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TWO INERTIAL EXTRAGRADIENT VISCOSITY ALGORITHMS FOR SOLVING VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

MUJAHID ABBAS[1,*], HIRA IQBAL[2]

[1]*Department of Mathematics, Government College University, Lahore 54000, Pakistan*
[2]*Department of Sciences and Humanities,*
National University of Computer and Emerging Sciences, Lahore Campus, Pakistan

Abstract. The aim of this paper is to propose two different kinds of self-adaptive algorithms for finding a common solution in the set of solutions of the variational inequality problem with a monotone and Lipschitz continuous operator and the set of fixed points of the mapping satisfying some condition in real Hilbert spaces. The algorithms are combinations of extragradient viscosity-type methods and inertial-type methods. Strong convergence theorems are established without the prior information of the Lipschitz constant of the operator. The proposed algorithms can be regarded as an enhancement of the previously known inertial-type gradient methods in each calculation step. Some numerical examples are also presented.

Keywords. Variational inequality; Extragradient method; Subgradient method; Viscosity method; Inertial method.

1. Introduction

Let E be a Hilbert space with inner product $\langle \cdot,\cdot \rangle$. Let C be a nonempty closed convex subset of E and let $\mathscr{A} : C \to E$ be a mpping. Recall the following classical variational inequality problem (VIP), which consists of finding a point $x^* \in C$ such that

$$\langle \mathscr{A} x^*, y \rangle \geq 0, \qquad \forall\, y \in C. \tag{1.1}$$

Denote the solution set of variational inequality (1.1) by $VI(\mathscr{A},C)$. Many problems of practical interest arising in optimization theory, operation research, economics, and engineering are equivalent to certain variational inequality problem; see, e.g., [3, 9, 16, 17, 19]. In general, there are two ways to solve the variational inequality: the regularization methods and the projection methods. The focus of this paper is the projection-based methods. The most basic projection method for solving VIPs is the gradient method, which performs only one projection onto the feasible set but this method involves a strong condition that the operators are either strongly monotone or inverse strongly monotone (see [12]). To avoid this strong condition, Korpelevich

*Corresponding author.
E-mail addresses: abbas.mujahid@gmail.com (M. Abbas), hira.iqbal@nu.edu.pk (H. Iqbal).
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[14] initiated the following extragradient method:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu \mathscr{A} x_n), \\ x_{n+1} = P_C(x_n - \mu \mathscr{A} y_n), \end{cases}$$

where $\mathscr{A} : E \to E$ is monotone and L-Lipschitz continuous operator, $\mu \in (0, \frac{1}{L})$ and P_C is the metric projection. Observe that, in each iteration, the extragradient method requires two projections onto C, which needs to solve two different optimization problems to generate each point of a sequence. This might take a considerable amount of computation time.

In order to control this, Censor, Gibali and Reich [4, 5, 6] modified Korpelevich's method and developed the subgradient extragradient algorithm, where one of the projections was computed explicitly onto a half-space instead of an arbitrary closed convex subset of a Hilbert space:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu \mathscr{A} x_n), \\ x_{n+1} = P_{C_n}(x_n - \mu \mathscr{A} y_n), \\ \text{such that} \quad C_n = \{x \in E : \langle x_n - \mu \mathscr{A} x_n - y_n, x - y_n \rangle \leq 0\}, \end{cases}$$

where $\mu \in (0, \frac{1}{L})$.

Tseng [26] proposed another method, which is known as Tseng's extragradient method. It reads

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu \mathscr{A} x_n), \\ x_{n+1} = y_n + \mu(\mathscr{A} x_n) - \mathscr{A} y_n)), \end{cases}$$

where μ is a constant in $(0, \frac{1}{L})$, to avoid the additional calculation of metric projections.

The inertial-type algorithm originated from solving the zero problem of a maximal monotone operator where the inertial technique was applied to obtain an inertial proximal method [1, 2]. Lately, interest in the study of inertial type algorithms has been growing among researchers, see [10, 11, 18, 20, 22] and references therein. These relevant results inspected the weak convergence of inertial type extrapolation algorithms and showed their enhanced computational performance. However, there are few inertial type strong convergence results available in the existing literature.

Let $T : C \to C$ be a mapping. A point $x^* \in C$ is called a fixed point of T if $x^* = Tx^*$. The fixed point set of T is denoted by $F(T)$. The fixed point problem (FPP) for T is defined as follows:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad x^* = Tx^*. \tag{1.2}$$

Recently, many algorithms for finding a common solution of VIP and FPP in a real Hilbert space have been studied (see, [7, 8, 24, 27, 28] and the references therein).

Tian and Tong [23] proposed an algorithm with a self-adaptive method for finding the solution of VIP involving monotone operators and the fixed point problem of a quasi-nonexpansive mapping. They studied the weak convergence of the algorithm under certain assumptions.

The organization of this paper is as follows. In Section 2, we recall some preliminaries and important results. In Section 3, two new algorithms for finding the common solution of $VI(C,A)$ and $F(T)$ are proposed. Moreover, strong convergence results are also obtained. Section 4 deals with the study the computational efficiency and advantage of the proposed algorithms over some known algorithms in the existing literature.

2. PRELIMINARIES

Let us recall some known concepts and results.

