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KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS AND DUALITY FOR SEMI-INFINITE PROGRAMMING PROBLEMS WITH VANISHING CONSTRAINTS

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Abstract. This paper is concerned with the semi-infinite programming problems with vanishing constraints. Both necessary and sufficient Karush-Kuhn-Tucker optimality conditions for the semi-infinite programming problems with vanishing constraints are established. We also formulate types of Wolfe and Mond-Weir dual problems and explore duality relations under convexity assumptions. Some examples are given to illustrate our results.

Keywords. Semi-infinite programming problems; Constraint qualifications; Karush-Kuhn-Tucker optimality conditions; Mond-Weir duality; Wolfe duality.

1. INTRODUCTION

The mathematical programming problems with vanishing constraints, presented in [1, 7], are able to be used in reformulating many problems from structural topology optimization. The papers [8, 9] considered numerous constraint qualifications and applied them to obtain corresponding Karush-Kuhn-Tucker (KKT) necessary optimality conditions. The concepts of stationary points of mathematical programming problems with vanishing constraints were studied in [3] under a topological point of view on critical point theory. Strong KKT necessary optimality conditions for multiobjective mathematical programming problems with vanishing constraints were discussed in [16]. The KKT necessary optimality conditions for mathematical programming problems with non-differentiable vanishing constraints were established in [14] via Clarke subdifferentials. Some results about duality for mathematical programming problems with vanishing constraints were obtained in [10, 15]. On the other hand, an optimization with an infinite number of constraints is called a semi-infinite programming problem. For some recent results in this direction; see, e.g., the papers [11, 12, 13, 18, 19, 21, 22, 23, 24] and references therein. In [5], KKT sufficient optimality conditions for semi-infinite programming problems with vanishing constraints were investigated. However, KKT necessary optimality conditions for semi-infinite programming problems with vanishing constraints have not considered yet in [5]. Moreover, to the best of our knowledge, there is no paper dealing with duality for semi-infinite programming problems with vanishing constraints.

Motivated by the above observations, in this paper, we establish Karush-Kuhn-Tucker optimality conditions and investigate duality problems for the semi-infinite programming problems with vanishing constraints. The outline of this paper is as follows. In Section 2, we recall basic

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concepts and some preliminaries. The KKT necessary and sufficient optimality conditions for the semi-infinite programming problems with vanishing constraints are discussed in Section 3. Section 4 is devoted to exploring Mond-Weir and Wolfe dual problems of the semi-infinite programming problems with vanishing constraints. Some examples are provided to illustrate the outcome of the paper.

2. PRELIMINARIES

The following notations and definitions will be used throughout the paper. Let $\mathbb{R}^n$ be an Euclidean space. The notation $\langle \cdot, \cdot \rangle$ is utilized to denote the inner product. $B(x,\delta)$ indicates the open ball centered at x with radius $\delta > 0$. For a given $\bar{x}$, $\mathscr{U}(\bar{x})$ is the system of the neighborhoods of $\bar{x}$. For $A \subseteq \mathbb{R}^n$, int$A$, cl$A$, aff$A$, span$A$ and co$A$ stand for its interior, closure, affine hull, linear hull, convex hull of A, respectively (resp). The cone and the convex cone (containing the origin) generated by A are denoted resp by cone$A$, pos$A$. Note that, for the given sets A_1, A_2 in $\mathbb{R}^n$, we can check that

$$\text{span}(A_1 \cup A_2) = \text{span}A_1 + \text{span}A_2 \ \text{ and } \ \text{pos}(A_1 \cup A_2) = \text{pos}A_1 + \text{pos}A_2.$$

The negative polar cone, the strictly negative polar cone and the orthogonal complement of A are defined resp by

$$A^- := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle \le 0, \ \forall x \in A\},$$

$$A^s := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle < 0, \ \forall x \in A \setminus \{0\}\},$$

$$A^\perp := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle = 0, \ \forall x \in A\}.$$

It is easy to check that $A^s \subset A^-$ and if $A^s \neq \emptyset$, then cl$A^s = A^-$. Moreover, the bipolar theorem (see, e.g., [2]) states that $A^{--} = $ cl cone$A$. For a given nonempty subset A of $\mathbb{R}^n$, the contingent cone [2] of A at $\bar{x} \in$ cl$A$ is

$$T(A,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \exists x_k \to x, \ \forall k \in \mathbb{N}, \bar{x} + \tau_k x_k \in A\}.$$

Note that if A is a convex set, then $T(A,\bar{x}) = \text{clcone}(A - \bar{x})$. If $\langle x^*, x\rangle \ge 0$ for all $x^* \in A^*$, where A^* is a subset of the dual space of $\mathbb{R}^n$, we write $\langle A^*, x\rangle \ge 0$. The notion $o(\tau^k)$, for $\tau > 0$ and $k \in \mathbb{N}$, denotes a moving point such that $o(\tau^k)/\tau^k \to 0$ as $\tau \to 0^+$. The cardinality of the index set I is denoted by $|I|$. For an index subset $I \subset \{1,...,n\}$, $x_I = 0 (x_I \ge 0)$ stands for $x_i = 0$ ($x_i \ge 0$, resp) for all $i \in I$.

In the line of [5], we consider the following semi-infinite programming with vanishing constraints (P):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_t(x) \le 0, t \in T, \\ & h_i(x) = 0, i = 1,...,q, \\ & H_i(x) \ge 0, i = 1,...,l, \\ & G_i(x)H_i(x) \le 0, i = 1,...,l, \end{array}$$

where f, $g_t(t \in T)$, $h_i(i = 1,...,q)$ and $G_i, H_i(i = 1,...,l)$ are continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$. The index set T is an arbitrary nonempty set, not necessary finite. Let us denote $I_h := \{1,...,q\}$ and $I_l := \{1,...,l\}$. The feasible solution set of (P) is

$$\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \le 0 (t \in T), h_i(x) = 0 (i \in I_h), H_i(x) \ge 0 (i \in I_l), G_i(x)H_i(x) \le 0 (i \in I_l)\}.$$

Definition 2.1. A point $\bar{x} \in \Omega$ is a local solution of (P), denoted by $\bar{x} \in \text{loc}\mathscr{S}(\text{P})$, if there exists a neighborhood $U \in \mathscr{U}(\bar{x})$ such that

$$f(\bar{x}) \leq f(x), \forall x \in \Omega \cap U.$$

If $U = \mathbb{R}^n$, the word "local" is omitted. In this case, the (global) solution set of (P) is denoted by $\mathscr{S}(P)$.

By $\mathbb{R}_+^{|T|}$, we indicate the collection of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x} \in \Omega$, we designate by $I_g(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|T|} | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $J \subset I_g(\bar{x})$ such that $\lambda_t > 0$ for all $t \in J$ and $\lambda_t = 0$ for all $t \in T \setminus J$. For each $\bar{x} \in \Omega$, define

$$I_+(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) > 0\}, I_0(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0\},$$
$$I_{+0}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) > 0, G_i(\bar{x}) = 0\},$$
$$I_{+-}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) > 0, G_i(\bar{x}) < 0\},$$
$$I_{0+}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0, G_i(\bar{x}) > 0\},$$
$$I_{00}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0, G_i(\bar{x}) = 0\},$$
$$I_{0-}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0, G_i(\bar{x}) < 0\}.$$

Definition 2.2. Let $\bar{x} \in \Omega$.

(i) The point $\bar{x}$ is called a strong stationary point of (P) iff there exists $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})} = 0$, $\lambda^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0$, $\lambda^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} = 0$ and $\lambda^G_{I_{+0}(\bar{x})} \geq 0$ such that

$$\nabla f(\bar{x}) + \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(\bar{x}) + \sum_{i \in I_l} \lambda_i^G \nabla G_i(\bar{x}) = 0. \tag{2.1}$$

(ii) The point $\bar{x}$ is said to be a VC-stationary point of (P) iff there exists $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})} = 0$, $\lambda^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0$, $\lambda^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{0-}(\bar{x})} = 0$ and $\lambda^G_{I_{+0}(\bar{x}) \cup I_{00}(\bar{x})} \geq 0$ such that

$$\nabla f(\bar{x}) + \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(\bar{x}) + \sum_{i \in I_l} \lambda_i^G \nabla G_i(\bar{x}) = 0. \tag{2.2}$$

It is easy to see that if $\bar{x} \in \Omega$ is a strong stationary point of (P), then $\bar{x}$ is a VC-stationary point of (P).

