ by definition of f_T, we get that $\text{co}(\text{cl}(T))(x) \subset G(F(T))(x)$ for all $x \in K$. Suppose that the converse statement is false, i.e., that there exist $x \in K$ and $u \in G(F(T))(x)$ such that $u \notin \text{co}(\text{cl}(T(x)))$. Invoking the separation theorem for convex sets, we conclude that there exist $d \in \mathbb{R}^n$ with $\|d\| = 1$ and $\delta \in \mathbb{R}$ such that $\langle v, d\rangle < \delta < \langle u, d\rangle$ for all $v \in \text{co}(\text{cl}(T))(x)$. Then, Taking $y = x + d$, we get that

$$f_T(x,y) = \sup_{v\in T(x)} \langle v, y-x\rangle = \sup_{v\in T(x)} \langle v, d\rangle \le \delta < \langle u, d\rangle,$$

in contradiction with the fact that $u \in \partial f_T(x,\cdot)(x)$. It follows that $\text{co}(\text{cl}(T))(x) = G(F(T))(x)$ for all $x \in \text{int}(\text{dom}(T))$. □

Next we prove that appropriate restrictions of F and G are mutual inverses.

Proposition 5.3. *The restriction of the mapping F to Γ and the restriction of the mapping G to $F(\Gamma)$ are bijections and mutual inverses, meaning that $(F\circ G)(f) = f$ for all $f \in F(\Gamma)$ and $(G\circ F)(T) = T$ for all $T \in \Gamma$.*

Proof. By the definitions of Γ and Θ, we have that $F(\Gamma) \subset \Theta$ and $G(\Theta) \subset \Gamma$. Then, we get from Lemma 5.1 that $\text{co}(\text{cl}(T(x))) = (G\circ F)(T)$. By definition of Γ, $T(x) = \text{co}(\text{cl}(T(x)))$ for all $x \in \text{dom}(T)$. So, $(G\circ F)(T) = T$ for all $T \in \Gamma$. Now, take $f \in F(\Gamma)$, so that there exists $T \in \Gamma$ satisfying $f = F(T)$. Hence, $(F\circ G)(f) = F((G\circ F)(T)) = F(T) = f$. □

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