Let E be a real Hilbert space and C a nonempty closed convex subset of E. Denote the strong convergence of the sequence $\{x_n\}$ to an element x by $x_n \to x$ and weak convergence of the sequence $\{x_n\}$ to x in E by $x_n \rightharpoonup x$.

A mapping $T : C \to C$ is said to be:

(1) monotone if $\langle Tx - Ty, x - y\rangle \geq 0$, for all $x, y \in C$,
(2) L-Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$, for all $x, y \in C$,
(3) contractive if there exists $k \in (0,1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$, for all $x, y \in C$,
(4) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$,
(5) a generalized nonexpansive mapping or a mapping satisfying Condition (C) if, for any $x, y \in C$, $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ implies that $\|Tx - Ty\| \leq \|x - y\|$.

It is known that, for each $x \in E$, there exists the unique nearest point $P_C x$ in C, that is, $P_C x$ in C satisfies the following:

$$\|P_C x - x\| = \min\{\|y - x\| : y \in C\}.$$

The mapping $P_C : E \to C$ is called the metric projection from E onto C. It is well known that, in a real Hilbert space E, the following assertions hold:

(1) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$ for all $x, y \in E$,
(2) $\langle x - P_C x, y - P_C x\rangle \leq 0$ for all $x \in E, y \in C$,
(3) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$ for all $x, y \in E$,
(4) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ for all $x, y \in E$,
(5) $\|\eta x + (1-\eta)y\|^2 = \eta\|x\|^2 + (1-\eta)\|y\|^2 - \eta(1-\eta)\|x - y\|^2$ for all $x, y \in E, \eta \in [0,1]$.

Lemma 2.1. [21] *Let C be any subset of a Hilbert space and let $T : C \to C$ be a mapping satisfying condition (C). If $x_n \rightharpoonup z$ and $\lim_{n\to\infty}\|x_n - Tz_n\| = 0$, then $Tz = z$.*

Lemma 2.2. [13] *Let $\mathscr{A} : E \to E$ be a monotone and L-Lipschitz continuous mapping on C. Let $S = P_C(I - \mu\mathscr{A})$, where $\mu > 0$. If $\{x_n\}$ is a sequence in E satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$, then $q \in VI(C, \mathscr{A}) = F(S)$.*

Lemma 2.3. [29] *Let $\{b_n\}$ be a sequence of non-negative real numbers such that $b_{n+1} \leq (1 - \delta_n)b_n + \delta_n c_n + s_n$, where $\delta_n \subset (0,1)$, c_n and s_n are sequences in $\mathbb{R}$ such that*

(1) $\sum_{n=0}^{\infty}\delta_n = \infty$*;*
(2) $\limsup_{n\to\infty} c_n \leq 0$*;*
(3) $\sum_{n=0}^{\infty}|s_n| < \infty$ *.*

Then $b_n \to 0$ as $n \to \infty$.

Lemma 2.4. [15] *Let $\{b_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{b_{n_i}\}$ of $\{b_n\}$ such that $b_{n_i} < b_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{t_j\} \subset \mathbb{N}$ such that $\lim_{j\to\infty} t_j = \infty$ and the following properties are satisfied for all (sufficiently large) number $j \in \mathbb{N}$:*

$$b_{t_j} \leq b_{t_j+1} \qquad \text{and} \qquad b_j \leq b_{t_j+1}.$$

In fact, t_j is the largest number in the set $\{1, 2, \ldots, j\}$ such that $b_n < b_{n+1}$.

3. MAIN RESULTS

In this section, we introduce two new iterative algorithms for finding an element in the intersection of the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous operator and the set of fixed points of a mapping satisfying Condition (C). Assume that $A : E \to E$ is monotone and Lipschitz continuous with unknown Lipschitz constant, $T : E \to E$ satisfies the Condition (C) and $S : E \to E$ is nonexpansive. Further, we assume that $\daleth = F(T) \cap VI(C,A)$ is nonempty.

3.1. **Inertial-type viscosity subgradient extragradient algorithm.** We first propose an inertial-type viscosity subgradient extragradient algorithm for solving VIP and FPP as follows:

Algorithm 1: Inertial-type viscosity subgradient extragradient algorithm

Initialization: Fix $\mu \in (0,1)$, $\lambda_0 > 0$, $\gamma_n \in [0,\gamma)$ for some $\gamma > 0$ and $\alpha_n \subset (0,1)$. Let $x_0, x_1 \in E$.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1: Set $w_n = x_n + \gamma_n(Sx_n - Sx_{n-1})$ and compute

$$y_n = P_C(w_n - \lambda_n \mathscr{A} w_n).$$

Step 2: Construct the half space $C_n = \{x \in E : \langle w_n - \lambda_n \mathscr{A} w_n - y_n, x - y_n\rangle \leq 0\}$ and compute

$$z_n = P_{C_n}(w_n - \lambda_n \mathscr{A} y_n),$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|}{\|\mathscr{A} w_n - \mathscr{A} y_n\|}, \lambda_n\}, & \text{if } \|\mathscr{A} w_n - \mathscr{A} y_n\| \neq 0 \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Step 3: Calculate

$$x_{n+1} = \alpha_n f z_n + (1-\alpha_n) T z_n.$$

Set $n := n+1$ and return to Step 1.

