J. Nonlinear Var. Anal. 4 (2020), No. 3, pp. 357-376
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.4.2020.3.03

A PARALLEL ITERATIVE METHOD FOR SOLVING A CLASS OF VARIATIONAL INEQUALITIES IN HILBERT SPACES

TRUONG MINH TUYEN[1,*], TRAN XUAN QUY[1], NGUYEN MINH TRANG[2]

[1] *Department of Mathematics and Informatics, Thai Nguyen University of Sciences, Thai Nguyen, Vietnam*
[2] *Faculty of International training, Thainguyen University of Technology, Thai Nguyen, Vietnam*

Abstract. The purpose of this paper is to introduce a new parallel iterative method for solving a variational inequality over the set of common fixed points of a finite family of sequences of nearly nonexpansive mappings. Solution theorems are established in a real Hilbert space.
Keywords. Variational inequality; Sequence of nearly nonexpansive mappings; Common fixed point; Common null point.

1. INTRODUCTION

Let C be a nonempty convex and closed subset of a real Hilbert space H. The classical variational inequality problem initially studied by Stampacchia [17] for a nonlinear operator $F: C \to H$ is the problem of finding an element $x^\dagger \in D$ such that

$$\langle Fx^\dagger, x^\dagger - y\rangle \leq 0, \ \forall y \in D, \tag{1.1}$$

where D is a nonempty convex closed subset of C. We denote by $\mathrm{VI}_D(C,F)$ the problem (1.1). If $C \equiv H$, we denote this problem by $\mathrm{VI}(D,F)$. The problem $\mathrm{VI}(D,F)$ is equivalent to the problem of finding a fixed point of the mapping $P_D(I-\lambda F)$, for all $\lambda > 0$, where P_D is the metric projection from H onto D. We know that if F is Lipschitzian and strongly monotone, then, for small $\lambda > 0$, the mapping $P_D(I-\lambda F)$ is a strict contraction. So, by the Banach contraction principle, the problem $\mathrm{VI}_D(C,F)$ has a unique solution $x^\dagger$ and the Picard iterative sequence $\{x_n\}$ defined by $x_{n+1} = P_D(I-\lambda F)x_n$ converge strongly to $x^\dagger$. The equivalence relation between the variational inequality problem and fixed point problem plays an important role in developing some efficient methods for solving variational inequality problems and related optimization problems; see. e,g., [2, 3, 5, 13] and the references therein. The problem of finding the fixed points of a nonexpansive mapping is the subject of the current interest related to variational inequality problems in functional analysis.

In 2000, Moudafi [10] proposed a viscosity approximation method for finding a fixed point of a nonexpansive mapping in Hilbert spaces, and then this method was considered by many authors. In 2001, Yamada [25] introduced a hybrid steepest-descent method for solving the

*Corresponding author.
E-mail addresses: tuyentm@tnus.edu.vn (T.M. Tuyen), quytx@tnus.edu.vn (T.X. Quy), nguyenminhtrang@tnut.edu.vn (N.M. Trang).
Received January 16, 2020; Accepted April 27, 2020.

problem (1.1), where $F : H \to H$ is a Lipschitz strongly monotone operator and D is the set of fixed points of a nonexpansive mapping $T : H \to H$, i.e., $D = Fix(T)$. Moreover, Yamada also considered the problem (1.1) in the case that D is the set of common fixed points of a finite family of nonexpansive mappings $T_1, T_2, ..., T_N$, i.e., $D = \cap_{i=1}^{N} Fix(T_i)$. Moreover, the problem of finding a fixed point of a nonexpansive mapping T when T is given by the perturbation mappings T_n, $n \geq 1$, has been studied by many authors; see, e.g., Combettes [4], Kim and Xu [8]. We know that, in some special cases, the sequence of perturbation mappings $\{T_n\}$ is a sequence of nearly nonexpansive mappings (see [16, Example 5.1] or Example 5.1 of Section 5). The theory of variational inequalities over the set of common fixed points of a finite family of sequences of nearly nonexpansive mappings is now a considerable research interest in optimization theory.

In 2012, inspired by Aoyama et al. [1], Ceng, Ansari and Yao [2], Wong, Sahu and Yao [23], Sahu, Kang and Sagar [16] introduced a strong convergence theorem for finding a solution of the variational inequality over the set of common fixed points of a sequence of nearly nonexpansive mappings. Recently, the results presented in [16] were further extended by several authors; see, e.g., Tuyen and Ha [20] and Tuyen [21, 22]. In 2018, Tuyen [21] introduced a cyclic iterative method for a more general problem and proved the following result.

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. Let $\mathscr{T}_i = \{T_{i,n}\}$, $i = 1,2,...,N$ be sequences of nearly nonexpansive mappings from C into itself with respect to the sequences $\{a_{i,n}\}$ such that $S = \cap_{i=1}^{N} Fix(\mathscr{T}_i) \neq \emptyset$ and T_i, $i = 1,2,...,N$, be mappings from C into itself defined by $T_i x = \lim_{n\to\infty} T_{i,n}x$ for all $x \in C$. Suppose that $\cap_{i=1}^{N} Fix(T_i) = \cap_{i=1}^{N} Fix(\mathscr{T}_i)$, $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be a sequence in C generated by the following iterative method:*

$$\begin{aligned} y_n^0 &= x_n, \\ y_n^i &= \beta_{i,n} y_n^{i-1} + (1-\beta_{i,n}) T_{i,n} y_n^{i-1},\ i = 1,2,...,N, \\ x_{n+1} &= P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) y_n^N],\ n \geq 0, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_{i,n}\}$, $i = 1,2,...,N$, are the sequences in $[a,b]$, with $a,b \in (0,1)$, which satisfy the following conditions:

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

ii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$;

iii) $\sum_{n=0}^{\infty} |\beta_{i,n+1} - \beta_{i,n}| < \infty$ *for all* $i = 1,2,...,N$;

iv) *either* $\sum_{n=0}^{\infty} \mathscr{D}_B(T_{i,n}, T_{i,n+1}) < \infty$ *or* $\lim_{n\to\infty} \mathscr{D}_B(T_{i,n}, T_{i,n+1})/\alpha_{n+1} = 0$ *for each* $B \in \mathscr{B}(C)$, *for all* $i = 1,2,...,N$;

v) $\lim_{n\to\infty} a_{i,n}/\alpha_n = 0$ *for all* $i = 1,2,...,N$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\dagger \in S$, which is the unique solution of the variational inequality $VI_S(C, \mu F - \gamma V)$.

Remark 1.1.

i) We denote by $\mathscr{B}(C)$ the collection of the bounded subsets of C and

$$\mathscr{D}_B(T_n, T_{n+1}) = \sup_{x\in B} \|T_{n+1}x - T_n x\|,$$

for all $B \in \mathscr{B}(C)$.

ii) Suppose $\sum_{n=0}^{\infty} \mathscr{D}_B(T_n, T_{n+1}) < \infty$ for each $B \in \mathscr{B}(C)$, where $\mathscr{T} = \{T_n\}$ is a sequence of nearly nonexpansive mappings from C into itself with respect to the sequence $\{a_n\}$.

From the assumption $\sum_{n=0}^{\infty} \mathscr{D}_B(T_n, T_{n+1}) < \infty$, we have $\sum_{n=1}^{\infty} \|T_n x - T_{n+1} x\| < \infty$ for all $x \in B$. Hence $\{T_n x\}$ is a Cauchy sequence for each $x \in B$. So, $\{T_n x\}$ converges strongly to some point for each $x \in B$. For each $x \in C$, since the set $B = \{x\}$ belongs to $\mathscr{B}(C)$, $\{T_n x\}$ converges to some point in C. Let $T : C \to C$ be a mapping defined by $Tx = \lim_{n\to\infty} T_n x$ for each $x \in C$. It is easy to see that T is a nonexpansive mapping. Now, for each $B \in \mathscr{B}(C)$, $x \in B$ and $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}\|T_n x - T_m x\| &\leq \sum_{k=n}^{m-1} \|T_k x - T_{k+1} x\| \\ &\leq \sum_{k=n}^{m-1} \mathscr{D}_B(T_k, T_{k+1}) \\ &\leq \sum_{k=n}^{\infty} \mathscr{D}_B(T_k, T_{k+1}).\end{aligned}$$

Thus,

$$\|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| \leq \sum_{k=n}^{\infty} \mathscr{D}_B(T_k, T_{k+1}),$$

for all $x \in B$. This implies that

$$\mathscr{D}_B(T_n, T) \leq \sum_{k=n}^{\infty} \mathscr{D}_B(T_k, T_{k+1}).$$

Therefore, $\lim_{n\to\infty} \mathscr{D}_B(T_n, T) = 0$.

iii) Suppose $\lim_{n\to\infty} \mathscr{D}_B(T_n, T) = 0$ for each $B \in \mathscr{B}(C)$, where T is a mapping from C into itself. Then, T is a nonexpansive mapping and $Fix(\mathscr{T}) \subset Fix(T)$.