For $\bar{x} \in \Omega$ and $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, define

$$I_g^+(\bar{x}) := \{t \in I_g(\bar{x}) \mid \lambda_t^g > 0\},$$
$$I_h^+(\bar{x}) := \{i \in I_h(\bar{x}) \mid \lambda_i^h > 0\}, I_h^-(\bar{x}) := \{i \in I_h(\bar{x}) \mid \lambda_i^h < 0\},$$

$$\hat{I}_+^+(\bar{x}) := \{i \in I_+(\bar{x}) \mid \lambda_i^H > 0\},$$
$$\hat{I}_0^+(\bar{x}) := \{i \in I_0(\bar{x}) \mid \lambda_i^H > 0\}, \hat{I}_0^-(\bar{x}) := \{i \in I_0(\bar{x}) \mid \lambda_i^H < 0\},$$

$$\hat{I}_{0+}^{+}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda_i^H > 0\}, \hat{I}_{0+}^{-}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda_i^H < 0\},$$
$$\hat{I}_{00}^{+}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^H > 0\}, \hat{I}_{00}^{-}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^H < 0\},$$
$$\hat{I}_{0-}^{+}(\bar{x}) := \{i \in I_{0-}(\bar{x}) \mid \lambda_i^H > 0\},$$

$$I_{+0}^{+}(\bar{x}) := \{i \in I_{+0}(\bar{x}) \mid \lambda_i^G > 0\}, I_{+0}^{-}(\bar{x}) := \{i \in I_{+0}(\bar{x}) \mid \lambda_i^G < 0\},$$
$$I_{+-}^{+}(\bar{x}) := \{i \in I_{+-}(\bar{x}) \mid \lambda_i^G > 0\},$$
$$I_{0+}^{+}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda_i^G > 0\}, I_{0+}^{-}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda_i^G < 0\},$$
$$I_{00}^{+}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G > 0\}, I_{00}^{-}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G < 0\},$$
$$I_{0-}^{+}(\bar{x}) := \{i \in I_{0-}(\bar{x}) \mid \lambda_i^G > 0\}.$$

Definition 2.3. [20] Let $X \subset \mathbb{R}^n$ be an open convex set and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\bar{x} \in X$.

(i) φ is convex at $\bar{x}$ if $\varphi(\lambda\bar{x} + (1-\lambda)x) \le \lambda\varphi(\bar{x}) + (1-\lambda)\varphi(x), \forall x \in X, \forall \lambda \in [0,1]$.

(ii) φ is quasiconvex at $\bar{x}$ if $\varphi(\lambda\bar{x} + (1-\lambda)x) \le \max\{\varphi(\bar{x}), \varphi(x)\}, \forall x \in X, \forall \lambda \in [0,1]$.

(iii) φ is pseudoconvex at $\bar{x}$ if, for all $x \in X$,
$$\langle \nabla\varphi(\bar{x}), x - \bar{x}\rangle \ge 0 \Rightarrow \varphi(x) \ge \varphi(\bar{x}).$$

(iv) φ is convex on X if φ is convex on each point of X. The other concepts here introduced can be defined on a set in a similar way.

Remark 2.1. [20] Let $X \subset \mathbb{R}^n$ be an open convex set and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\bar{x} \in X$.

(i) If φ is convex at $\bar{x}$, then
$$\langle \nabla\varphi(\bar{x}), x - \bar{x}\rangle \le \varphi(x) - \varphi(\bar{x}), \text{for all } x \in X.$$

(ii) If φ is quasiconvex at $\bar{x}$, then, for all $x \in X$,
$$\varphi(x) \le \varphi(\bar{x}) \Rightarrow \langle \nabla\varphi(\bar{x}), x - \bar{x}\rangle \le 0.$$

(iii) If φ is convex at $\bar{x}$ then φ is pseudoconvex at $\bar{x}$. If φ is pseudoconvex at $\bar{x}$ then φ is quasiconvex at $\bar{x}$.

Lemma 2.1. [20] *Let $\{C_t | t \in \Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K = \text{pos}\left(\bigcup_{t\in\Gamma} C_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

Lemma 2.2. [4] *Suppose that S, T, P are arbitrary (possibly infinite) index sets, $a_s = a(s) = (a_1(s), ..., a_n(s))$ maps S onto $\mathbb{R}^n$, and so do a_t and a_p. Suppose that the set* $\text{co}\{a_s, s \in S\} + \text{pos}\{a_t, t \in T\} + \text{span}\{a_p, p \in P\}$ *is closed. Then the following statements are equivalent:*

$$I: \quad \begin{cases} \langle a_s, x\rangle < 0, s \in S, S \neq \emptyset \\ \langle a_t, x\rangle \le 0, t \in T \\ \langle a_p, x\rangle = 0, p \in P \end{cases} \quad \textit{has no solution } x \in \mathbb{R}^n;$$
$$II: \quad 0 \in \text{co}\{a_s, s \in S\} + \text{pos}\{a_t, t \in T\} + \text{span}\{a_p, p \in P\}.$$

Lemma 2.3. [6] *If A is a nonempty compact subset of $\mathbb{R}^n$, then,*

(i) co$A$ *is a compact set;*

(ii) *if $0 \notin \text{co}A$, then* pos$A$ *is a closed cone.*

3. KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

In this section, we write the index set I_g instead of $I_g(\bar{x})$ for the sake of convenience. The other index sets are described similarly. Now, we establish the KKT necessary optimality condition for local solutions of (P) under the following constraint qualifications:
(i) (ACQ):

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-\cap(\bigcup_{i\in I_h}\nabla h_i(\bar{x}))^\perp\cap(\bigcup_{i\in I_{0+}}\nabla H_i(\bar{x}))^\perp\cap(\bigcup_{i\in I_{00}\cup I_{0-}}-\nabla H_i(\bar{x}))^-\cap(\bigcup_{i\in I_{+0}}\nabla G_i(\bar{x}))^- \subseteq T(\Omega,\bar{x}),$$

(ii) (VC-ACQ):

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-\cap(\bigcup_{i\in I_h}\nabla h_i(\bar{x}))^\perp\cap(\bigcup_{i\in I_{0+}}\nabla H_i(\bar{x}))^\perp\cap(\bigcup_{i\in I_{00}\cup I_{0-}}-\nabla H_i(\bar{x}))^-\cap(\bigcup_{i\in I_{+0}\cup I_{00}}\nabla G_i(\bar{x}))^- \subseteq T(\Omega,\bar{x}).$$

It is evident that (ACQ) implies (VC-ACQ). The following example shows that the inversion is not true in general.

Example 3.1. Let $n=2$ and $l=1$. Consider the following (P):

$$\begin{aligned} \min \quad & f(x)=x_1^2+x_2^2,\\ \text{s.t.} \quad & g_t(x)=-tx_1\le 0, t\in T=\mathbb{N},\\ & H_1(x)=x_1+2x_2\ge 0,\\ & G_1(x)H_1(x)=x_1(x_1+2x_2)\le 0. \end{aligned}$$

Then,

$$\Omega=\{x\in\mathbb{R}^2 \mid x_1\ge 0, x_1+2x_2=0\}\cup\{x\in\mathbb{R}^2 \mid x_1=0, x_2\ge 0\}.$$

For $\bar{x}=(0,0)\in\Omega$, one has

$$T(\Omega,\bar{x})=\Omega, I_g=\mathbb{N}, \nabla g_t(\bar{x})=\{(-t,0)\}, t\in T,$$

$$I_+=I_{0+}=I_{0-}=\emptyset, I_{00}=\{1\}, \nabla G_1(\bar{x})=\{(1,0)\}, \nabla H_1(\bar{x})=\{(1,2)\},$$

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-=\{x\in\mathbb{R}^2 \mid x_1\ge 0\},$$

$$(\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x}))^-=(-\nabla H_1(\bar{x}))^-=\{x\in\mathbb{R}^2 \mid x_1+2x_2\ge 0\},$$

$$(\bigcup_{i\in I_{00}}\nabla G_i(\bar{x}))^-=(\nabla G_1(\bar{x}))^-=\{x\in\mathbb{R}^2 \mid x_1\le 0\},$$

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-\cap(\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})))^-=\{x\in\mathbb{R}^2 \mid x_1\ge 0, x_1+2x_2\ge 0\},$$

and

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-\cap(\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})))^-\cap(\bigcup_{i\in I_{00}}\nabla G_i(\bar{x}))^-=\{x\in\mathbb{R}^2 \mid x_1=0, x_2\ge 0\}.$$

Hence,

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-\cap(\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})))^-\not\subseteq T(\Omega,\bar{x}),$$

and

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-\cap(\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})))^-\cap(\bigcup_{i\in I_{00}}\nabla G_i(\bar{x}))^-\subset T(\Omega,\bar{x}).$$

Thus, (ACQ) does not hold at $\bar{x}$ and (VC-ACQ) holds at $\bar{x}$.

Proposition 3.1. *Let* $\bar{x}\in \mathrm{loc}\mathscr{S}(\mathrm{P})$.