Lemma 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then,*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - z_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - w_n\|^2 - 2\lambda_n \langle \mathscr{A} p, y_n - p\rangle.$$

Proof. Since $p \in C_n, VI(C, \mathscr{A})$, we have

$$\begin{aligned}\|z_n - p\|^2 =& \|P_{C_n}(w_n - \lambda_n \mathscr{A} y_n) - p\|^2 \\ \leq& \langle z_n - p, w_n - \lambda_n \mathscr{A} y_n - p\rangle \\ =& \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|w_n - z_n\|^2 - \langle z_n - p, \lambda_n \mathscr{A} y_n\rangle.\end{aligned}$$

This implies that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - z_n\|^2 - 2\langle z_n - p, \lambda_n \mathscr{A} y_n\rangle. \tag{3.1}$$

Since $\mathscr{A}$ is monotone and $\lambda_n > 0$, we have

$$2\lambda_n \langle \mathscr{A} y_n - \mathscr{A} p, y_n - p\rangle \geq 0.$$

So, (3.1) becomes

$$\begin{aligned}\|z_n - p\|^2 \leq & \|w_n - p\|^2 - \|w_n - z_n\|^2 - 2\langle z_n - p, \lambda_n \mathscr{A} y_n\rangle + 2\lambda_n \langle \mathscr{A} y_n - \mathscr{A} p, y_n - p\rangle \\ \leq & \|w_n - p\|^2 - \|w_n - z_n\|^2 + 2\lambda_n \Big(\langle \mathscr{A} y_n, y_n - z_n\rangle - \langle \mathscr{A} p, y_n - p\rangle \Big) \\ \leq & \|w_n - p\|^2 - \|w_n - z_n\|^2 \\ & + 2\lambda_n \Big(\langle \mathscr{A} y_n - \mathscr{A} w_n, y_n - z_n\rangle + \langle \mathscr{A} w_n, y_n - z_n\rangle - \langle \mathscr{A} p, y_n - p\rangle \Big). \end{aligned} \tag{3.2}$$

From the definition of $\{\lambda_n\}$, we have

$$\begin{aligned} 2\lambda_n \langle \mathscr{A} y_n - \mathscr{A} w_n, y_n - z_n\rangle \leq & 2\lambda_n \|\mathscr{A} y_n - \mathscr{A} w_n\| \|y_n - z_n\| \\ \leq & 2\mu \frac{\lambda_n}{\lambda_{n+1}} \|y_n - w_n\| \|y_n - z_n\| \\ \leq & \mu \frac{\lambda_n}{\lambda_{n+1}} \|y_n - w_n\|^2 + \mu \frac{\lambda_n}{\lambda_{n+1}} \|y_n - z_n\|^2. \end{aligned} \tag{3.3}$$

Since $z_n \in C_n$, we obtain that

$$\langle w_n - \lambda_n \mathscr{A} w_n - y_n, z_n - y_n\rangle \leq 0.$$

Thus, in order to estimate $2\lambda_n \langle \mathscr{A} w_n, y_n - z_n\rangle$, we get

$$\begin{aligned} 2\lambda_n \langle \mathscr{A} w_n, y_n - z_n\rangle \leq & 2\langle y_n - w_n, z_n - y_n\rangle \\ = & \|z_n - w_n\|^2 - \|y_n - w_n\|^2 - \|z_n - y_n\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we obtain the required inequality immediately. □

Theorem 3.1. *Assume that the following conditions are satisfied:*

(1) $\lim_{n\to\infty} \alpha_n = 0$,
(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(3) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$,
(4) $\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm* (1) *converges strongly to an element of* $\daleth$.

Proof. Observe that $\daleth$ is closed and convex and $P_\daleth \circ f : E \to E$ is a contraction mapping. From the Banach contraction theorem, $P_\daleth \circ f$ has a unique fixed point $p \in E$, that is, $p = P_\daleth \circ f(p)$. In particular,

$$\langle fp - p, y - p\rangle \leq 0, \quad y \in \daleth. \tag{3.5}$$

Since $\{\lambda_n\}$ is bounded and monotonically nonincreasing, $\lim_{n\to\infty} \lambda_n$ exists, we have

$$\lim_{n\to\infty} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu \frac{\lambda_n}{\lambda_{n+1}} > 0 \text{ for all } n \geq n_0. \tag{3.6}$$

Let $p \in VI(C,\mathscr{A})$. Then $\langle \mathscr{A}p, y_n - p\rangle \geq 0$. It follows from Lemma 3.1 that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - z_n\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - w_n\|^2$$

which together with (3.6) gives

$$\|z_n - p\| \leq \|w_n - p\| \text{ for all } n \geq n_0. \tag{3.7}$$

Further, as $w_n = x_n + \gamma_n(Sx_n - Sx_{n-1})$, we have

$$\begin{aligned}\|w_n - p\| =&\|x_n + \gamma_n(Sx_n - Sx_{n-1}) - p\|\\ \leq&\|x_n - p\| + \gamma_n\|x_n - x_{n-1}\|\\ \leq&\|x_n - p\| + \alpha_n\cdot\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\|.\end{aligned} \tag{3.8}$$

From $\lim_{n\to\infty}\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, we find that there exists $M_1 > 0$ such that

$$\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1 \text{ for all } n \geq 0.$$

Therefore, (3.7) and (3.8) imply that

$$\|z_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \alpha_n\cdot M_1. \tag{3.9}$$

Claim 1. Show that $\{x_n\}$ is bounded.