In this paper, we give a parallel iterative method for finding a solution of the variational inequality $\mathrm{VI}_S(C, \mu F - \gamma V)$, where S is the set of the common fixed points of a finite family of sequences of nearly nonexpansive mappings. We prove the strong convergence theorem under the following conditions (see, Theorem 3.1):

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
ii) $\mathscr{D}_B(T_{i,n}, T_i) \to 0$ for each $B \in \mathscr{B}(C)$, for all $i = 1, 2, ..., N$;
iii) $\lim_{n\to\infty} a_{i,n}/\alpha_n = 0$ for all $i = 1, 2, ..., N$.

Next, in Section 4, we give some applications of the main theorem to the problem of finding a common fixed point of nonexpansive mappings or nonexpansive semigroups and the problem of finding a common null point of monotone operators in a real Hilbert space. Finally, in Section 5, we give a numerical example to illustrate the obtained results and show its performance.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let C be a nonempty convex closed subset of H. We know that, for each $x \in H$, there is a unique $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u \in C} \|x - u\|, \tag{2.1}$$

and the mapping $P_C : H \to C$ defined by (2.1) is called the metric projection from H onto C.

Recall that a mapping $T: C \to H$ is said to be:

a) monotone if $\langle Tx - Ty, x-y\rangle \geq 0$, $\forall x, y \in C$;

b) η-strongly monotone if there exists a positive real number η such that

$$\langle Tx - Ty, x-y\rangle \geq \eta \|x-y\|^2, \ \forall x, y \in C;$$

c) k-Lipschitzian if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x-y\|, \ \forall x, y \in C.$$

If $k = 1$, then T is called a nonexpansive mapping, and if $k \in [0,1)$, then T is called a strict contraction.

Let $\mathscr{T} = \{T_n\}$ be a sequence of mappings from C into itself. We denote by $Fix(\mathscr{T})$ the set of common fixed points of the sequence $\mathscr{T}$, that is, $Fix(\mathscr{T}) = \cap_{n=1}^{\infty} Fix(T_n)$. Fix a sequence $\{a_n\} \subset [0,1)$ such that $\lim_{n\to\infty} a_n = 0$, and let $\{T_n\}$ be a sequence of mappings from C into H. Then, $\{T_n\}$ is called a sequence of nearly nonexpansive mappings [23] with respect to the sequence $\{a_n\}$ if $\|T_n x - T_n y\| \leq \|x-y\| + a_n$, for all $x, y \in C$ and for all $n \in \mathbb{N}$.

We know that if C is a bounded set and $T: C \to C$ is an asymptotically nonexpansive mapping, that is, there is a sequence $\{k_n\}$ such that, $k_n \geq 1$ for all n, $\lim_{n\to\infty} k_n = 1$, $\|T^n x - T^n y\| \leq k_n \|x-y\|$ for all $x, y \in C$, then $\mathscr{T} = \{T^n\}$ is a sequence of nearly nonexpansive mappings. Indeed, for all $x, y \in C$, we have

$$\begin{aligned}\|T^n x - T^n y\| &\leq k_n \|x-y\| \\ &= \|x-y\| + (k_n - 1)\|x-y\| \\ &\leq \|x-y\| + (k_n-1)\operatorname{diam}(C).\end{aligned}$$

Hence, $\mathscr{T}$ is a sequence of nearly nonexpansive mappings with respect to sequence $\{(k_n - 1)\operatorname{diam}(C)\}$.

For an operator $A: H \to 2^H$, we define its domain, range and graph as follows:

$$\begin{aligned}D(A) &= \{x \in H : \ Ax \neq \emptyset\}, \\ R(A) &= \cup\{Az : \ z \in D(A)\},\end{aligned}$$

and

$$G(A) = \{(x,y) \in H \times H : \ x \in D(A), \ y \in Ax\},$$

respectively. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y \text{ if and only if } y \in Ax.$$

The operator A is said to be monotone if, for each $x, y \in D(A)$, $\langle u - v, x - y\rangle \geq 0$ for all $u \in Ax$ and $v \in Ay$. We denote by I the identity operator on H. A monotone operator A is said to be maximal monotone if there is no proper monotone extension of A or $R(I + \lambda A) = H$ for all $\lambda > 0$ (see [15]). If A is monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A : R(I+\lambda A) \to D(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$, which is called the resolvent of A. A monotone operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A.

Remark 2.1. If A is a maximal monotone, then it satisfies the range condition.

The following lemmas will be needed in the sequel for the proof of main results in this paper.

Lemma 2.1. *Let H be a real Hilbert space. For all $x, y \in H$ and $t \in [0,1]$, we have*

a) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
b) $\|(1-t)x+ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2$.

Lemma 2.2. [6, 7] *The metric projection mapping P_C is characterized by the following properties:*

a) $P_C x \in C$ *for all* $x \in H$;
b) $\langle x - P_C x, P_C x - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;
c) $\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ *for all* $x \in H$ *and* $y \in C$;
d) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$ *for all* $x, y \in H$.

Lemma 2.3. [2] *Let $V : C \to H$ be an L-Lipschitzian mapping and $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator. Then, for $0 \leq \gamma L < \mu\eta$,*

$$\langle x-y, (\mu F - \gamma V)x - (\mu F - \gamma V)y\rangle \geq (\mu\eta - \gamma L)\|x-y\|^2, \ \forall x, y \in C, \tag{2.2}$$

that is, $\mu F - \gamma V$ is strongly monotone with coefficient $\mu\eta - \gamma L$.

Lemma 2.4. [25] *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator on C. Define the mapping $G : C \to H$ by $Gx = (I - \lambda\mu F)x$, $\forall x \in C$. Then G is a strict contraction provided $\mu < 2\eta/k^2$. More precisely, for $\mu \in (0, 2\eta/k^2)$,*

$$\|Gx - Gy\| \leq (1 - \lambda\tau)\|x-y\|, \ \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.5. [6] *Let T be a nonexpansive self-map defined on a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$.*

Lemma 2.6. [18] *Let $A : D(A) \to 2^H$ be a monotone operator. For each λ, $\mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, it holds*

$$\|J_\lambda^A x - J_\mu^A x\| \leq \frac{|\lambda - \mu|}{\lambda}\|x - J_\lambda^A x\|.$$

Lemma 2.7. [9] *Let $\{s_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_k}\}$ so that $s_{n_k} \leq s_{n_k+1}$, $\forall k \geq 0$. For every $n > n_0$ define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : \ s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ for all $n > n_0$, $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.

Lemma 2.8. [24] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n + \sigma_n,$$

where $\{b_n\}$, $\{c_n\}$, and $\{\sigma_n\}$ are the sequences of real numbers such that

i) $\{b_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} b_n = +\infty$,
ii) $\limsup_{n\to\infty} c_n \leq 0$,
iii) $\sum_{n=0}^{\infty} \sigma_n < \infty$ *or* $\limsup_{n\to\infty} \sigma_n/b_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. Let $\mathscr{T}_i = \{T_{i,n}\}$, $i = 1,2,...,N$ be sequences of nearly nonexpansive mappings from C into itself with respect to the sequences $\{a_{i,n}\}$ such that $S = \cap_{i=1}^N Fix(\mathscr{T}_i) \neq \emptyset$ and let T_i, $i = 1,2,...,N$, be mappings from C into itself such that $\lim_{n\to\infty} \mathscr{D}_B(T_{i,n}, T_i) = 0$. Suppose that $\cap_{i=1}^N Fix(T_i) = \cap_{i=1}^N Fix(\mathscr{T}_i)$ and $0 \le \gamma L < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$. We consider the following problem:

$$\text{Find an element } x^\dagger \in S, \tag{3.1}$$

such that $x^\dagger$ is a unique solution of the variational inequality $\mathrm{VI}_S(C, \mu F - \gamma V)$.