(i) If (ACQ) holds at $\bar{x}$ *and the set*

$$\Delta := \mathrm{pos}\left(\bigcup_{t\in I_g}\nabla g_t(\bar{x})\cup\bigcup_{i\in I_{00}\cup I_{0-}}(-\nabla H_i(\bar{x}))\cup\bigcup_{i\in I_{+0}}\nabla G_i(\bar{x})\right) + \mathrm{span}\left(\bigcup_{i\in I_h}\nabla h_i(\bar{x})\cup\bigcup_{i\in I_{0+}}\nabla H_i(\bar{x})\right)$$

is closed, then $\bar{x}$ *is a strong stationary point of (P).*

(ii) If (VC-ACQ) holds at $\bar{x}$ *and the set*

$$\Delta_1 := \mathrm{pos}\left(\bigcup_{t\in I_g}\nabla g_t(\bar{x})\cup\bigcup_{i\in I_{00}\cup I_{0-}}(-\nabla H_i(\bar{x}))\cup\bigcup_{i\in I_{+0}\cup I_{00}}\nabla G_i(\bar{x})\right) + \mathrm{span}\left(\bigcup_{i\in I_h}\nabla h_i(\bar{x})\cup\bigcup_{i\in I_{0+}}\nabla H_i(\bar{x})\right)$$

is closed, then $\bar{x}$ *is a VC-stationary point of (P).*

Proof. Due to the similarity, we only prove (ii). Since $\bar{x}\in \mathrm{loc}\mathscr{S}(\mathrm{P})$, there exists $U\in\mathscr{U}(\bar{x})$ such that there is no $x\in\Omega\cap U$ satisfying

$$f(x)<f(\bar{x}). \tag{3.1}$$

First, we prove that

$$(\nabla f(\bar{x}))^s\cap T(\Omega,\bar{x})=\emptyset. \tag{3.2}$$

Suppose to the contrary that there exists $d\in(\nabla f(\bar{x}))^s\cap T(\Omega,\bar{x})$. Then, one has

$$\langle\nabla f(\bar{x}),d\rangle<0.$$

By $d\in T(\Omega,\bar{x})$, there exist $\tau_k\downarrow 0$ and $d_k\to d$ such that $\bar{x}+\tau_k d_k\in\Omega$ for all k. Since f is continuously differentiable at $\bar{x}$, one gets

$$f(\bar{x}+\tau_k d_k)=f(\bar{x})+\tau_k\langle\nabla f(\bar{x}),d_k\rangle+o(\tau_k\|d_k\|).$$

Consequently,

$$\frac{f(\bar{x}+\tau_k d_k)-f(\bar{x})}{\tau_k}=\langle\nabla f(\bar{x}),d_k\rangle+\frac{o(\tau_k\|d_k\|)}{\tau_k\|d_k\|}.\|d_k\|\to\langle\nabla f(\bar{x}),d\rangle<0,\text{ when }k\to\infty.$$

Thus, there exists $\bar{k}>0$ such that $\dfrac{f(\bar{x}+\tau_k d_k)-f(\bar{x})}{\tau_k}<0$, for all $k>\bar{k}$. Hence, we assure the existence of $k>\bar{k}$ large enough such that $\bar{x}+\tau_k d_k\in\Omega\cap U$ and

$$f(\bar{x}+\tau_k d_k)<f(\bar{x}),$$

which contradicts (3.1). Therefore, (3.2) holds. We conclude from (3.2) and (VC-ACQ) that

$$(\nabla f(\bar{x}))^s \cap (\bigcup_{t\in I_g} \nabla g_t(\bar{x}))^- \cap (\bigcup_{i\in I_h} \nabla h_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{0+}} \nabla H_i(\bar{x}))^\perp$$
$$\cap (\bigcup_{i\in I_{00}\cup I_{0-}} -\nabla H_i(\bar{x}))^- \cap (\bigcup_{i\in I_{+0}\cup I_{00}} \nabla G_i(\bar{x}))^- = \emptyset.$$

This leads that there is no $d \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \nabla f(\bar{x}), d\rangle < 0, & \\ \langle \nabla g_t(\bar{x}), d\rangle \le 0, & \forall t \in I_g, \\ \langle \nabla h_i(\bar{x}), d\rangle = 0, & \forall i \in I_h, \\ \langle \nabla H_i(\bar{x}), d\rangle = 0, & \forall i \in I_{0+}, \\ \langle -\nabla H_i(\bar{x}), d\rangle \le 0, & \forall i \in I_{00}\cup I_{0-}, \\ \langle \nabla G_i(\bar{x}), d\rangle \le 0, & \forall i \in I_{+0}\cup I_{00}. \end{cases}$$

On the other hand, as $\{\nabla f(\bar{x})\}$ is a compact set, $\{\nabla f(\bar{x})\} + \Delta_1$ is closed. According to Lemma 2.2, one has

$$0 \in \nabla f(\bar{x}) + \text{pos}\left(\bigcup_{t\in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i\in I_{00}\cup I_{0-}} (-\nabla H_i(\bar{x})) \cup \bigcup_{i\in I_{+0}\cup I_{00}} \nabla G_i(\bar{x})\right)$$
$$+ \text{span}\left(\bigcup_{i\in I_h} \nabla h_i(\bar{x}) \cup \bigcup_{i\in I_{0+}} \nabla H_i(\bar{x})\right).$$

Consequently,

$$0 \in \nabla f(\bar{x}) + \text{pos} \bigcup_{t\in I_g} \nabla g_t(\bar{x}) + \text{span} \bigcup_{i\in I_h} \nabla h_i(\bar{x}) + \text{span} \bigcup_{i\in I_{0+}} \nabla H_i(\bar{x})$$
$$+ \text{pos} \bigcup_{i\in I_{00}\cup I_{0-}} (-\nabla H_i(\bar{x})) + \text{pos} \bigcup_{i\in I_{+0}\cup I_{00}} \nabla G_i(\bar{x}).$$

From Lemma 2.1, there exists $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\lambda^H_{I_+} = 0$, $\lambda^H_{I_{00}\cup I_{0-}} \ge 0$, $\lambda^G_{I_{+-}\cup I_{0+}\cup I_{0-}} = 0$ and $\lambda^G_{I_{+0}\cup I_{00}} \ge 0$ such that

$$\nabla f(\bar{x}) + \sum_{t\in T} \lambda^g_t \nabla g_t(\bar{x}) + \sum_{i\in I_h} \lambda^h_i \nabla h_i(\bar{x}) - \sum_{i\in I_l} \lambda^H_i \nabla H_i(\bar{x}) + \sum_{i\in I_l} \lambda^G_i \nabla G_i(\bar{x}) = 0.$$

Hence, $\bar{x}$ is a VC-stationary point of (P). □

Proposition 3.2. *Let $\bar{x}$ be a strong stationary point of (P). Suppose that f is pseudo-convex at $\bar{x}$, $g_t(t \in I_g), h_i(i \in I^+_h), -h_i(i \in I^-_h), G_i(i \in I^+_{+0}), H_i(i \in \hat{I}^-_{0+}), -H_i(i \in \hat{I}^+_{0+} \cup \hat{I}^+_{00} \cup \hat{I}^+_{0-})$ are quasi-convex at $\bar{x}$. Then, the following statements hold:*

(i) $\bar{x}$ is a local solution of (P);
(ii) if $\hat{I}^-_{0+} \cup I^+_{+0} = \emptyset$, then $\bar{x}$ is a global solution of (P).

Proof. Since $\bar{x}$ is a strong stationary point of (P), there exists $(\lambda^g_J, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{|J|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, where J is a finite subset of I_g, with $\lambda^H_{I_+} = 0$, $\lambda^H_{I_{00}\cup I_{0-}} \ge 0$, $\lambda^G_{I_{+-}\cup I_{0+}\cup I_{00}\cup I_{0-}} = 0$ and

$\lambda_{I_{+0}}^G \geq 0$ such that

$$\nabla f(\bar{x}) + \sum_{t\in J} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i\in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i\in I_{0+}\cup I_{00}\cup I_{0-}} \lambda_i^H \nabla H_i(\bar{x}) + \sum_{i\in I_{+0}} \lambda_i^G \nabla G_i(\bar{x}) = 0. \quad (3.3)$$

For an arbitrary $x \in \Omega$, one gets that $g_t(x) \leq 0 = g_t(\bar{x})$ for each $t \in I_g$. Thus, by the quasiconvexity at $\bar{x}$ of $g_t(t \in I_g)$, one has

$$\langle \nabla g_t(\bar{x}), x - \bar{x} \rangle \leq 0, \forall t \in J,$$

which together with $\lambda_J^g \in \mathbb{R}_+^{|J|}$ leads to

$$\left\langle \sum_{t\in J} \lambda_t^g \nabla g_t(\bar{x}), x - \bar{x} \right\rangle \leq 0. \quad (3.4)$$

We deduce from $x, \bar{x} \in \Omega$ that $h_i(x) = h_i(\bar{x}) = 0, \forall i \in I_h$. Hence,

$$h_i(x) \leq h_i(\bar{x}), \forall i \in I_h^+ \text{ and } -h_i(x) \leq -h(\bar{x}), \forall i \in I_h^-.$$

The above inequalities along with the quasiconvexity at $\bar{x}$ of $h_i(i \in I_h^+)$ and $-h_i(i \in I_h^-)$ implies that

$$\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_h^+ \quad \text{and} \quad \langle -\nabla h_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_h^-.$$