Since T is a generalized nonexpansive mapping, we have that

$$\frac{1}{2}\|p - Tp\| \leq \|z_n - p\|$$

implies

$$\|Tz_n - Tp\| = \|Tz_n - p\| \leq \|z_n - p\|.$$

Consequently, it follows from Algorithm 1 that

$$\begin{aligned}\|x_{n+1} - p\| \leq&\alpha_n\|fz_n - p\| + (1-\alpha_n)\|z_n - p\|\\ \leq&\alpha_n\Big(\|fz_n - fp\| + \|fp - p\|\Big) + (1-\alpha_n)\|z_n - p\|\\ \leq&k\alpha_n\|z_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|z_n - p\|\\ \leq&(1-(1-k)\alpha_n)\|z_n - p\| + \alpha_n\|fp - p\|.\end{aligned}$$

From (3.9), we obtain that

$$\begin{aligned}\|x_{n+1} - p\| \leq&(1-(1-k)\alpha_n)\|w_n - p\| + \alpha_n\|fp - p\|\\ \leq&(1-(1-k)\alpha_n)\|x_n - p\| + \alpha_n\cdot M_1 + \alpha_n\|fp - p\|\\ \leq&(1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{M_1 + \|fp - p\|}{1-k}\\ \leq&\max\left\{\|x_n - p\|, \frac{M_1 + \|fp - p\|}{1-k}\right\}\\ \leq&\cdots\leq\max\left\{\|x_0 - p\|, \frac{M_1 + \|fp - p\|}{1-k}\right\}.\end{aligned}$$

This shows the sequence $\{x_n\}$ is bounded. As a consequence, sequences $\{w_n\}$, $\{fx_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{Tz_n\}$ are bounded.

Claim 2. Show that

$$\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-w_n\|^2+\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4,$$

where $M_4>0$ is some positive real number.

Indeed,

$$\begin{aligned}\|x_{n+1}-p\|^2 \le&\alpha_n\|fz_n-p\|^2+(1-\alpha_n)\|Tz_n-p\|^2\\ \le&\alpha_n\Big(\|fz_n-fp\|+\|fp-p\|\Big)^2+(1-\alpha_n)\|z_n-p\|^2\\ \le&\alpha_n\Big(k\|z_n-p\|+\|fp-p\|\Big)^2+(1-\alpha_n)\|z_n-p\|^2\\ \le&\|z_n-p\|^2+\alpha_n\Big(2\|fp-p\|\|z_n-p\|+\|fp-p\|^2\Big)\\ \le&\|z_n-p\|^2+\alpha_n M_2,\end{aligned}$$

where $M_2>0$ is some constant. It follows from Lemma 3.1 that

$$\|x_{n+1}-p\|^2 \le \|w_n-p\|^2-\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2-\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-w_n\|^2+\alpha_n M_2. \tag{3.10}$$

Now, (3.9) implies that

$$\begin{aligned}\|w_n-p\|^2 \le&\Big(\|x_n-p\|+\alpha_n M_1\Big)^2\\ =&\|x_n-p\|^2+2\alpha_n M_1\|x_n-p\|+(\alpha_n M_1)^2\\ =&\|x_n-p\|^2+\alpha_n M_3,\end{aligned} \tag{3.11}$$

where $M_3=\alpha_n M_1^2+M_1\|x_n-p\|$. Thus, (3.10) and (3.11) imply that

$$\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-z_n\|^2+\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n-w_n\|^2 \le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4,$$

where $M_4=M_2+M_3$.

Claim 3. Show that

$$\begin{aligned}\|x_{n+1}-p\|^2 \le&(1-(1-k)\alpha_n)\|x_n-p\|^2+(1-k)\alpha_n\cdot\\ &\left(\frac{2}{1-k}\langle fp-p,x_{n+1}-p\rangle+3M\frac{\gamma_n}{\alpha_n}\|x_n-x_{n-1}\|\right).\end{aligned}$$

Indeed,

$$\begin{aligned}\|w_n - p\|^2 =&\|(x_n - p) + \gamma_n(Sx_n - Sx_{n-1})\|^2 \\ \leq&\|x_n - p\|^2 + \gamma_n^2\|x_n - x_{n-1}\|^2 + 2\gamma_n\|x_n - p\|\|x_n - x_{n-1}\|.\end{aligned} \tag{3.12}$$

By (3.9), we obtain that

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq&\alpha_n\|fz_n - fp\|^2 + (1-\alpha_n)\|Tz_n - p\|^2 + 2\alpha_n\langle fp - p, x_{n+1} - p\rangle \\ \leq&\alpha_n k^2\|z_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle fp - p, x_{n+1} - p\rangle \\ \leq&\alpha_n k\|z_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle fp - p, x_{n+1} - p\rangle \\ \leq&(1-(1-k)\alpha_n)\|w_n - p\|^2 + 2\alpha_n\langle fp - p, x_{n+1} - p\rangle.\end{aligned} \tag{3.13}$$