In order to solve Problem (3.1), we introduce the following parallel iterative method.

Algorithm 3.1. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be a sequence in C generated by the following iterative method:

$$\begin{aligned} &y_n^i = \beta_{i,n} x_n + (1-\beta_{i,n}) T_{i,n} x_n, \ i = 1,2,...,N, \\ &\text{Chosse } i_n \text{ such that } \|y_n^{i_n} - x_n\| = \max_{i=1,2,...,N} \|y_n^i - x_n\|, \text{ let } y_n = y_n^{i_n}, \\ &x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) y_n], \ n \ge 0, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_{i,n}\}$, $i = 1,2,...,N$, are the sequences in $[a,b]$, with $a, b \in (0,1)$.

We begin with the following proposition.

Proposition 3.1. *In Algorithm 3.1, the sequence $\{x_n\}$ is bounded.*

Proof. Taking $p \in S$ for each $i \in \{1,2,...,N\}$, we have

$$\begin{aligned} \|y_n^i - p\| &\le \beta_{i,n}\|x_n - p\| + (1-\beta_{i,n})\|T_{i,n}x_n - T_{i,n}p\| \\ &\le \beta_{i,n}\|x_n - p\| + (1-\beta_{i,n})\|x_n - p\| + a_{i,n} \\ &= \|x_n - p\| + a_{i,n}. \end{aligned} \tag{3.3}$$

Next, from (3.3), we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n \gamma L \|x_n - p\| + \alpha_n \|(\gamma V - \mu F)p\| + (1-\alpha_n \tau)\|y_n - p\| \\ &\le \alpha_n \gamma L \|x_n - p\| + \alpha_n \|(\gamma V - \mu F)p\| + (1-\alpha_n \tau)(\|x_n - p\| + a_{i_n,n}) \\ &\le [1 - \alpha_n(\tau - \gamma L)]\|x_n - p\| + \alpha_n \|(\gamma V - \mu F)p\| + a_{i_n,n}. \end{aligned} \tag{3.4}$$

Since $\lim_{n\to\infty} a_{i,n}/\alpha_n = 0$ for all $i = 1,2,\ldots,N$, we find that there exists $K_1 > 0$ such that

$$\frac{\alpha_n \|(\gamma V - \mu F)p\| + a_{i,n}}{\alpha_n} \le K_1, \ \forall n \ge 0,$$

for all $i = 1, 2, \ldots, N$. Thus, from (3.4), we get that

$$\begin{aligned}\|x_{n+1} - p\| &\leq [1 - \alpha_n(\tau - \gamma L)]\|x_n - p\| + \alpha_n K_1 \\ &\leq \max\{\frac{K_1}{\tau - \gamma L}, \|x_n - p\|\} \\ &\vdots \\ &\leq \max\{\frac{K_1}{\tau - \gamma L}, \|x_0 - p\|\},\end{aligned}$$

which implies that $\{x_n\}$ is bounded, so are $\{T_{i,n}x_n\}$, $\{Fy_n^i\}$, $i = 1, 2, ..., N$, and $\{Vx_n\}$. This completes the proof. ☐

Proposition 3.2. *In Algorithm 3.1, if*

$$s_n = \|x_n - x^*\|^2,\ b_n = \frac{2\alpha_n(\tau - \gamma L)}{1 + \alpha_n(\tau - \gamma L)},$$

and

$$c_n = \frac{1}{2(\tau - \gamma L)}\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle,$$

$$\sigma_n = \frac{a(1-b)(1 - \alpha_n\tau)}{1 + \alpha_n(\tau - \gamma L)}\|T_{i_n,n}x_n - x_n\|^2, \sigma_n' = \frac{K_2}{1 + \alpha_n(\tau - \gamma L)}a_{i_n,n},$$

for each $x^* \in S$, then

$$s_{n+1} \leq (1 - b_n)s_n + b_n c_n - \sigma_n + \sigma_n'. \tag{3.5}$$

Proof. Let $x^* \in S$ and $z_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu F)y_n$. Since $x_{n+1} = P_C(z_n)$, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \langle z_n - x^*, x_{n+1} - x^*\rangle + \langle P_C z_n - z_n, P_C z_n - x^*\rangle \\ &\leq \langle z_n - x^*, x_{n+1} - x^*\rangle \\ &= \langle \alpha_n(\gamma Vx_n - \mu Fx^*) + [(I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)x^*], x_{n+1} - x^*\rangle \\ &= \alpha_n\gamma\langle Vx_n - Vx^*, x_{n+1} - x^*\rangle + \alpha_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\ &\quad + \langle(I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)x^*, x_{n+1} - x^*\rangle \\ &\leq \alpha_n\gamma L\|x_n - x^*\|.\|x_{n+1} - x^*\| + \alpha_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\ &\quad + (1 - \alpha_n\tau)\|y_n - x^*\|.\|x_{n+1} - x^*\| \\ &\leq \frac{\alpha_n\gamma L}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \frac{(1 - \alpha_n\tau)}{2}(\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\quad + \alpha_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}[1 + \alpha_n(\tau - \gamma L)]\|x_{n+1} - x^*\|^2 \leq\ &\alpha_n\gamma L\|x_n - x^*\|^2 + (1 - \alpha_n\tau)\|y_n - x^*\|^2 \\ &+ 2\alpha_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle.\end{aligned} \tag{3.6}$$

Now, it follows from Lemma 2.1 that

$$\begin{aligned}\|y_n-x^*\|^2 &= \|\beta_{i_n,n}(x_n-x^*)+(1-\beta_{i_n,n})(T_{i_n,n}x_n-x^*)\|^2\\ &= \beta_{i_n,n}\|x_n-x^*\|^2+(1-\beta_{i_n,n})\|T_{i_n,n}x_n-x^*\|^2-\beta_{i_n,n}(1-\beta_{i_n,n})\|T_{i_n,n}x_n-x_n\|^2\\ &\le \beta_{i_n,n}\|x_n-x^*\|^2+(1-\beta_{i_n,n})(\|x_n-x^*\|+a_{i_n,n})^2-\beta_{i_n,n}(1-\beta_{i_n,n})\|T_{i_n,n}x_n-x_n\|^2\\ &= \|x_n-x^*\|^2+(2\|x_n-x^*\|^2+a_{i_n,n})a_{i_n,n}-a(1-b)\|T_{i_n,n}x_n-x_n\|^2\\ &\le \|x_n-x^*\|^2+K_2a_{i_n,n}-a(1-b)\|T_{i_n,n}x_n-x_n\|^2,\end{aligned} \tag{3.7}$$

where

$$K_2=\max_{i=1,2,\dots,N}\left\{\sup_n\{2\|x_n-x^*\|^2+a_{i,n}\|\}\right\}<\infty.$$

From (3.6) and (3.7), we get

$$\begin{aligned}[1+\alpha_n(\tau-\gamma L)]\|x_{n+1}-x^*\|^2 \le [1-\alpha_n(\tau-\gamma L)\|x_n-x^*\|^2+K_2(1-\alpha_n\tau)a_{i_n,n}\\ -a(1-b)(1-\alpha_n\tau)\|T_{i_n,n}x_n-x_n\|^2\\ +2\alpha_n\langle(\gamma V-\mu F)x^*,x_{n+1}-x^*\rangle,\end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le [1-\frac{2\alpha_n(\tau-\gamma L)}{1+\alpha_n(\tau-\gamma L)}]\|x_n-x^*\|^2+\frac{K_2(1-\alpha_n\tau)}{1+\alpha_n(\tau-\gamma L)}a_{i_n,n}\\ -\frac{a(1-b)(1-\alpha_n\tau)}{1+\alpha_n(\tau-\gamma L)}\|T_{i_n,n}x_n-x_n\|^2\\ +\frac{2\alpha_n(\tau-\gamma L)}{1+\alpha_n(\tau-\gamma L)}\frac{1}{\tau-\gamma L}\langle(\gamma V-\mu F)x^*,x_{n+1}-x^*\rangle.\end{aligned} \tag{3.9}$$