It is immediate from the above inequalities and the definitions of I_h^+, I_h^- that

$$\left\langle \sum_{i\in I_h} \lambda_i^h \nabla h_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \quad (3.5)$$

Again, we derive from $x \in \Omega$ that $-H_i(x) \leq 0, \forall i \in I_l$. Thus, $-H_i(x) \leq -H_i(\bar{x}), i \in \hat{I}_{0+}^+ \cup \hat{I}_{00}^+ \cup \hat{I}_{0-}^+$. Therefore, by the quasiconvexity of $-H_i, i \in \hat{I}_{0+}^+ \cup \hat{I}_{00}^+ \cup \hat{I}_{0-}^+$ at $\bar{x}$, one arrives at

$$\langle -\nabla H_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in \hat{I}_{0+}^+ \cup \hat{I}_{00}^+ \cup \hat{I}_{0-}^+. \quad (3.6)$$

(i) Now, for $i \in \hat{I}_{0+}^-$, we deduce from $G_i(\bar{x}) > 0$ and the continuity of G_i that there exists a neighborhood $U_1 \in \mathscr{U}(\bar{x})$ satisfying $G_i(x) > 0$ for all $x \in U_1$. This leads to $H_i(x) = 0, \forall i \in \hat{I}_{0+}^-, \forall x \in U_1$. Hence, $H_i(x) \leq H_i(\bar{x}), i \in \hat{I}_{0+}^-$. By invoking the quasiconvexity of $H_i(i \in \hat{I}_{0+}^-)$, we have

$$\langle \nabla H_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in \hat{I}_{0+}^-. \quad (3.7)$$

Taking into account (3.6), (3.7) and the definitions of the occurring index sets, we obtain

$$\left\langle -\sum_{i\in I_l} \lambda_i^H \nabla H_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \quad (3.8)$$

Moreover, for $i \in I_{+0}^+$, the continuity of H_i at $\bar{x}$ together with $H_i(\bar{x}) > 0$ justifies the existence of $U_2 \in \mathscr{U}(\bar{x})$ such that $H_i(x) > 0$ for all $x \in U_2$. Therefore, $G_i(x) \leq 0 = G_i(\bar{x}), \forall i \in I_{+0}^+, \forall x \in U_2$. By employing the quasiconvexity of $G_i(i \in I_{+0}^+)$, one has $\langle \nabla G_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_{+0}^+$. Since $\lambda_i^G > 0, i \in I_{+0}^+$, one obtains that

$$\left\langle \sum_{i\in I_{+0}} \lambda_i^G \nabla G_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \quad (3.9)$$

It follows from (3.3)-(3.9) that

$$\langle \nabla f(\bar{x}), x-\bar{x}\rangle = -\left\langle \sum_{t\in T}\lambda_t^g \nabla g_t(\bar{x}) + \sum_{i\in I_h}\lambda_i^h \nabla h_i(\bar{x}) - \sum_{i\in I_l}\lambda_i^H \nabla H_i(\bar{x}) + \sum_{i\in I_l}\lambda_i^G \nabla G_i(\bar{x}), x-\bar{x}\right\rangle$$
$$\geq 0,$$

for all $x\in\Omega\cap U$, where $U=U_1\cap U_2$. The above inequality together with the pseudo-convexity of f at $\bar{x}$ implies that $f(x)\geq f(\bar{x})$ for all $x\in\Omega\cap U$.
(ii) If $\hat{I}_{0+}^{-}\cup I_{+0}^{+}=\emptyset$, then we deduce from (3.3) - (3.6) and (3.8) that

$$\langle \nabla f(\bar{x}), x-\bar{x}\rangle = -\left\langle \sum_{t\in T}\lambda_t^g \nabla g_t(\bar{x}) + \sum_{i\in I_h}\lambda_i^h \nabla h_i(\bar{x}) - \sum_{i\in I_l}\lambda_i^H \nabla H_i(\bar{x}) + \sum_{i\in I_l}\lambda_i^G \nabla G_i(\bar{x}), x-\bar{x}\right\rangle$$
$$\geq 0,$$

for all $x\in\Omega$. By the pseudo-convexity of f at $\bar{x}$, one has $f(x)\geq f(\bar{x})$ for all $x\in\Omega$. □

Proposition 3.3. *Let $\bar{x}$ be a VC-stationary point of (P). Suppose that f is pseudo-convex at $\bar{x}$, $g_t(t\in I_g), h_i(i\in I_h^+), -h_i(i\in I_h^-), G_i(i\in I_{+0}^+), H_i(i\in \hat{I}_{0+}^-), -H_i(i\in \hat{I}_{0+}^+\cup\hat{I}_{00}^+\cup\hat{I}_{0-}^+)$ are quasiconvex at $\bar{x}$. Then, the following statements hold:*

(i) if $I_{00}^+=\emptyset$, $\bar{x}$ is a local solution of (P);
(ii) if $\hat{I}_{0+}^-\cup I_{+0}^+\cup I_{00}^+=\emptyset$, then $\bar{x}$ is a global solution of (P).

Proof. Since $\bar{x}$ is a VC-stationary point of (P), there exists $(\lambda_J^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}_+^{|J|}\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$, where J is a finite subset of I_g, with $\lambda_{I_+}^H=0$, $\lambda_{I_{00}\cup I_{0-}}^H\geq 0$, $\lambda_{I_{+-}\cup I_{0+}\cup I_{0-}}^G=0$ and $\lambda_{I_{+0}\cup I_{00}}^G\geq 0$ such that

$$\nabla f(\bar{x}) + \sum_{t\in J}\lambda_t^g\nabla g_t(\bar{x}) + \sum_{i\in I_h}\lambda_i^h\nabla h_i(\bar{x}) - \sum_{i\in I_{0+}\cup I_{00}\cup I_{0-}}\lambda_i^H\nabla H_i(\bar{x}) + \sum_{i\in I_{+0}\cup I_{00}}\lambda_i^G\nabla G_i(\bar{x}) = 0. \tag{3.10}$$

Since $I_{00}^+=\emptyset$, (3.10) implies (3.3). The proof is continued just as in the proof of Proposition 3.2. □

Example 3.2. Let $n=2$ and $l=1$. Consider the following (P):

$$\begin{aligned} \min\quad & f(x)=x_1^2+x_2^2+2x_1,\\ \text{s.t.}\quad & g_t(x)=-tx_1\leq 0, t\in\mathbb{N},\\ & H_1(x)=x_1-x_2\geq 0,\\ & G_1(x)H_1(x)=x_1(x_1-x_2)\leq 0. \end{aligned}$$

Then, $\Omega=\{x\in\mathbb{R}^2\mid x_1\geq 0, x_1-x_2=0\}\cup\{x\in\mathbb{R}^2\mid x_1=0, x_2\leq 0\}$. For $\bar{x}=(0,0)\in\Omega$, one has

$$T(\Omega,\bar{x})=\Omega, \nabla f(\bar{x})=\{(2,0)\}, I_g=\mathbb{N}, \nabla g_t(\bar{x})=\{(-t,0)\}, t\in T,$$

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^-=\{x\in\mathbb{R}^2\mid x_1\geq 0\},$$

$$I_+=I_{0+}=I_{0-}=\emptyset, I_{00}=\{1\}, \nabla G_1(\bar{x})=\{(1,0)\}, \nabla H_1(\bar{x})=\{(1,-1)\},$$

$$(\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})))^-=\{x\in\mathbb{R}^2\mid x_1-x_2\geq 0\}, (\bigcup_{i\in I_{00}}\nabla G_i(\bar{x}))^-=\{x\in\mathbb{R}^2\mid x_1\leq 0\}.$$

Hence,

$$(\bigcup_{t\in I_g}\nabla g_t(\bar{x}))^- \cap (\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})))^- \cap (\bigcup_{i\in I_{00}}\nabla G_i(\bar{x}))^- \subset T(\Omega,\bar{x}),$$

i.e., (VC-ACQ) holds at $\bar{x}$. Moreover,

$$\Delta_1 = \text{pos}\left(\bigcup_{t\in I_g}\nabla g_t(\bar{x}) \cup \bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x})) \cup \bigcup_{i\in I_{00}}\nabla G_i(\bar{x})\right) = \{x\in\mathbb{R}^2 \mid x_2 \geq 0\}$$

is closed. Since $f(x)-f(\bar{x})\geq 0$ for all $x\in\Omega$, $\bar{x}$ is a solution of (P). Thus, all assumptions in Proposition 3.1 (ii) are satisfied. Now, let $\lambda_1^H = 0$, $\lambda_1^G = 0$ and $\lambda^g : T\to\mathbb{R}$ be defined by

$$\lambda^g(t) = \begin{cases} 1, & \text{if } t = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$(2,0) + \sum_{t\in T}\lambda_t^g(-t,0) - \lambda_1^H(1,-1) + \lambda_1^G(1,0) = (0,0),$$

i.e., $\bar{x}$ is a VC-stationary point of (P). Notice that, for the above $(\lambda^g,\lambda_1^H,\lambda_1^G)$, $\bar{x}$ is also a strong stationary point of (P). Furthermore, we can check that $f, g_t (t\in T), H_1, -H_1, G_1$ are convex at $\bar{x}$. Hence, all assumptions in Proposition 3.2 and Proposition 3.3 are fulfilled. Then, it follows that $\bar{x}$ is a solution of (P). Note that if we choose $\tilde{\lambda}_1^H = 0$, $\tilde{\lambda}_1^G = 1$ and

$$\tilde{\lambda}^g(t) = \begin{cases} 1, & \text{if } t = 3, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$(2,0) + \sum_{t\in T}\tilde{\lambda}_t^g(-t,0) - \tilde{\lambda}_1^H(1,-1) + \tilde{\lambda}_1^G(1,0) = (0,0),$$

i.e., $\bar{x}$ is a VC-stationary point of (P). But, $\bar{x}$ is not a strong stationary point of (P) in this case.