Inequalities (3.12) and (3.13) imply that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ \leq&(1-(1-k)\alpha_n)\|x_n - p\|^2 + 2\gamma_n\|x_n - p\|\|x_n - x_{n-1}\| \\ &+ \gamma_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle fp - p, x_{n+1} - p\rangle \\ =&(1-(1-k)\alpha_n)\|x_n - p\|^2 + (1-k)\alpha_n \cdot \frac{2}{1-k}\langle fp - p, x_{n+1} - p\rangle \\ &+ \gamma_n\left(2\|x_n - p\|\|x_n - x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\|^2\right) \\ \leq&(1-(1-k)\alpha_n)\|x_n - p\|^2 + (1-k)\alpha_n \cdot \frac{2}{1-k}\langle fp - p, x_{n+1} - p\rangle \\ &+ \gamma_n\left(2\|x_n - p\|\|x_n - x_{n-1}\| + \gamma\|x_n - x_{n-1}\|^2\right) \\ \leq&(1-(1-k)\alpha_n)\|x_n - p\|^2 + (1-k)\alpha_n\left(\frac{2}{1-k}\langle fp - p, x_{n+1} - p\rangle + 3M\frac{\gamma_n}{\alpha_n} \cdot \|x_n - x_{n-1}\|\right),\end{aligned} \tag{3.14}$$

where $M = \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \gamma\|x_n - x_{n-1}\|\}$.

Claim 4. Show that $\{x_n\}$ converges strongly to some $p \in \daleth$.

Case 1: Assume that there exists $N \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$ for all $n \geq N$. Thus, $\lim_{n\to\infty}\|x_n - p\|^2$ exists. Consequently, from Claim 2, we get

$$\lim_{n\to\infty}\|y_n - w_n\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.15}$$

Taking limit as $n \to \infty$, we have

$$\|w_n - z_n\| \leq \|w_n - y_n\| + \|y_n - z_n\| \to 0. \tag{3.16}$$

From the definition of $\{w_n\}$ and the assumptions, we have

$$\begin{aligned}\|x_n - w_n\| \leq& \gamma_n \|x_n - x_{n-1}\| \\ =& \alpha_n \cdot \frac{\gamma_n}{\alpha_n} \|x_n - x_{n-1}\|, \text{ which approaches to } 0\end{aligned} \tag{3.17}$$

as $n \to \infty$. Inequalities (3.16) and (3.17) imply that

$$\|x_n - z_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Note that

$$\|x_{n+1} - Tz_n\| = \alpha_n \|fz_n - Tz_n\| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

We now claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.20}$$

Consider

$$\begin{aligned}&\|x_{n+2} - x_{n+1}\| \\ =& \|(1-\alpha_{n+1})(Tz_{n+1} - Tz_n) + \alpha_{n+1}(fz_{n+1} - fz_n) + (fz_n - Tz_n)(\alpha_{n+1} - \alpha_n)\| \\ \leq& (1-\alpha_{n+1})\|Tz_{n+1} - Tz_n\| + \alpha_{n+1}k\|z_{n+1} - z_n\| + \|fz_n - Tz_n\||\alpha_{n+1} - \alpha_n|.\end{aligned} \tag{3.21}$$

Observe that

$$\|z_n - Tz_n\| \leq \|z_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\|$$

and

$$\|x_n - x_{n+1}\| \leq \|x_n - z_n\| + \|z_n - z_{n+1}\| + \|z_{n+1} - x_{n+1}\|.$$

Then, (3.18) and (3.19) imply that

$$\lim_{n\to\infty} \|z_n - Tz_n\| \leq \lim_{n\to\infty} \|x_n - x_{n+1}\|$$

and

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| \leq \lim_{n\to\infty} \|z_n - z_{n+1}\|.$$

As a result, we have

$$\lim_{n\to\infty} \|z_n - Tz_n\| \leq \lim_{n\to\infty} \|z_n - z_{n+1}\|.$$

It follows that there exists $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2}\|z_n - Tz_n\| \leq \|z_n - z_{n+1}\|, \quad \forall n \geq n_1.$$

From the definition of T, we have

$$\|Tz_{n+1} - Tz_n\| \leq \|z_n - z_{n+1}\|, \quad \forall n \geq n_1.$$

Therefore, it follows from (3.21) that

$$\|x_{n+2} - x_{n+1}\| \leq (1-(1-k)\alpha_{n+1})\|z_{n+1} - z_n\| + \|fz_n - Tz_n\||\alpha_{n+1} - \alpha_n|. \tag{3.22}$$

Note that

$$\begin{aligned}\|z_{n+1} - z_n\| \leq& \|w_{n+1} - w_n\| + \|\lambda_n \mathscr{A} y_n - \lambda_{n+1} \mathscr{A} y_{n+1}\| \\ \leq& \|w_{n+1} - w_n\| + \lambda_n \|\mathscr{A} y_n\| + \lambda_{n+1} \|\mathscr{A} y_{n+1}\| \\ \leq& \|w_{n+1} - w_n\| + \lambda_0(\|\mathscr{A} y_n\| + \|\mathscr{A} y_{n+1}\|) \\ \leq& \|w_{n+1} - w_n\| + \lambda_0 \cdot L_1,\end{aligned}$$

where $L_1 = \sup_{n\in\mathbb{N}}(\|\mathscr{A}y_n\| + \|\mathscr{A}y_{n+1}\|)$. Since

$$\|w_n - w_{n+1}\| \leq \|x_n - x_{n+1}\| + \gamma_n\|x_n - x_{n-1}\| + \gamma_{n+1}\|x_n - x_{n+1}\|,$$