Put

$$s_n=\|x_n-x^*\|^2,\ b_n=\frac{2\alpha_n(\tau-\gamma L)}{1+\alpha_n(\tau-\gamma L)},$$

and

$$c_n=\frac{1}{2(\tau-\gamma L)}\langle(\gamma V-\mu F)x^*,x_{n+1}-x^*\rangle,$$

$$\sigma_n=\frac{a(1-b)(1-\alpha_n\tau)}{1+\alpha_n(\tau-\gamma L)}\|T_{i_n,n}x_n-x_n\|^2,\sigma_n'=\frac{K_2}{1+\alpha_n(\tau-\gamma L)}a_{i_n,n}.$$

Then, inequality (3.9) can be rewritten as

$$s_{n+1}\le(1-b_n)s_n+b_nc_n-\sigma_n+\sigma_n'.$$

This completes the proof. □

Theorem 3.1. *Suppose that the following conditions hold*

i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$;
ii) $\lim_{n\to\infty}a_{i,n}/\alpha_n=0$ *for all* $i=1,2,\dots,N$.

Then, the sequence $\{x_n\}$ *generated in Algorithm 3.1 converges strongly to* $x^\dagger\in S$*, which is the unique solution of the variational inequality* $VI_S(C,\mu F-\gamma V)$*, that is,*

$$\langle(\mu F-\gamma V)x^\dagger,x^\dagger-y\rangle\le 0,\ \forall y\in S. \tag{3.10}$$

Proof. Let $x^\dagger \in S$ be the unique solution of variational inequality (3.10), we next show that $s_n \to 0$ by considering two possible cases.

Case 1. $\{s_n\}$ is eventually decreasing, i.e. there exists $N_0 \geq 0$ such that $\{s_n\}$ is decreasing for $n \geq N_0$ and thus $\{s_n\}$ must be convergent. It then follows from (3.5) that

$$0 \leq \sigma_n \leq (s_n - s_{n+1}) + b_n(c_n - s_n) + \sigma_n' \to 0.$$

Hence $\|x_n - T_{i_n,n}x_n\| \to 0$ as $n \to \infty$. We further have

$$\|y_n^{i_n} - x_n\| = \beta_{i_n,n}\|x_n - T_{i_n,n}x_n\| \leq b\|x_n - T_{i_n,n}x_n\| \to 0,$$

which implies that

$$\|y_n^{i_n} - x_n\| = \|y_n - x_n\| \to 0. \tag{3.11}$$

It follows from the definition of y_n that

$$\|y_n^i - x_n\| \to 0, \tag{3.12}$$

for all $i = 1, 2, \ldots, N$. Thus, from (3.2) and the condition $\{\beta_{i_n}\} \subset [a,b] \subset (0,1)$, we obtain

$$\|x_n - T_{i,n}x_n\| = \frac{1}{\beta_{i,n}}\|y_n^i - x_n\| \to 0, \tag{3.13}$$

for all $i = 1, 2, \ldots, N$.

Next, letting $B = \{x_n\}$, for each $i \in \{1, 2, ..., N\}$, we have

$$\begin{aligned}\|x_n - T_i x_n\| &\leq \|x_n - T_{i,n}x_n\| + \|T_{i,n}x_n - T_i x_n\| \\ &\leq \|x_n - T_{i,n}x_n\| + \mathscr{D}_B(T_{i,n}, T_i) \to 0,\end{aligned}$$

which implies that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i = 1, 2, ..., N$.

Next, we show that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_{n+1} - x^\dagger\rangle \leq 0.$$

Since $T_i x = \lim_{n\to\infty} T_{i,n}x$ for all $x \in C$, T_i is a nonexpansive mapping for all $i = 1, 2, ..., N$. Let $T = \frac{1}{N}\sum_{i=1}^N T_i$. It is clear that T is a nonexpansive mapping and $S = \cap_{i=1}^N F(T_i) = F(T)$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_n - x^\dagger\rangle = \lim_{k\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_{n_k} - x^\dagger\rangle. \tag{3.14}$$

Without loss of generality, one may assume that $x_{n_k} \rightharpoonup z \in C$. By Lemma 2.5, and

$$\|x_n - Tx_n\| \leq \frac{1}{N}\|x_n - T_i x_n\| \to 0,$$

as $n \to \infty$, we get that $z \in F(T) = S$. Hence, from (3.14) and Lemma 2.2 b), we have

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_n - x^\dagger\rangle = \langle(\gamma V - \mu F)x^\dagger, z - x^\dagger\rangle \leq 0. \tag{3.15}$$

Now, from the nonexpansiveness of P_C, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|P_C[\alpha_n\gamma V x_n + (I - \alpha_n\mu F)y_n] - P_C(x_n)\| \\ &\leq \|\alpha_n\gamma V x_n + (I - \alpha_n\mu F)y_n - x_n\| \\ &\leq \|(I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)x_n\| + \alpha_n\|(\gamma V - \mu F)x_n\| \\ &\leq (1 - \alpha_n\tau)\|y_n - x_n\| + K_3\alpha_n,\end{aligned} \tag{3.16}$$

where $K_3 = \sup_n\{\|(\gamma V - \mu F)x_n\|\} < \infty$. So, from the condition i) and (3.11), we obtain that $\|x_{n+1} - x_n\| \to 0$. Combining with (3.11), we get

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_{n+1} - x^\dagger\rangle \le 0, \tag{3.17}$$

that is, $\limsup_{n\to\infty} c_n \le 0$. From (3.5) with $x^* = x^\dagger$, we have

$$s_{n+1} \le (1 - b_n)s_n + b_n c_n + \sigma'_n.$$

Hence, all conditions of Lemma 2.6 are satisfied. Therefore, we immediately deduce that $s_n \to 0$ or $x_n \to x^\dagger$.

Case 2. $\{s_n\}$ is not a monotone sequence. Then, from Lemma 2.7, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \le n : \ s_k < s_{k+1}\}.$$

Moreover, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $s_{\tau(n)} < s_{\tau(n+1)}$ for all $n \ge n_0$. From (3.5), we have

$$0 < s_{\tau(n)+1} - s_{\tau(n)} \le b_{\tau(n)}c_{\tau(n)} + \sigma'_{\tau(n)}.$$

Since $b_{\tau(n)} \to 0$, $\sigma'_{\tau(n)} \to 0$ and the boundedness of $\{c_{\tau(n)}\}$, we derive

$$\lim_{n\to\infty}(s_{\tau(n)+1} - s_{\tau(n)}) = 0. \tag{3.18}$$

By a similar argument to Case 1, we obtain

$$\|x_{\tau(n)} - T_i x_{\tau(n)}\| \to 0, \tag{3.19}$$

for all $i = 1, 2, \ldots, N$. Also, we get

$$s_{\tau(n)+1} \le (1 - b_{\tau(n)})s_n + b_{\tau(n)}c_{\tau(n)} + \sigma'_{\tau(n)},$$

where $\limsup_{n\to\infty} c_{\tau(n)} \le 0$. Since $s_{\tau(n)+1} > s_{\tau(n)}$ and $b_{\tau(n)} > 0$, we have

$$0 \le s_{\tau(n)} \le c_{\tau(n)} + \frac{\sigma'_{\tau(n)}}{b_{\tau(n)}}.$$

Thus, from $\limsup_{n\to\infty} c_{\tau(n)} \le 0$ and the condition iii), we get $\lim_{n\to\infty} s_{\tau(n)} = 0$. Combining with (3.18), it implies that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$. Now, we have

$$0 \le s_n \le \max\{s_{\tau(n)}, s_n\} \le s_{\tau(n)+1} \to 0.$$

Therefore, $s_n \to 0$, that is, $\{x_n\}$ converges strongly to $x^\dagger$. This completes the proof. □

Next, we have the following corollary.