4. The duality

In this section, we consider the Wolfe [25] and Mond-Weir [17] duality schemes for (P). For $\bar{x}\in\Omega$, the index sets with respect to $\bar{x}$ are denoted identically to Section 3.

4.1. **The Wolfe type duality.** For $\bar{x}\in\Omega$, $(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}_+^{|T|}\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})}\geq 0$, $\lambda^G_{I_{0+}(\bar{x})}\leq 0$ and $\lambda^G_{I_{+-}(\bar{x})\cup I_{0-}(\bar{x})}\geq 0$, we define

$$L(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) := f(u) + \sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) - \sum_{i\in I_l}\lambda_i^H H_i(u) + \sum_{i\in I_l}\lambda_i^G G_i(u).$$

In the line of [15], we consider the Wolfe type dual problem as follows:

$$\begin{aligned}(\mathrm{D}_W(\bar{x})):\ \max L(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) = f(u) &+ \sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) \\ &- \sum_{i\in I_l}\lambda_i^H H_i(u) + \sum_{i\in I_l}\lambda_i^G G_i(u)\end{aligned}$$

$$\begin{aligned}\text{s.t. } & \nabla f(u) + \sum_{t\in T}\lambda_t^g\nabla g_t(u) + \sum_{i\in I_h}\lambda_i^h\nabla h_i(u) - \sum_{i\in I_l}\lambda_i^H\nabla H_i(u) + \sum_{i\in I_l}\lambda_i^G\nabla G_i(u) = 0, \\ & \lambda^H_{I_+(\bar{x})}\geq 0,\ \lambda^G_{I_{0+}(\bar{x})}\leq 0,\ \lambda^G_{I_{+-}(\bar{x})\cup I_{0-}(\bar{x})}\geq 0, \\ & (u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}_+^{|T|}\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l.\end{aligned}$$

The feasible set of $(\mathrm{D}_W(\bar{x}))$ is defined by

$$\begin{aligned}\Omega_W(\bar{x}) := \Big\{ & (u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \mathbb{R}^n \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l \mid \lambda^H_{I_+(\bar{x})} \geq 0, \\ & \lambda^G_{I_{0+}(\bar{x})} \leq 0, \lambda^G_{I_{+-}(\bar{x})\cup I_{0-}(\bar{x})} \geq 0, \\ & \nabla f(u) + \sum_{t\in T}\lambda_t^g \nabla g_t(u) + \sum_{i\in I_h}\lambda_i^h \nabla h_i(u) - \sum_{i\in I_l}\lambda_i^H \nabla H_i(u) + \sum_{i\in I_l}\lambda_i^G \nabla G_i(u) = 0 \Big\}.\end{aligned}$$

We designate by

$$\mathrm{pr}_{\mathbb{R}^n}\Omega_W(\bar{x}) := \{u \in \mathbb{R}^n \mid (u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_W(\bar{x})\}$$

the projection of the set $\Omega(\bar{x})$ on $\mathbb{R}^n$. The other Wolfe type duality problem of (P), which is not dependent on $\bar{x}$, is

$$\begin{aligned}(\mathrm{D}_W):\ \max L(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) &= f(y) + \sum_{t\in T}\lambda_t g_t(y) + \sum_{i\in I_h}\lambda_i^h h_i(y) \\ &\quad - \sum_{i\in I_l}\lambda_i^H H_i(y) + \sum_{i\in I_l}\lambda_i^G G_i(y) \\ \text{s.t. } (y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) &\in \Omega_W := \bigcap_{\bar{x}\in\Omega}\Omega_W(\bar{x}).\end{aligned}$$

Definition 4.1. Let $\bar{x} \in \Omega$.

(i) $(\bar{u},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in \Omega_W(\bar{x})$ is a local solution of $(\mathrm{D}_W(\bar{x}))$, denoted by $(\bar{u},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in loc\mathscr{S}(\mathrm{D}_W(\bar{x}))$, if there exists $U \in \mathscr{N}(\bar{u})$ such that

$$L(\bar{u},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \geq L(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H),\ \forall (u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_W(\bar{x}) \cap U.$$

(ii) $(\bar{y},\bar{\alpha},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in \Omega_W$ is a local solution of (D_W), denoted by $(\bar{y},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in loc\mathscr{S}(\mathrm{D}_W)$, if there exists $U \in \mathscr{N}(\bar{y})$ such that

$$L(\bar{y},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \geq L(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H),\ \forall (y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_W \cap U.$$

If $U = \mathbb{R}^n$, the word "local" is omitted.

The following propositions describe weak duality relations between (P) and the dual problems $(\mathrm{D}_W(\bar{x}))$ and (D_W).

Proposition 4.1. *(Weak duality) Let $x \in \Omega$ and $(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_W$. If f is convex at y, $g_t(t \in I_g^+(x)), h_i(i \in I_h^+(x)), -h_i(i \in I_h^-(x)), H_i(i \in \hat{I}_0^-(x)), -H_i(i \in \hat{I}_+^+(x) \cup \hat{I}_0^+(x)), G_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x)), -G_i(i \in I_{+0}^-(x) \cup I_{0+}^-(x) \cup I_{00}^-(x))$ are convex at y, then*

$$f(x) \geq L(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H).$$

Proof. For $x \in \Omega$ and $(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_W = \bigcap_{\bar{x}\in\Omega}\Omega_W(\bar{x})$, one gets

$$g_t(x) \leq 0(t \in T), h_i(x) = 0(i \in I_h), H_i(x) \geq 0(i \in I_l), G_i(x)H_i(x) \leq 0(i \in I_l), \tag{4.1}$$

and

$$\nabla f(y) + \sum_{t\in T}\lambda_t^g \nabla g_t(y) + \sum_{i\in I_h}\lambda_i^h \nabla h_i(y) - \sum_{i\in I_l}\lambda_i^H \nabla H_i(y) + \sum_{i\in I_l}\lambda_i^G \nabla G_i(y) = 0 \tag{4.2}$$

with $\lambda^H_{I_+(x)} \geq 0, \lambda^G_{I_{0+}(x)} \leq 0, \lambda^G_{I_{+-}(x)\cup I_{0-}(x)} \geq 0$.

Therefore, we infer from (4.1), the convexity of $f, g_t(t\in I_g^+(x)), h_i(i\in I_h^+(x)), -h_i(i\in I_h^-(x))$, $H_i(i\in \hat{I}_0^-(x))$, $-H_i(i\in \hat{I}_+^+(x)\cup\hat{I}_0^+(x)), G_i(i\in I_{+0}^+(x)\cup I_{+-}^+(x)\cup I_{00}^+(x)\cup I_{0-}^+(x)), -G_i(i\in I_{+0}^-(x)\cup I_{0+}^-(x)\cup I_{00}^-(x))$ at y and the definitions of the index sets that

$$f(y)+\langle\nabla f(y),x-y\rangle\le f(x),$$

$$g_t(y)+\langle\nabla g_t(y),x-y\rangle\le g_t(x)\le 0,\lambda_t^g>0,\forall t\in I_g^+(x),$$

$$h_i(y)+\langle\nabla h_i(y),x-y\rangle\le h_i(x)=0,\lambda_i^h>0,\forall i\in I_h^+(x),$$

$$-h_i(y)+\langle-\nabla h_i(y),x-y\rangle\le -h_i(x)=0,\lambda_i^h<0,\forall i\in I_h^-(x),$$

$$H_i(y)+\langle\nabla H_i(y),x-y\rangle\le H_i(x)=0,\lambda_i^H<0,\forall i\in \hat{I}_0^-(x),$$

$$-H_i(y)+\langle-\nabla H_i(y),x-y\rangle\le -H_i(x)<0,\lambda_i^H>0,\forall i\in \hat{I}_+^+(x),$$

$$-H_i(y)+\langle-\nabla H_i(y),x-y\rangle\le -H_i(x)=0,\lambda_i^H>0,\forall i\in \hat{I}_0^+(x),$$

$$G_i(y)+\langle\nabla G_i(y),x-y\rangle\le G_i(x)=0,\lambda_i^G>0,\forall i\in I_{+0}^+(x)\cup I_{00}^+(x),$$

$$G_i(y)+\langle\nabla G_i(y),x-y\rangle\le G_i(x)<0,\lambda_i^G>0,\forall i\in I_{+-}^+(x)\cup I_{0-}^+(x),$$

$$-G_i(y)+\langle-\nabla G_i(y),x-y\rangle\le -G_i(x)=0,\lambda_i^G<0,\forall i\in I_{+0}^-(x)\cup I_{00}^-(x),$$

and

$$-G_i(y)+\langle-\nabla G_i(y),x-y\rangle\le -G_i(x)<0,\lambda_i^G<0,\forall i\in I_{0+}^-(x).$$