we get that

$$\|z_{n+1} - z_n\| \leq (1+\gamma_{n+1})\|x_n - x_{n+1}\| + \gamma_n\|x_n - x_{n-1}\| + \lambda_0 \cdot L_1. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
\leq&(1-(1-k)\alpha_{n+1})(1+\gamma_{n+1})\|x_n - x_{n+1}\| + \gamma_n\|x_n - x_{n-1}\| \\
&+ \lambda_0 \cdot L_1 + L_2 \cdot |\alpha_{n+1} - \alpha_n| \\
=&(1-(1-k)\alpha_{n+1})\|x_n - x_{n+1}\| + (1-k)\alpha_{n+1}\left(\frac{\gamma_{n+1}}{(1-k)\alpha_{n+1}} \cdot \|x_n - x_{n+1}\|\right. \\
&\left.+ \frac{\alpha_n}{(1-k)\alpha_{n+1}} \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| + L_2 \cdot \frac{|\alpha_{n+1} - \alpha_n|}{(1-k)\alpha_{n+1}}\right) + \lambda_0 \cdot L_1,
\end{aligned}$$

where $L_2 = \sup_{n\to\infty}\|fz_n - Tz_n\|$. Note that

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\frac{\gamma_{n+1}}{\alpha_{n+1}} \cdot \|x_n - x_{n+1}\| + \frac{\alpha_n}{\alpha_{n+1}} \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| + L_2 \cdot \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}}\right) \\
=&\limsup_{n\to\infty}\left(\frac{\gamma_{n+1}}{\alpha_{n+1}} \cdot \|x_n - x_{n+1}\| + \frac{\alpha_n}{\alpha_{n+1}} \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| + L_2 \cdot \left|1 - \frac{\alpha_n}{\alpha_{n+1}}\right|\right) = 0.
\end{aligned}$$

Therefore, all the assumptions of Lemma 2.3 are satisfied and we obtain (3.20). In view of the boundedness of $\{x_n\}$, we find that there exists $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle fp - p, x_n - p\rangle = \limsup_{i\to\infty}\langle fp - p, x_{n_i} - p\rangle. \tag{3.24}$$

In view of $x_{n_i} \rightharpoonup x^* \in E$, we have that (3.24) becomes

$$\limsup_{n\to\infty}\langle fp - p, x_n - p\rangle = \limsup_{i\to\infty}\langle fp - p, x^* - p\rangle.$$

From (3.17), we obtain that $w_{n_i} \rightharpoonup x^*$. From (3.15) and Lemma 2.2, we have $x^* \in VC(I, \mathscr{A})$. Again, from (3.18), we obtain $z_{n_i} \rightharpoonup x^*$. Also, equations (3.18), (3.19) and (3.20) yield $z_n - Tz_n \to 0$. It follows from Lemma 2.1 that $x^* \in F(T)$. Therefore, $x^* \in \daleth$. Consequently, (3.5) and (3.20) yield

$$\begin{aligned}
\limsup_{n\to\infty}\langle fp - p, x_{n+1} - p\rangle \leq&\limsup_{n\to\infty}\langle fp - p, x_n - p\rangle \\
=&\langle fp - p, x^* - p\rangle \\
\leq&0.
\end{aligned} \tag{3.25}$$

Finally, from Claim 3 and Lemma 2.3, we have $\lim_{n\to\infty}\|x_n - p\| = 0$.

Case 2: There exists a subsequence $\{\|x_{n_i} - p\|\}$ of $\{\|x_n - p\|\}$ such that $\{\|x_{n_i} - p\|\} < \{\|x_{n_i+1} - p\|\}$ for all $i \in \mathbb{N}$. By Lemma 2.4, there exists a nondecreasing sequence $\{t_i\} \subset \mathbb{N}$

such that $\lim_{i\to\infty} t_i = \infty$ and the following hold for all $i \in \mathbb{N}$

$$\begin{cases} \|x_{t_i} - p\|^2 \le & \|x_{t_i+1} - p\|^2, \\ \|x_i - p\|^2 \le & \|x_{t_i} - p\|^2. \end{cases} \tag{3.26}$$

Thus, it follows from Claim 2 that

$$\begin{aligned}\left(1-\mu\frac{\lambda_{t_i}}{\lambda_{t_i+1}}\right)\|y_{t_i}-z_{t_i}\|^2+\left(1-\mu\frac{\lambda_{t_i}}{\lambda_{t_i+1}}\right)\|y_{t_i}-w_{t_i}\|^2 \le& \|x_{t_i}-p\|^2-\|x_{t_i+1}-p\|^2+\alpha_{t_i}M_4\\ \le& \alpha_{t_i}M_4 \longrightarrow 0\end{aligned}$$

as $n \to \infty$, which further implies that

$$\lim_{i\to\infty} \|y_{t_i} - z_{t_i}\| = \|y_{t_i} - w_{t_i}\| = 0.$$

As in Case 1, $\lim_{i\to\infty} \|x_{t_i} - x_{t_i+1}\| = 0$ and $\limsup_{i\to\infty} \langle fp - p, x_{t_i+1} - p\rangle \le 0$. From Claim 3 and (3.26), we have

$$\|x_i - p\|^2 \le \frac{2}{1-k}\langle fp - p, x_{t_i+1} - p\rangle + 3M\frac{\gamma_{t_i}}{\alpha_{t_i}} \cdot \|x_{t_i} - x_{t_i-1}\|.$$