Corollary 3.1. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian η-strongly monotone operator and $V : C \longrightarrow H$ be an L-Lipschitzian mapping. Let $\mathscr{T} = \{T_n\}$ be a sequence of nearly nonexpansive mappings from C into itself with respect to the sequence $\{a_n\}$ such that $S = Fix(\mathscr{T}) \ne \emptyset$ and let T be a mapping from C into itself such that* $\lim_{n\infty} \mathscr{D}_B(T_n, T) = 0$ *for each $B \in \mathscr{B}(C)$. Suppose that $Fix(T) = Fix(\mathscr{T})$,*

$0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be a sequence in C generated by the following iterative method:

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n) T_n x_n, \\ x_{n+1} &= P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) y_n],\ n \geq 0, \end{aligned} \tag{3.20}$$

where $\{\alpha_n\} \in (0,1)$ and $\{\beta_n\} \in [a,b]$, with $a, b \in (0,1)$, satisfy the following conditions:

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

ii) $\lim_{n\to\infty} a_n/\alpha_n = 0$.

Then, $\{x_n\}$ converges strongly to $x^\dagger \in S$, which is the unique solution of the variational inequality $VI_S(C, \mu F - \gamma V)$.

If $Vx = 0$ for all x in Algorithm 3.1, then we have the following result for solving Problem (3.1).

Theorem 3.2. *For an arbitrary $x_0 \in C$, let $\{x_n\}$ be a sequence in C generated by the following iterative method:*

$$\begin{aligned} & y_n^i = \beta_{i,n} x_n + (1-\beta_{i,n}) T_{i,n} x_n,\ i = 1,2,\dots,N, \\ & \textit{Chosse } i_n \textit{ such that } \|y_n^{i_n} - x_n\| = \max_{i=1,2,\dots,N} \|y_n^i - x_n\|,\ \textit{let } y_n = y_n^{i_n}, \\ & x_{n+1} = P_C[(I - \alpha_n \mu F) y_n],\ n \geq 0, \end{aligned} \tag{3.21}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_{i,n}\}$, $i = 1,2,\dots,N$ are the sequences in $[a,b]$, with $a, b \in (0,1)$, which satisfy the following conditions:

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

ii) $\lim_{n\to\infty} a_{i,n}/\alpha_n = 0$ *for all* $i = 1,2,\dots,N$.

Then, $\{x_n\}$ converges strongly to $x^\dagger \in S$, which is the unique solution of the variational inequality $VI_S(C, F)$.

4. Applications

4.1. **Common fixed points of nonexpansive mappings.** Now, we give an application of Theorem 3.1 to the problem of finding a common fixed point of a finite family of nonexpansive mappings in a real Hilbert space. We have the following theorem.

Theorem 4.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. Let T_i, $i = 1,2,\dots,N$, be nonexpansive mappings from C into itself such that $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. Suppose that $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be a sequence in C generated by the following iterative method:*

$$\begin{aligned} & y_n^i = \beta_{i,n} x_n + (1-\beta_{i,n}) T_i x_n,\ i = 1,2,\dots,N, \\ & \textit{Chosse } i_n \textit{ such that } \|y_n^{i_n} - x_n\| = \max_{i=1,2,\dots,N} \|y_n^i - x_n\|,\ \textit{let } y_n = y_n^{i_n}, \\ & x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) y_n],\ n \geq 0, \end{aligned} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_{i,n}\}$, $i = 1,2,\dots,N$, are the sequences in $[a,b]$, with $a, b \in (0,1)$, which satisfy the following condition:

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

Then, $\{x_n\}$ converges strongly to $x^\dagger \in S$, which is the unique solution of the variational inequality $VI_S(C,\mu F-\gamma V)$.

Proof. Applying Theorem 3.1 to $T_{i,n}=T_i$ for all $i=1,2,...,N$ and for all $n\in\mathbb{N}$, we obtain the result this theorem immediately. □

If $N=1$ in Theorem 4.1, then we have the following result.

Corollary 4.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F:C\to H$ be a k-Lipschitzian and η-strongly monotone operator and $V:C\to H$ be an L-Lipschitzian mapping. Let T be a nonexpansive mapping from C into itself such that $S=Fix(T)\neq\emptyset$. Suppose that $0<\mu<2\eta/k^2$ and $0\le\gamma L<\tau$, where $\tau=1-\sqrt{1-\mu(2\eta-\mu k^2)}$. For an arbitrary $x_0\in C$, let $\{x_n\}$ be a sequence in C generated by the following iterative method:*

$$\begin{aligned} y_n &= \beta_n x_n+(1-\beta_n)Tx_n,\\ x_{n+1} &= P_C[\alpha_n\gamma Vx_n+(I-\alpha_n\mu F)y_n],\ n\ge 0, \end{aligned} \tag{4.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}\subset[a,b]$ with $a,b\in(0,1)$, which satisfy the following condition:

i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^\infty\alpha_n=\infty$;

Then, $\{x_n\}$ converges strongly to $x^\dagger\in S$, which is the unique solution of the variational inequality $VI_S(C,\mu F-\gamma V)$.

Next, we give a strong convergence theorem for finding a common fixed point of a sequence of nonexpansive mappings.

Theorem 4.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F:C\to H$ be a k-Lipschitzian and η-strongly monotone operator and $V:\ C\longrightarrow H$ be an L-Lipschitzian mapping. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $S=\cap_{n=0}^\infty Fix(T_n)\neq\emptyset$. Let T be a mapping from C into itself such that $\lim_{n\to\infty}\mathscr{D}_B(T_n,T)=0$ for each $B\in\mathscr{B}(C)$ and $Fix(T)=\cap_{n=0}^\infty Fix(T_n)$. Let $\{x_n\}$ be a sequence in C generated by $x_0\in C$ and*

$$\begin{aligned} y_n &= \beta_n x_n+(1-\beta_n)T_nx_n,\\ x_{n+1} &= P_C[\alpha_n\gamma Vx_n+(I-\alpha_n\mu F)y_n],\forall n\ge 0, \end{aligned} \tag{4.3}$$

where $0<\mu<2\eta/k^2$, $0\le\gamma L<\tau$ with $\tau=1-\sqrt{1-\mu(2\eta-\mu k^2)}$ and $\{\alpha_n\}\subset(0,1)$ and $\{\beta_n\}\subset[a,b]$ with $a,b\in(0,1)$, which satisfy the following condition:

i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^\infty\alpha_n=\infty$;

Then, the sequence $\{x_n\}$ converges strongly to $x^\dagger\in S$ which is the unique solution of the variational inequality $VI_S(C,\mu F-\gamma V)$.

Proof. Applying Corollary 3.1 to the case $N=1$, we obtain the proof of this theorem immediately. □

4.2. **Common fixed points of nonexpansive semigroups.** Let C be a nonempty convex closed subset of a Hilbert space H. Motivated by Nakajo and Takahashi [11], Takahashi, Takeuchi and

Kubota [19] gave the following definition: Let $\{T_n\}$ and $\mathscr{T}$ be two families of nonexpansive mappings of C into itself such that

$$Fix(\mathscr{T}) = \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset,$$

where $Fix(T_n)$ is the set of all fixed points of T_n and $Fix(\mathscr{T})$ is the set of all common fixed points of $\mathscr{T}$. Then, $\{T_n\}$ is said to satisfy the NST-condition (I) (NST stands for Nakajo–Shimoji–Takahashi) with $\mathscr{T}$ if, for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0,$$

which implies that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ for all $T \in \mathscr{T}$.

A family $\mathscr{T} = \{T(s) : \ 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions (see [14]):

i) $T(0)x = x$ for all $x \in C$;
ii) $T(s+t) = T(s)T(t)$ for all $s,t \geq 0$;
iii) $\|T(s)x - T(s)y\| \leq \|x-y\|$ for all $s \geq 0$ and $x,y \in C$;
iv) for any $x \in C$, $s \mapsto T(s)x$ is continuous.