The above inequalities imply that

$$\begin{aligned}&f(y)+\sum_{t\in T}\lambda_t g_t(y)+\sum_{i\in I_h}\lambda_i^h h_i(y)-\sum_{i\in I_l}\lambda_i^H H_i(y)+\sum_{i\in I_l}\lambda_i^G G_i(y)\\&+\left\langle\nabla f(y)+\sum_{t\in T}\lambda_t^g\nabla g_t(y)+\sum_{i\in I_h}\lambda_i^h\nabla h_i(y)-\sum_{i\in I_l}\lambda_i^H\nabla H_i(y)+\sum_{i\in I_l}\lambda_i^G\nabla G_i(y),x-y\right\rangle\le f(x),\end{aligned}$$

which together with (4.2) leads to

$$f(x)\ge L(y,\alpha,\lambda^g,\lambda^h,\lambda^G,\lambda^H).$$

The proof is complete. □

Corollary 4.1. *(Weak duality) Let $\bar{x}\in\Omega$ and $(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Omega_W(\bar{x})$. If f is convex at u, $g_t(t\in I_g^+(\bar{x})), h_i(i\in I_h^+(\bar{x})), -h_i(i\in I_h^-(\bar{x})), H_i(i\in\hat{I}_0^-(\bar{x})), -H_i(i\in\hat{I}_+^+(\bar{x})\cup\hat{I}_0^+(\bar{x})), G_i(i\in I_{+0}^+(\bar{x})\cup I_{+-}^+(\bar{x})\cup I_{00}^+(\bar{x})\cup I_{0-}^+(\bar{x})), -G_i(i\in I_{+0}^-(\bar{x})\cup I_{0+}^-(\bar{x})\cup I_{00}^-(\bar{x}))$ are convex at u, then*

$$f(\bar{x})\ge L(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H).$$

Proposition 4.2. *(Strong duality) Let $\bar{x}\in\Omega$ be a local solution of (P). If (VC-ACQ) holds at $\bar{x}$ and the set Δ_1 is closed, then there exists $(\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)\in\mathbb{R}_+^{|T|}\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\bar{\lambda}^H_{I_+(\bar{x})}=0$, $\bar{\lambda}^H_{I_{00}(\bar{x})\cup I_{0-}(\bar{x})}\ge 0$, $\bar{\lambda}^G_{I_{+-}(\bar{x})\cup I_{0+}(\bar{x})\cup I_{0-}(\bar{x})}=0$ and $\bar{\lambda}^G_{I_{+0}(\bar{x})\cup I_{00}(\bar{x})}\ge 0$ such that $(\bar{x},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)\in\Omega_W(\bar{x})$ and $f(\bar{x})=L(\bar{x},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)$. Moreover, if f is convex at $\bar{x}$, $g_t(t\in I_g^+(\bar{x})), h_i(i\in I_h^+(\bar{x})), -h_i(i\in I_h^-(\bar{x})), H_i(i\in\hat{I}_0^-(\bar{x})), -H_i(i\in\hat{I}_+^+(\bar{x})\cup\hat{I}_0^+(\bar{x})), G_i(i\in I_{+0}^+(\bar{x})\cup I_{+-}^+(\bar{x})\cup I_{00}^+(\bar{x})\cup I_{0-}^+(\bar{x})), -G_i(i\in I_{+0}^-(\bar{x})\cup I_{0+}^-(\bar{x})\cup I_{00}^-(\bar{x}))$ are convex at $\bar{x}$, then $(\bar{x},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)$ is a solution of $D_W(\bar{x})$.*

Proof. By Proposition 3.1 (ii), there exists $(\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)\in\Lambda(\bar x)\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\bar\lambda^H_{I_+(\bar x)}=0$, $\bar\lambda^H_{I_{00}(\bar x)\cup I_{0-}(\bar x)}\geq 0$, $\bar\lambda^G_{I_{+-}(\bar x)\cup I_{0+}(\bar x)\cup I_{0-}(\bar x)}=0$ and $\bar\lambda^G_{I_{+0}(\bar x)\cup I_{00}(\bar x)}\geq 0$ such that

$$\nabla f(\bar x)+\sum_{t\in T}\bar\lambda^g_t\nabla g_t(\bar x)+\sum_{i\in I_h}\bar\lambda^h_i\nabla h_i(\bar x)-\sum_{i\in I_l}\bar\lambda^H_i\nabla H_i(\bar x)+\sum_{i\in I_l}\bar\lambda^G_i\nabla G_i(\bar x)=0.$$

Since $\bar\lambda^g\in\Lambda(\bar x)$, one has $\bar\lambda^g_t g_t(\bar x)=0$ for all $t\in T$, and thus, $\sum_{t\in T}\bar\lambda^g_t g_t(\bar x)=0$. The fact that $\bar x\in\Omega$ guarantees that $\sum_{i\in I_h}\bar\lambda^h_i h_i(\bar x)=0$. Moreover, we deduce from $\bar\lambda^H_{I_+(\bar x)}=0$ and $H_i(\bar x)=0$ for all $i\in I_0(\bar x)$ that $\sum_{i\in I_l}\bar\lambda^H_i H_i(\bar x)=0$. Analogously, since $\bar\lambda^G_{I_{+-}(\bar x)\cup I_{0+}(\bar x)\cup I_{0-}}(\bar x)=0$ and $G_i(\bar x)=0$ for all $i\in I_{+0}(\bar x)\cup I_{00}(\bar x)$, one has $\sum_{i\in I_l}\bar\lambda^G_i G_i(\bar x)=0$. Thus, $(\bar x,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)\in\Omega_W(\bar x)$ and

$$\sum_{t\in T}\lambda_t g_t(\bar x)+\sum_{i\in I_h}\lambda^h_i h_i(\bar x)-\sum_{i\in I_l}\lambda^H_i H_i(\bar x)+\sum_{i\in I_l}\lambda^G_i G_i(\bar x)=0,$$

which implies $f(\bar x)=L(\bar x,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)$.

Now, we suppose to the contrary that $(\bar x,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)$ is not a solution of $D_W(\bar x)$. By definition, there exists $(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Omega_W(\bar x)$ such that

$$L(\bar x,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)<L(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H),$$

Consequently,

$$f(\bar x)<L(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H).$$

which contradicts with Corollary 4.1. So, $(\bar x,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)$ is a solution of $(D_W(\bar x))$. □

Example 4.1. Let $n=2$ and $l=1$. Consider the following (P):

$$\begin{array}{ll}\min & f(x)=x_1^2+x_2^2+2x_1,\\ \text{s.t.} & g_t(x)=-tx_1\leq 0, t\in T=\mathbb{N},\\ & H_1(x)=x_1-x_2\geq 0,\\ & G_1(x)H_1(x)=x_1(x_1-x_2)\leq 0.\end{array}$$

Then, $\Omega=\{x\in\mathbb{R}^2\mid x_1\geq 0, x_1-x_2=0\}\cup\{x\in\mathbb{R}^2\mid x_1=0, x_2\leq 0\}$. For any $\bar x\in\Omega$,

$(\mathrm{D}_W(\bar x))$: $\max L(u,\lambda^g,\lambda^G,\lambda^H)=u_1^2+u_2^2+2u_1-\sum_{t\in T}\lambda_t t u_1-\lambda^H_1(u_1-u_2)+\lambda^G_1 u_1$

s.t. $(2u_1+2,u_2)+\sum_{t\in T}\lambda^g_t(-t,0)-\lambda^H_1(1,-1)+\lambda^G_1(1,0)=(0,0)$,

$$\lambda^H_1\begin{cases}\geq 0, & \text{if } 1\in I_+(\bar x),\\ \in\mathbb{R}, & \text{if } 1\in I_0(\bar x),\end{cases}\quad \lambda^G_1\begin{cases}\leq 0, & \text{if } 1\in I_{0+}(\bar x),\\ \geq 0, & \text{if } 1\in I_{+-}(\bar x)\cup I_{0-}(\bar x),\\ \in\mathbb{R}, & \text{if } 1\in I_{+0}(\bar x)\cup I_{00}(\bar x),\end{cases}$$

$(u,\lambda^g,\lambda^G_1,\lambda^H_1)\in\mathbb{R}^2\times\mathbb{R}^{|T|}_+\times\mathbb{R}\times\mathbb{R}$.