Consequently, $\limsup_{i\to\infty} \|x_i - p\| \le 0$, that is, x_i converges to p. This completes the proof. □

3.2. **Inertial-type viscosity Tseng's extragradient algorithm.** Now, we propose viscosity-type Tseng's extragradient algorithm for solving VIP and FPP as follows:

Algorithm 2: Inertial-type viscosity Tseng's extragradient algorithm

---

Initialization: Fix $\mu \in (0,1)$, $\lambda_0 > 0$, $\gamma > 0$, $\alpha_n \in (0,1)$ and $\beta_n \subset (0,\beta)$. Let $x_0, x_1 \in E$.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1: Set $w_n = x_n + \gamma_n(Sx_n - Sx_{n-1})$ and compute

$$y_n = P_C(w_n - \lambda_n \mathscr{A} w_n).$$

Step 2: Compute

$$t_n = (1-\beta_n)z_n + \beta_n T z_n,$$

where $z_n = y_n - \lambda_n(\mathscr{A} y_n - \mathscr{A} w_n)$
Step 3:

$$x_{n+1} = \alpha_n f z_n + (1-\alpha_n)t_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\mu\frac{\|y_n - w_n\|}{\|\mathscr{A} y_n - \mathscr{A} w_n\|}, \lambda_n\}, & \text{if } \|\mathscr{A} y_n - \mathscr{A} w_n\| \ne 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and return to Step 1.

Lemma 3.2. *Let $\{z_n\}$ be a sequence generated by Algorithm 2. Then, we have*

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2$$

and

$$\|z_n - y_n\| \le \mu\frac{\lambda_n}{\lambda_{n+1}}\|y_n - w_n\|^2.$$

Proof. Note that

$$\begin{aligned}
\|z_n - p\|^2 \leq & \|y_n - \lambda_n(\mathscr{A}y_n - \mathscr{A}w_n) - p\|^2 \\
= & \|y_n - p\|^2 + \lambda_n^2\|\mathscr{A}y_n - \mathscr{A}w_n\|^2 - 2\lambda_n\langle \mathscr{A}y_n - \mathscr{A}w_n, y_n - p\rangle \\
= & \|y_n - w_n\|^2 + \|w_n - p\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \lambda_n^2\|\mathscr{A}y_n - \mathscr{A}w_n\|^2 \\
& - 2\lambda_n\langle \mathscr{A}y_n - \mathscr{A}w_n, y_n - p\rangle. \qquad (3.27)
\end{aligned}$$

Since $y_n = P_C(w_n - \lambda_n\mathscr{A}w_n)$, we have

$$\langle y_n - w_n + \lambda_n\mathscr{A}w_n, y_n - p\rangle \leq 0.$$

Thus,

$$\lambda_n\langle \mathscr{A}w_n, y_n - p\rangle \leq -\langle y_n - w_n, y_n - p\rangle. \qquad (3.28)$$

By (3.27) and (3.28), we have

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 - 2\lambda_n\langle \mathscr{A}y_n, y_n - p\rangle.$$

Using the fact that $\langle \mathscr{A}y_n, y_n - p\rangle \geq 0$, we obtain that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2,$$

and

$$\|z_n - y_n\| = \lambda_n\|\mathscr{A}y_n - \mathscr{A}w_n\| \leq \frac{\lambda_n}{\lambda_{n+1}}\|y_n - w_n\|.$$

□

Theorem 3.2. *Assume that the following conditions are satisfied:*

(1) $\lim_{n\to\infty}\alpha_n = 0$,
(2) $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(3) $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$,
(4) $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,
(5) $\lim_{n\to\infty}\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 2 converges strongly to an element of* $\daleth$.

Proof. **Claim 1.** Show that $\{x_n\}$ is bounded.

Since T is a generalized nonexpansive mapping, we have that

$$\frac{1}{2}\|p - Tp\| \leq \|z_n - p\|$$

implies

$$\|Tz_n - Tp\| = \|Tz_n - p\| \leq \|z_n - p\|$$

Thus,

$$\begin{aligned}
\|t_n - p\|^2 = & (1 - \beta_n)\|z_n - p\|^2 + \beta_n\|Tz_n - p\|^2 - \beta_n(1 - \beta_n)\|z_n - Tz_n\|^2 \\
\leq & \|z_n - p\|^2 - \beta_n(1 - \beta_n)\|z_n - Tz_n\|^2. \qquad (3.29)
\end{aligned}$$

From Lemma 3.2, we get that

$$\|t_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2. \tag{3.30}$$

Since $\lim_{n\to\infty} 1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} = 1 - \mu^2 > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \text{for all} \quad n \ge n_0.$$

Consequently, it follows from Lemma 3.2 that

$$\|z_n - p\| \le \|w_n - p\| \quad \text{and} \quad \|t_n - p\| \le \|w_n - p\| \text{ for all } n \ge n_0. \tag{3.31}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - p\| \le& \alpha_n\|fz_n - p\| + (1-\alpha_n)\|t_n - p\|\\ \le& \alpha_n k\|z_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|t_n - p\|\\ \le& (1-(1-k)\alpha_n)\|w_n - p\| + \alpha_n\|fp - p\|.\end{aligned}$$