We need the following lemma.

Lemma 4.1. [12] *Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathscr{T} = \{T(s) : \ 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C with $Fix(\mathscr{T}) \neq \emptyset$. Let $\{t_n\}$ be a sequence of real numbers with $0 < t_n < \infty$ such that $\lim_{n\to\infty} t_n = \infty$. For $n \in \mathbb{N}$, define a mapping T_n of C into itself by*

$$T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds, \text{ for all } x \in C.$$

Then, $\{T_n\}$ satisfies the NST-condition (I) with $\mathscr{T} = \{T(s) : \ 0 \leq s < \infty\}$.

Theorem 4.3. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $\mathscr{T}_i = \{T_i(s) : \ 0 \leq s < \infty\}$, $i = 1,2,\ldots,N$, be one-parameter nonexpansive semigroups on C with $S = \cap_{i=1}^{N} Fix(\mathscr{T}_i) \neq \emptyset$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. For an arbitrary $x_0 \in C$, $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$ with $\tau = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$, let $\{x_n\}$ be a sequence in C defined by*

$$\begin{aligned} &y_n^i = \beta_{i,n}x_n + (1-\beta_{i,n})T_{i,n}x_n, \ i = 1,2,\ldots,N, \\ &\textit{Chosse } i_n \textit{ such that } \|y_n^{i_n} - x_n\| = \max_{i=1,2,\ldots,N} \|y_n^i - x_n\|, \textit{ let } y_n = y_n^{i_n}, \\ &x_{n+1} = P_C[\alpha_n\gamma V x_n + (I - \alpha_n\mu F)y_n], \ n \geq 0, \end{aligned} \tag{4.4}$$

with $T_{i,n}x = \dfrac{1}{t_n}\int_0^{t_n} T_i(s)xds$, for all $i = 1,2,\ldots,N$ and $x \in C$, where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\beta_{i,n}\}$, $i = 1,2,\ldots,N$ are the sequences in $[a,b]$, with $a,b \in (0,1)$, and $\{t_n\}$ is a sequence in $(0,\infty)$, which satisfy the following conditions:

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
ii) $\lim_{n\to\infty} t_n = \infty$ and $\sum_{n=0}^{\infty} \dfrac{|t_{n+1} - t_n|}{t_{n+1}} < \infty$.

Then, $\{x_n\}$ converges strongly to $x^\dagger \in S$, which is the unique solution of the variational inequality $VI_S(C, \mu F - \gamma V)$.

Proof. We will apply the proof of Theorem 3.1. First, for any $B \in \mathscr{B}(C)$, since $T_{i,n}(s)$ is nonexpansive for all $s \geq 0$ and for all $i = 1,2,\ldots,N$, we have that

$$K = \max_{i=1,2,\ldots,N}\{\sup_{s\geq 0, n\geq 1, x\in B}\{\|T_{i,n}(s)x\|\}\} < \infty.$$

Thus, for each $i = 1,2,\ldots,N$, we have

$$\begin{aligned}\|T_{i,n}x - T_{i,n+1}x\| &= \|\frac{1}{t_n}\int_0^{t_n} T_{i,n}(s)xds - \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T_{i,n+1}(s)xds\| \\ &\leq \frac{|t_{n+1}-t_n|}{t_n t_{n+1}}\|\int_0^{t_n} T_{i,n}(s)xds\| + \frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}} T_{i,n+1}(s)xds\| \\ &\leq 2K\frac{|t_{n+1}-t_n|}{t_{n+1}}.\end{aligned}$$

So, if $\{t_n\}$ satisfies the condition $\sum_{n=0}^{\infty}\dfrac{|t_{n+1}-t_n|}{t_{n+1}} < \infty$, then we obtain that $\sum_{n=0}^{\infty}\mathscr{D}_B(T_n, T_{n+1}) < \infty$ for each $B \in \mathscr{B}(C)$. Thus, it follows from Remark 1.1 that $\lim_{n\to\infty}\mathscr{D}_B(T_n, T) = 0$ for each $B \in \mathscr{B}(C)$.

Now, by an argument similar to the proof of Theorem 3.1, we get that

$$\lim_{n\to\infty}\|x_n - T_{i,n}x_n\| = 0, \tag{4.5}$$

for all $i = 1,2,\ldots,N$. So, by use of Lemma 4.1, we have

$$\lim_{n\to\infty}\|x_n - T_i(s)x_n\| = 0,$$

for all $i = 1,2,\ldots,N$ and $s \geq 0$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_n - x^\dagger\rangle = \lim_{k\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_{n_k} - x^\dagger\rangle, \tag{4.6}$$

where $x^\dagger$ is a unique solution of the variational inequality $\mathrm{VI}_S(C, \mu F - \gamma V)$. Without loss of generality, one may assumes that $x_{n_k} \rightharpoonup z \in C$. By Lemma 2.5 and (4.5), one obtains that $z \in Fix(T_i(s))$ for all $s \geq 0$ and $i = 1,2,\ldots,N$. Thus, $z \in S$. Hence, from (4.6), we have

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^\dagger, x_n - x^\dagger\rangle = \langle(\gamma V - \mu F)x^\dagger, z - x^\dagger\rangle \leq 0. \tag{4.7}$$

The rest of the proof follows the pattern of Theorem 3.1. □

Remark 4.1. The sequences $\alpha_n = \dfrac{1}{n}$, $t_n = \sqrt{n}$, and $\beta_{i,n} = \dfrac{1}{2}$ satisfy all the conditions i)-ii) in Theorem 4.3.

We also have the following result.

Corollary 4.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $\mathscr{T} = \{T(s) : 0 \leq s < \infty\}$ be one-parameter nonexpansive semigroup on C with $S = Fix(\mathscr{T}) \neq \emptyset$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be* *an L-Lipschitzian mapping. For an arbitrary* $x_0 \in C$, $0 < \mu < 2\eta/k^2$ *and* $0 \leq \gamma L < \tau$ *with* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$, *let* $\{x_n\}$ *be a sequence in C defined by*

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) T_n x_n, \\ x_{n+1} &= P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) y_n],\ n \geq 0, \end{aligned} \tag{4.8}$$

with $T_n x = \frac{1}{t_n} \int_0^{t_n} T(s)xds$, *for all* $x \in C$, *where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$, $\{\beta_n\}$ *is the sequence in* $[a,b]$, *with* $a,b \in (0,1)$, *and* $\{t_n\}$ *is a sequence in* $(0,\infty)$, *which satisfy the following conditions:*

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
ii) $\lim_{n\to\infty} t_n = \infty$ *and* $\sum_{n=0}^{\infty} \frac{|t_{n+1} - t_n|}{t_{n+1}} < \infty$.

Then, $\{x_n\}$ *converges strongly to* $x^\dagger \in S$, *which is the unique solution of the variational inequality* $VI_S(C, \mu F - \gamma V)$.

4.3. **Common null points of monotone operators.** Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. Let $A_i :\ \overline{D(A_i)} \subset C \longrightarrow 2^H$, $i = 1,2,...,N$ be monotone operators such that $S = \cap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$ and $\overline{D(A_i)} \subset C \subset \cap_{r>0} R(I + rA_i)$ for all $i = 1,2,...,N$.

Now, to find an element $x^* \in S$, we introduce the following iterative method:

$$\begin{aligned} &y_n^0 = x_n, \\ &y_n^i = \beta_{i,n} x_n + (1 - \beta_{i,n}) J_{i,n} x_n,\ J_{i,n} = (I + r_{i,n} A_i)^{-1},\ i = 1,2,...,N, \\ &\text{Chosse } i_n \text{ such that } \|y_n^{i_n} - x_n\| = \max_{i=1,2,...,N} \|y_n^i - x_n\|, \text{ let } y_n = y_n^{i_n}, \\ &x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) y_n^N],\ n \geq 0, \end{aligned} \tag{4.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_{i,n}\}$, $i = 1,2,...,N$ are the sequences in $[a,b]$, with $a,b \in (0,1)$, and $\{r_{i,n}\}$ are the sequences of positive numbers for all $i = 1,2,...,N$.