By taking $\bar x=(0,0)\in\Omega$, we invoke from Example 3.2 that all hypotheses of Proposition 4.2 are fulfilled. Now, if we select $\bar\lambda^H_1=0$, $\bar\lambda^G_1=0$ and

$$\bar\lambda^g(t)=\begin{cases}1, & \text{if } t=2,\\ 0, & \text{otherwise},\end{cases}$$

then

$$(2,0)+\sum_{t\in T}\bar\lambda^g_t(-t,0)-\bar\lambda^H_1(1,-1)+\bar\lambda^G_1(1,0)=(0,0),$$

i.e., $(\bar{x},\bar{\lambda}^g,\bar{\lambda}_1^G,\bar{\lambda}_1^H)\in\Omega_W(\bar{x})$ and $f(\bar{x})=L(\bar{x},\bar{\lambda}^g,\bar{\lambda}_1^G,\bar{\lambda}_1^H)$. Moreover, we can check that f is convex at $\bar{x}$ and $g_t(t\in T),G_1,H_1,-H_1$ are convex at $\bar{x}$. Hence, Proposition 4.2 asserts that $(\bar{x},\bar{\lambda}^g,\bar{\lambda}_1^G,\bar{\lambda}_1^H)$ is a solution of $(\mathrm{D}_W(\bar{x}))$.

We can check directly that $(\bar{x},\bar{\lambda}^g,\bar{\lambda}_1^G,\bar{\lambda}_1^H)$ is a solution of $(\mathrm{D}_W(\bar{x}))$ as follows. Since $\bar{x}=(0,0)$, one has $I_+(\bar{x})=I_{0+}(\bar{x})=I_{0-}=\emptyset,I_{00}(\bar{x})=\{1\}$. Hence,

$$\Omega_W(\bar{x})=\Big\{(u,\lambda^g,\lambda_1^G,\lambda_1^H)\in\mathbb{R}^2\times\mathbb{R}_+^{|T|}\times\mathbb{R}\times\mathbb{R}\mid\lambda_1^H\in\mathbb{R},\lambda_1^G\in\mathbb{R},$$

$$(2u_1+2,u_2)+\sum_{t\in T}\lambda_t^g(-t,0)-\lambda_1^H(1,-1)+\lambda_1^G(1,0)=(0,0)\Big\}.$$

For an arbitrary $u\in\Omega_W(\bar{x})$, we deduce from the convexity of $f,g_t(t\in I_g^+(\bar{x})),G_1,H_1,-H_1$ at u and the definitions of the index sets that

$$f(u)+\langle(2u_1+2,u_2),\bar{x}-u\rangle\le f(\bar{x}),$$

$$g_t(u)+\langle(-t,0),\bar{x}-u\rangle\le g_t(\bar{x})\le 0,\lambda_t^g>0,\forall t\in I_g^+(\bar{x}),$$

$$H_1(u)+\langle(1,-1),\bar{x}-u\rangle\le H_1(\bar{x})=0,\lambda_1^H<0,\text{if }1\in\hat{I}_{00}^-(\bar{x}),$$

$$-H_1(u)+\langle-(1,-1),\bar{x}-u\rangle\le -H_1(\bar{x})=0,\lambda_i^H>0,\text{if }1\in\hat{I}_{00}^+(\bar{x}),$$

$$G_1(u)+\langle(1,0),\bar{x}-u\rangle\le G_1(\bar{x})=0,\lambda_1^G>0,\text{if }1\in I_{00}^+(\bar{x}),$$

and

$$-G_1(u)+\langle-(1,0),\bar{x}-u\rangle\le -G_1(\bar{x})=0,\lambda_1^G<0,\text{if }1\in I_{00}^-(\bar{x}).$$

The above inequalities imply that

$$f(u)+\sum_{t\in T}\lambda_t g_t(u)-\sum_{i\in I_l}\lambda_i^H H_i(u)+\sum_{i\in I_l}\lambda_i^G G_i(u)$$

$$+\left\langle(2u_1+2,u_2)+\sum_{t\in T}\lambda_t^g(-t,0)-\lambda_1^H(1,-1)+\lambda_1^G(1,0),\bar{x}-u\right\rangle\le f(\bar{x}),$$

which together with $u\in\Omega_W(\bar{x})$ yields that

$$L(u,\lambda^g,\lambda_1^G,\lambda_1^H)\le f(\bar{x})=L(\bar{x},\bar{\lambda}^g,\bar{\lambda}_1^G,\bar{\lambda}_1^H).$$

4.2. **The Mond-Weir type duality.** For $\bar{x}\in\Omega$, $(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}_+^{|T|}\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})}\ge 0$, $\lambda^G_{I_{0+}(\bar{x})}\le 0$ and $\lambda^G_{I_{+-}(\bar{x})\cup I_{0-}(\bar{x})}\ge 0$, we define

$$\widetilde{L}(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H):=f(u).$$

In the line of [15], we consider the Mond-Weir type dual problem as follows:

$(\mathrm{D}_{MW}(\bar{x}))$: $\max\widetilde{L}(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)=f(u)$

s.t. $\nabla f(u)+\sum\limits_{t\in T}\lambda_t^g\nabla g_t(u)+\sum\limits_{i\in I_h}\lambda_i^h\nabla h_i(u)-\sum\limits_{i\in I_l}\lambda_i^H\nabla H_i(u)+\sum\limits_{i\in I_l}\lambda_i^G\nabla G_i(u)=0,$

$\lambda_t^g g_t(u)\ge 0(t\in T),\lambda_i^h h_i(u)=0(i\in I_h),-\lambda_i^H H_i(u)\ge 0(i\in I_l),\lambda_i^G G_i(u)\ge 0(i\in I_l),$

$\lambda^H_{I_+(\bar{x})}\ge 0,\ \lambda^G_{I_{0+}(\bar{x})}\le 0,\ \lambda^G_{I_{+-}(\bar{x})\cup I_{0-}(\bar{x})}\ge 0,$

$(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}_+^{|T|}\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l.$

The feasible set of ($D_{MW}(\bar{x})$) is defined by

$$\Omega_{MW}(\bar{x}) := \Big\{(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \mathbb{R}^n \times \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l \mid \lambda^H_{I_+(\bar{x})} \geq 0,$$
$$\lambda^G_{I_{0+}(\bar{x})} \leq 0, \lambda^G_{I_{+-}(\bar{x})\cup I_{0-}(\bar{x})} \geq 0,$$
$$\nabla f(u) + \sum_{t\in T} \lambda_t^g \nabla g_t(u) + \sum_{i\in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i\in I_l} \lambda_i^H \nabla H_i(u) + \sum_{i\in I_l} \lambda_i^G \nabla G_i(u) = 0,$$
$$\lambda_t^g g_t(u) \geq 0 \ (t \in T), \lambda_i^h h_i(u) = 0 \ (i \in I_h),$$
$$-\lambda_i^H H_i(u) \geq 0 \ (i \in I_l), \lambda_i^G G_i(u) \geq 0 \ (i \in I_l)\Big\}.$$

We designate by

$$\mathrm{pr}_{\mathbb{R}^n}\Omega_{MW}(\bar{x}) := \{u \in \mathbb{R}^n \mid (u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_{MW}(\bar{x})\}$$

the projection of the set $\Omega_{MW}(\bar{x})$ on $\mathbb{R}^n$. The other Mond-Weir type duality problem of (P), which is not dependent on $\bar{x}$, is

$$(D_{MW}): \ \max \widetilde{L}(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) = f(y)$$
$$\text{s.t. } (y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_{MW} := \bigcap_{\bar{x}\in\Omega} \Omega_{MW}(\bar{x}).$$

Definition 4.2. Let $\bar{x} \in \Omega$.

(i) $(\bar{u},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$ is a local solution of ($D_{MW}(\bar{x})$), denoted by $(\bar{u},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in loc\mathscr{S}(D_{MW}(\bar{x}))$, if there exists $U \in \mathscr{N}(\bar{u})$ such that

$$\widetilde{L}(\bar{u},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \geq \widetilde{L}(u,\lambda^g,\lambda^h,\lambda^G,\lambda^H), \ \forall (u,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_{MW}(\bar{x}) \cap U.$$

(ii) $(\bar{y},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in \Omega_{MW}$ is a local solution of (D_{MW}), denoted by $(\bar{y},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \in loc\mathscr{S}(D_{MW})$, if there exists $U \in \mathscr{N}(\bar{y})$ such that

$$\widetilde{L}(\bar{y},\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H) \geq \widetilde{L}(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H), \ \forall (y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_{MW} \cap U.$$

If $U = \mathbb{R}^n$, then the word "local" is dropped.