Theorem 4.4. *If the sequences* $\{r_{i,n}\}$, $\{\beta_{i,n}\}$, $i = 1,2,...,N$, *and* $\{\alpha_n\}$ *satisfy the following conditions:*

i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
ii) $\min_{i=1,2,...,N}\{\inf_n\{r_{i,n}\}\} \geq r > 0$, $\sum_{n=1}^{\infty} |r_{i,n+1} - r_{i,n}| < \infty$ *for all* $i = 1,2,...,N$,

then the sequence $\{x_n\}$ *defined by* (4.9) *converges strongly to* $x^\dagger \in S$, *which is the unique solution of variational inequality* $VI_S(C, \mu F - \gamma V)$.

Proof. Let $T_{i,n} = J_{i,n}$ for all $i = 1,2,...,N$ and $n \in \mathbb{N}$. Then, $\mathscr{T}_i = \{T_{i,n}\}$ are sequences of nonexpansive mappings for all $i = 1,2,...,N$ and $S = \cap_{i=1}^{N} Fix(\mathscr{T}_i) \neq \emptyset$.

First, we show that $\sum_{n=0}^{\infty} \mathscr{D}_B(T_{i,n+1}, T_{i,n}) < \infty$ for all $i = 1,2,...,N$ and for all $B \in \mathscr{B}(C)$. Let $B \in \mathscr{B}(C)$, and set $K = \max_{i=1,2,...,N}\{\sup_n\{\|z - J_{i,n+1}z\| :\ z \in B\}\} < \infty$. From Lemma 2.6, for each $i \in \{1,2,...,N\}$, we have

$$\begin{aligned}\mathscr{D}_B(T_{i,n+1},T_{i,n}) &= \sup\{\|J_{i,n+1}z - J_{i,n}z\| :\ z\in B\}\\ &\le \sup\{\frac{|r_{i,n+1}-r_{i,n}|}{r_{i,n+1}}\|z-J_{i,n+1}z\| :\ z\in B\}\\ &\le K\frac{|r_{i,n+1}-r_{i,n}|}{r}.\end{aligned}$$

Hence, $\sum_{n=0}^{\infty}\mathscr{D}_B(T_{i,n+1},T_{i,n}) < \infty$ for all $i=1,2,...,N$. Since $\sum_{n=1}^{\infty}|r_{i,n+1}-r_{i,n}| < \infty$, we have that $\{r_{i,n}\}$ are Cauchy sequences for all $i=1,2,...,N$. Suppose $\lim_{n\to\infty} r_{i,n} = r_i \ge r$ for all $i=1,2,...,N$. Setting $T_i = J_{r_i}^{A_i}$ for all $i=1,2,...,N$, we have

$$\|T_{i,n}x - T_ix\| = \|J_{i,n}x - J_{r_i}^{A_i}x\| \le \frac{|r_{i,n}-r_i|}{r_i}\|x-T_ix\|,\ \forall x\in C,$$

which implies that $T_ix = \lim_{n\to\infty} T_{i,n}x$ for all $x\in C$. So, by use of Theorem 3.1, we have that $\{x_n\}$ converges strongly to $x^\dagger \in S$, which is the unique solution of variational inequality $\mathrm{VI}_S(C,\mu F-\gamma V)$. □

Corollary 4.3. *Let C be a nonempty convex closed subset of a Hilbert space H. Let F : CrightarrowH be a k-Lipschitzian and η-strongly monotone operator and V : CrightarrowH be an L-Lipschitzian mapping. Let $A:\ D(A)\subset C \longrightarrow 2^H$ be a monotone operator such that $S=A^{-1}(0)\neq\emptyset$ and $\overline{D(A)}\subset C\subset \cap_{r>0}R(I+rA)$. If the sequences $\{r_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:*

i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$;
ii) $\inf_n\{r_n\}\ge r>0$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$,

then the sequence $\{x_n\}$ defined by $x_0\in C$ and

$$x_{n+1} = P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu F)J_{r_n}^{A}x_n]$$

converges strongly to $x^\dagger\in S$, which is the unique solution of the variational inequality $VI_S(C,\mu F-\gamma V)$.

5. Numerical experiments

In this section, we will verify the proposed algorithm in Theorem 3.2. The algorithms are implemented in MATLAB 7.0 running on a HP Compaq 510, Core(TM) 2 Duo CPU. T5870 with 2.0 GHz and 2GB RAM.

Example 5.1. Consider the problem of finding an element $x^\dagger\in S$ such that

$$\varphi(x^\dagger) = \min_{x\in S}\varphi(x), \tag{5.1}$$

where

$$\begin{aligned}&\varphi(x) = (x_1+1)^2+(x_2-1)^2+x_3^2 \text{ for all } x=(x_1,x_2,x_3)\in\mathbb{R}^3,\\ &S=\cap_{i=1}^{100}C_i\neq\emptyset,\\ &C_i=\{(x_1,x_2,x_3):\ (x_1-1/i)^2+x_2^2+(x_3+1/i)^2\le 2\},\ i=1,2,...,100.\end{aligned}$$

It is easy to show that φ is a convex function, $F = \bigtriangledown\varphi$ is a 2-Lipschitz and 2-strongly monotone operator, and $x^\dagger = (-1,1,0)$ is the minimum point of φ on S.

Let $T_i = P_{C_i}$, $i = 1,2,...,100$, where P_{C_i} is metric projections from $\mathbb{R}^3$ onto C_i. Suppose that P_{C_i} is given by the perturbation operator $P^n_{C_i}$, defined by

$$P^n_{C_i}x = \begin{cases} P_{C_i}x, & \text{if } x \in C_i, \\ P_{C_i}x + a_{i,n}e, & \text{if } x \notin C_i, \end{cases} \tag{5.2}$$

where $\{a_{i,n}\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} a_{i,n} = 0$, $\sum_{n=0}^{\infty} |a_{i,n} - a_{i,n+1}| < \infty$ and $e \in \mathbb{R}^3$ such that $\|e\| \le 1$ for all $i = 1,2,...,100$.

Let $T_{i,n} = P^n_{C_i}$ for all $i = 1,2,...,N$. Then $\mathscr{T}_i = \{T_{i,n}\}$ is a sequence of nearly nonexpansive mappings with respect to the sequence $\{a_{i,n}\}$. Indeed, for all $x, y \in \mathbb{R}^3$, we consider the following cases

Case 1. If $x, y \in C_i$ or $x, y \in \mathbb{R}^3 \setminus C_i$, then

$$\|T_{i,n}x - T_{i,n}y\| = \|P_{C_i}x - P_{C_i}y\| \le \|x - y\|. \tag{5.3}$$

Case 2. If $x \in C_i$ and $y \in \mathbb{R}^3 \setminus C_i$, then

$$\|T_{i,n}x - T_{i,n}y\| = \|P_{C_i}x - P_{C_i}y - a_{i,n}e\| \le \|x - y\| + a_{i,n}. \tag{5.4}$$

From (5.3) and (5.4), we deduce that $\{\mathscr{T}_i\}$ is the sequence of nearly nonexpansive mappings with respect to the sequence $\{a_{i,n}\}$, for all $i = 1,2,...,100$.

It is easy to see that

$$S = \cap_{i=1}^{100} F(\mathscr{T}_i) = \cap_{i=1}^{100} C_i \neq \emptyset.$$

Now, for each $x \in \mathbb{R}^3$, we have $\|T_{i,n}x - T_ix\| \le a_{i,n} \to 0$, as $n \to \infty$, which implies that $\lim_{n\to\infty} \mathscr{T}_{i,n}x = T_ix$, for all $i = 1,2,...,100$.

We know that problem (5.1) is equivalent to the following variational inequality problem.

Find an element $x^\dagger \in S$ such that

$$\langle Fx^\dagger, x^\dagger - y\rangle \le 0, \ \forall y \in S. \tag{5.5}$$

Suppose $a_{i,n} = 1/n$ and $e = (1,0,0)$ for all $i = 1,2,...,100$. Applying the iterative methods (1.2) and (3.14) to $\mu = 9/10$, $\beta_{i,n} = 1/2$ for all $i = 1,2,...,100$, and $x^0 = (1,2,3)$, we obtain the following table of results.