Proposition 4.3. *(Weak duality) Let $x \in \Omega$ and $(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_{MW}$. If f is pseudoconvex at y, $g_t(t \in I_g^+(x)), h_i(i \in I_h^+(x)), -h_i(i \in I_h^-(x)), H_i(i \in \hat{I}_0^-(x)), -H_i(i \in \hat{I}_+^+(x) \cup \hat{I}_0^+(x)), G_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x)), -G_i(i \in I_{+0}^-(x) \cup I_{0+}^-(x) \cup I_{00}^-(x))$ are quasiconvex at y, then*

$$f(x) \geq \widetilde{L}(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H).$$

Proof. For $x \in \Omega$ and $(y,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Omega_{MW} = \bigcap_{\bar{x}\in\Omega} \Omega_{MW}(\bar{x})$, one gets

$$g_t(x) \leq 0 (t \in T), h_i(x) = 0 (i \in I_h), H_i(x) \geq 0 (i \in I_l), G_i(x)H_i(x) \leq 0 (i \in I_l), \tag{4.3}$$

$$\nabla f(y) + \sum_{t\in T} \lambda_t^g \nabla g_t(y) + \sum_{i\in I_h} \lambda_i^h \nabla h_i(y) - \sum_{i\in I_l} \lambda_i^H \nabla H_i(y) + \sum_{i\in I_l} \lambda_i^G \nabla G_i(y) = 0, \tag{4.4}$$

and

$$\lambda_t^g g_t(y) \geq 0 (t \in T), \lambda_i^h h_i(y) = 0 (i \in I_h), -\lambda_i^H H_i(y) \geq 0 (i \in I_l), \lambda_i^G G_i(y) \geq 0 (i \in I_l), \tag{4.5}$$

with $\lambda^H_{I_+(x)} \geq 0$, $\lambda^G_{I_{0+}(x)} \leq 0$, and $\lambda^G_{I_{+-}(x)\cup I_{0-}(x)} \geq 0$. It follows from the above inequalities that

$$g_t(x) \leq 0 \leq g_t(y), \forall t \in I_g^+(x),$$

$$h_i(x) = h_i(y) = 0, \forall i \in I_h^+(x) \cup I_h^-(x),$$

$$H_i(x) = 0 \leq H_i(y), \forall i \in \hat{I}_0^-(x),$$

$$-H_i(x) \leq 0 \leq -H_i(y), \forall i \in \hat{I}_+^+(x) \cup \hat{I}_0^+(x),$$

$$G_i(x) \leq 0 \leq G_i(y), \forall i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x),$$

and

$$-G_i(x) \leq 0 \leq -G_i(y), \forall i \in I_{+0}^-(x) \cup I_{0+}^-(x) \cup I_{00}^-(x).$$

Therefore, we deduce from the quasiconvexity of $g_t(t \in I_g^+(x)), h_i(i \in I_h^+(x)), -h_i(i \in I_h^-(x)), H_i(i \in \hat{I}_0^-(x)), -H_i(i \in \hat{I}_+^+(x) \cup \hat{I}_0^+(x)), G_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x)), -G_i(i \in I_{+0}^-(x) \cup I_{0+}^-(x) \cup I_{00}^-(x))$ at y and the definitions of the index sets that

$$\langle \nabla g_i(y), x-y \rangle \leq 0, \lambda_i^g > 0, \forall i \in I_g^+(x),$$

$$\langle \nabla h_i(y), x-y \rangle \leq 0, \lambda_i^h > 0, \forall i \in I_h^+(x),$$

$$\langle -\nabla h_i(y), x-y \rangle \leq 0, \lambda_i^h < 0, \forall i \in I_h^-(x),$$

$$\langle \nabla H_i(y), x-y \rangle \leq 0, \lambda_i^H < 0, \forall i \in \hat{I}_0^-(x),$$

$$\langle -\nabla H_i(y), x-y \rangle \leq 0, \lambda_i^H > 0, \forall i \in \hat{I}_+^+(x) \cup \hat{I}_0^+(x),$$

$$\langle \nabla G_i(y), x-y \rangle \leq 0, \lambda_i^G > 0, \forall i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x),$$

and

$$\langle -\nabla G_i(y), x-y \rangle \leq 0, \lambda_i^G < 0, \forall i \in I_{+0}^-(x) \cup I_{0+}^-(x) \cup I_{00}^-(x).$$

We derive from the above inequalities and (4.4) that

$$\langle \nabla f(y), x-y \rangle = -\left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(y) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(y) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(y) + \sum_{i \in I_l} \lambda_i^G \nabla G_i(y), x-y \right\rangle \geq 0,$$

which along with the pseudoconvexity of f at y implies that

$$f(x) \geq f(y) = \widetilde{L}(y, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

The conclusion is obtained. □

Corollary 4.2. *(Weak duality) Let $\bar{x} \in \Omega$ and $(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$. If f is pseudoconvex at u, $g_t(t \in I_g^+(\bar{x})), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), H_i(i \in \hat{I}_0^-(\bar{x})), -H_i(i \in \hat{I}_+^+(\bar{x}) \cup \hat{I}_0^+(\bar{x})), G_i(i \in I_{+0}^+(\bar{x}) \cup I_{+-}^+(\bar{x}) \cup I_{00}^+(\bar{x})), -G_i(i \in I_{+0}^-(\bar{x}) \cup I_{0+}^-(\bar{x}) \cup I_{00}^-(\bar{x}))$ are quasiconvex at u, then*

$$f(\bar{x}) \geq \widetilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

Proposition 4.4. *(strong duality) Let $\bar{x} \in \Omega$ be a local solution of (P). If (VC-ACQ) holds at $\bar{x}$ and the set Δ_1 is closed, then there exists $(\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\bar{\lambda}^H_{I_+(\bar{x})} = 0$, $\bar{\lambda}^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0$, $\bar{\lambda}^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{0-}(\bar{x})} = 0$ and $\bar{\lambda}^G_{I_{+0}(\bar{x}) \cup I_{00}(\bar{x})} \geq 0$ such that $(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$. Moreover, if f is pseudoconvex at $\bar{x}$, $g_t(t \in I_g^+(\bar{x})), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), H_i(i \in \hat{I}_0^-(\bar{x})), -H_i(i \in \hat{I}_+^+(\bar{x}) \cup \hat{I}_0^+(\bar{x})), G_i(i \in I_{+0}^+(\bar{x}) \cup I_{+-}^+(\bar{x}) \cup I_{00}^+(\bar{x})), -G_i(i \in I_{+0}^-(\bar{x}) \cup I_{0+}^-(\bar{x}) \cup I_{00}^-(\bar{x}))$ are quasiconvex at $\bar{x}$, then $(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is a solution of $D_{MW}(\bar{x})$.*

Proof. By Proposition 3.1 (ii), there exists $(\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\bar{\lambda}^H_{I_+(\bar{x})} = 0$, $\bar{\lambda}^H_{I_{00}(\bar{x})\cup I_{0-}(\bar{x})} \geq 0$, $\bar{\lambda}^G_{I_{+-}(\bar{x})\cup I_{0+}(\bar{x})\cup I_{0-}(\bar{x})} = 0$ and $\bar{\lambda}^G_{I_{+0}(\bar{x})\cup I_{00}(\bar{x})} \geq 0$ such that

$$\nabla f(\bar{x}) + \sum_{t\in T} \bar{\lambda}^g_t \nabla g_t(\bar{x}) + \sum_{i\in I_h} \bar{\lambda}^h_i \nabla h_i(\bar{x}) - \sum_{i\in I_l} \bar{\lambda}^H_i \nabla H_i(\bar{x}) + \sum_{i\in I_l} \bar{\lambda}^G_i \nabla G_i(\bar{x}) = 0.$$

Since $\bar{\lambda}^g \in \Lambda(\bar{x})$, one has $\bar{\lambda}^g_t g_t(\bar{x}) = 0$ for all $t \in T$. It follows from $\bar{x} \in \Omega$ that $\lambda^h_i h_i(\bar{x}) = 0, \forall i \in I_h$. Moreover, we deduce from $\bar{\lambda}^H_{I_+(\bar{x})} = 0$ and $H_i(\bar{x}) = 0$ for all $i \in I_0(\bar{x})$ that $-\bar{\lambda}^H_i H_i(\bar{x}) = 0$ for all $i \in I_l$. Similarly, since $\bar{\lambda}^G_{I_{+-}(\bar{x})\cup I_{0+}(\bar{x})\cup I_{0-}(\bar{x})} = 0$ and $G_i(\bar{x}) = 0$ for all $i \in I_{+0}(\bar{x}) \cup I_{00}(\bar{x})$, one has $\bar{\lambda}^G_i G_i(\bar{x}) = 0$ for all $i \in I_l$. Thus, $(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$ and $f(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$.

Arguing by contradiction, we suppose that $(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is not a solution of $D_{MW}(\bar{x})$. By denotation, there exists $(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$ such that

$$f(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) < \widetilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

which contradicts Corollary 4.2. So, $(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is a solution of $D_{MW}(\bar{x})$. This completes the proof. □

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