Stop condition: $\text{TOL}_n = \|x_n - x^\dagger\| < \text{err}$

Algorithm (3.21) with $\alpha_n = n^{-0.25}$			Algorithm (1.2) with $\alpha_n = 1/n$		
err	TOL_n	n	err	TOL_n	n
10^{-7}	1.407897e − 008	10	10^{-7}	9.994859e − 008	1784
10^{-8}	1.635852e − 010	11	10^{-8}	9.998582e − 009	6409
10^{-9}	1.635852e − 010	11	10^{-9}	9.999927e − 010	23030
10^{-10}	5.379840e − 012	12	10^{-10}	9.999882e − 011	82779

TABLE 1. Table of numerical results for Algorithm (3.21) and Algorithm (1.2).

Now, we applying our algorithm (Algorithm (3.21)) with the sequence $\{\alpha_n\}$ defined by

$$\alpha_n = \begin{cases} n^{-0.25}, & \text{if } n \text{ odd}, \\ n^{-0.2}, & \text{if } n \text{ even}. \end{cases} \tag{5.6}$$

We obtain the following table of numerical results.

Stop condition: $\mathrm{TOL}_n = \|x_n - x^\dagger\| < \mathrm{err}$		
err	TOL_n	n
10^{-7}	$1.635852e-009$	11
10^{-8}	$1.635852e-009$	11
10^{-9}	$9.357268e-010$	12
10^{-10}	$5.379840e-011$	13

TABLE 2. Table of numerical results for Algorithm (3.21) with $\{\alpha_n\}$ defined by (5.6).

The behaviours of the functions TOL_n for our algorithm and Algorithm (1.2) in the above cases after 100 iteration steps are presented in Figure 1.

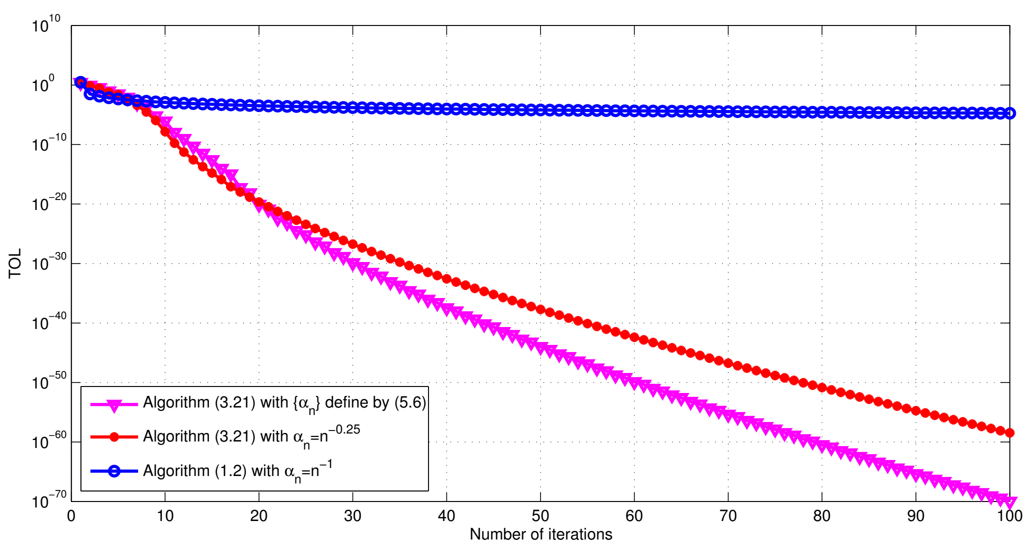

FIGURE 1. The behaviour of the function TOL_n.

Remark 5.1.

i) In this section, we use the symbol TOL_n to denote the error between the approximate solution x^n and the exact solution $x^\dagger$, that is, $\mathrm{TOL}_n = \|x^n - x^\dagger\|$.

ii) In this example (for our algorithm), the sequence $\{\alpha_n\}$ defined by $\alpha_n = n^{-0.25}$ satisfies the condition i) of Theorem 3.2 but do not satisfy the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and the sequence $\{\alpha_n\}$ defined by (5.6) satisfies the conditions i) of Theorem 3.2 but do not satisfy the conditions i) and ii) in Theorem 1.1.

Acknowledgements
The third author was supported by the Science and Technology Fund of Thai Nguyen University of Technology (TNUT), Vietnam.

REFERENCES

[1] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007), 2350-2360.
[2] L.C. Ceng, Q.H. Ansari, J.C. Yao, Some iterative methods for finding fixed points and for solving constrained convex minimization problems, Nonlinear Anal. 74 (2011), 5286-5302.
[3] S.Y. Cho, X. Qin, On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems, Appl. Math. Comput. 235 (2014), 430-438.
[4] P. L. Combettes, On the numerical robustness of the parallel projection method in signal synthesis, IEEE Signal Process. Lett. 8 (2001), 45-47.
[5] P. Duan, X. Zheng, Bounded pertrbation resilience of a viscosity iterative method for split feasibility problems, J. Nonlinear Funct. Anal. 2019 (2019), Article ID 1.
[6] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., vol. 28, Cambridge Univ. Press, Cambridge, UK 1990.
[7] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.
[8] T. H. Kim and H. K. Xu, Robustness of Mann's algorithm for nonexpansive mappings, J. Math. Anal. Appl. 327 (2007), 1105-1115.
[9] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. (2008), 899-912.
[10] A. Moudafi, Vicosity approximation methods for fixed point problems, J. Math. Anal. Appl. 241 (2000), 45-55.
[11] K. Nakajo, W. Takahashi, Strong convergence theorem for nonexpansive mappings and nonexpansive semigroup, J. Math. Anal. Appl. 279 (2003), 372-379.
[12] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings in Banach spaces, J. Nonlinear Convex Anal. 8 (2007), 11-34.
[13] X. Qin, L. Wang, J.C. Yao, Inertial splitting method for maximal monotone mappings, J. Nonlinear Convex Anal. 21 (2020), 2325-2333.
[14] S. Reich, Product formulas, nonlinear semigroups, and accretive operators, J. Funct. Anal. 36 (1980), 147-168.
[15] S. Reich, Extension problems for accretive sets in Banach spaces, J. Funct. Anal. 26 (1977), 378-395.
[16] D. R. Sahu, S.M. Kang, V. Sagar, Approximation of common fixed points of a sequence of nearly nonexpansive mappings and solutions of variational inequality problems, J. Appl. Math. 2012 (2012), Article ID 902437.
[17] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris 258 (1964), 4413-4416.
[18] W. Takahashi, Nonlinear Functional Analysis, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, 2000.
[19] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008), 276-286.
[20] T.M. Tuyen, N.S. Ha, Parallel iterative method for a finite family of sequences of nearly nonexpansive mappings in Hilbert spaces, Comput. Appl. Math. 37 (2018), 3093-3117.
[21] T.M. Tuyen, A cyclic iterative method for solving a class of variational inequalities in Hilbert spaces, Optimization, 67 (2018), 1769-1796.
[22] T.M. Tuyen, Strong convergence theorems for a finite family of sequences of nearly nonexpansive mappings in Hilbert spaces, Numer. Funct. Anal. Optim. 39 (2018), 1034-1053.
[23] N.C. Wong, D.R. Sahu, J.C. Yao, A generalized hybrid steepest-descent method for variational inequalities in Banach spaces, Fixed Point Theory Appl. 2011 (2011), Article ID 754702.
[24] Z. Xue, H. Zhou, Y.J. Cho, Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces, J. Nonlinear Convex Anal. 1 (2000), 313-320.

[25] Y. Yamada, The hybrid steepest-descent method for variational inequalities problems over the intersection of the fixed point sets of nonexpansive mappings, In: Butnariu, D., Censor, Y., Reich, S. (eds.) Inhently Parallel Algorithms in Feasibility and Optimization and Their Applications, pp. 473-504, North-Holland, Amsterdam, 2001.