Note that

$$\begin{aligned}\|w_n - p\| \le& \|x_n - p\| + \gamma_n\|x_n - x_{n-1}\|\\ =& \|x_n - p\| + \alpha_n \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\|.\end{aligned}$$

From $\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$, we find that there exists $n_1 \in \mathbb{N}$ such that

$$\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1, \quad \text{for all} \quad n \ge n_1,$$

which implies

$$\|w_n - p\| \le \|x_n - p\| + \alpha_n \cdot M_1. \tag{3.32}$$

Furthermore, for all $n \ge n_1$, we have

$$\begin{aligned}\|x_{n+1} - p\| \le& (1-(1-k)\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\| + \alpha_n \cdot M_1\\ =& (1-(1-k)\alpha_n)\|x_n - p\| + \alpha_n(1-k)\left(\frac{M_1 + \|fp - p\|}{1-k}\right)\\ \le& \max\left\{\|x_n - p\|, \frac{M_1 + \|fp - p\|}{1-k}\right\}\\ \le& \max\left\{\|x_0 - p\|, \frac{M_1 + \|fp - p\|}{1-k}\right\}.\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{z_n\}$, $\{fz_n\}$ and $\{y_n\}$ are bounded as well.

Claim 2. Show that

$$(1-(1-k)\alpha_n)\left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 + \beta_n(1-\beta_n)(1-\alpha_n)\|z_n - Tz_n\|^2$$
$$\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.$$

From (3.30), we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
=&\alpha_n\|fz_n-p\|^2+(1-\alpha_n)\|t_n-p\|^2-\alpha_n(1-\alpha_n)\|fz_n-t_n\|^2\\
\le&\alpha_n\|fz_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2-(1-\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\\
&-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2\\
\le&\alpha_n\left(\|fz_n-fp\|^2+\|fp-p\|^2+2\|fz_n-fp\|\|fp-p\|\right)+(1-\alpha_n)\|w_n-p\|^2\\
&-(1-\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2\\
\le&\alpha_n\left(k\|z_n-p\|^2+\|fp-p\|^2+2k\|z_n-p\|\|fp-p\|\right)+(1-\alpha_n)\|w_n-p\|^2\\
&-(1-\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2\\
\le&\alpha_nk\left(\|w_n-p\|^2-\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\right)+\alpha_n\|fp-p\|^2\\
&+2\alpha_nk\|z_n-p\|\|fp-p\|+(1-\alpha_n)\|w_n-p\|^2\\
&-(1-\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2-\beta_n(1-\beta_n)\|(1-\alpha_n)\|z_n-Tz_n\|^2\\
\le&(1-(1-k)\alpha_n)\|w_n-p\|^2-(1-(1-k)\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\\
&-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2+\alpha_nM_2
\end{aligned}
$$

where $M_2=\|fp-p\|^2+2k\|z_n-p\|\|fp-p\|$. Then, (3.32) implies

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le&(1-(1-k)\alpha_n)\left(\|x_n-p\|^2+(\alpha_nM_1)^2+2\alpha_nM_1\|x_n-p\|\right)+\alpha_nM_2\\
&-(1-(1-k)\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2\\
\le&(1-(1-k)\alpha_n)\|x_n-p\|^2-(1-(1-k)\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\\
&-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2+\alpha_nM_3+\alpha_nM_2\\
\le&(1-(1-k)\alpha_n)\|x_n-p\|^2-(1-(1-k)\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\\
&-\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2+\alpha_nM_4,
\end{aligned}
$$

where $M_3=\gamma_nM_1^2+2M_1\|x_n-p\|$ and $M_4=M_2+M_3$.

Finally, we arrive at

$$(1-(1-k)\alpha_n)\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2+\beta_n(1-\beta_n)(1-\alpha_n)\|z_n-Tz_n\|^2$$
$$\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4.$$

Claim 3. Show that

$$\|x_{n+1}-p\|^2 \leq (1-(1-k)\alpha_n)\|x_n-p\|^2 + \alpha_n(1-k)\left[\frac{\alpha_n M_1 - k\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)}{1-k}\|y_n-w_n\|+2\langle fp-p,x_{n+1}-p\rangle\right].$$

From Algorithm 2, Lemma 3.2, (3.31) and (3.32), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|\alpha_n(fz_n-fp)+(1-\alpha_n)(t_n-p)\|^2+2\alpha_n\langle fp-p,x_{n+1}-p\rangle\\ &\leq \alpha_n k\|z_n-p\|^2+(1-\alpha_n)\|t_n-p\|^2+2\alpha_n\langle fp-p,x_{n+1}-p\rangle\\ &\leq \alpha_n k\left(\|w_n-p\|^2-\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\right)\\ &\quad +(1-\alpha_n)\|w_n-p\|^2+2\alpha_n\langle fp-p,x_{n+1}-p\rangle\\ &\leq (1-(1-k)\alpha_n)k\|x_n-p\|^2+\alpha_n^2M_1^2-\alpha_n k\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\\ &\quad +2\alpha_n\langle fp-p,x_{n+1}-p\rangle.\end{aligned}$$

This implies that

$$\|x_{n+1}-p\|^2 \leq (1-(1-k)\alpha_n)\|x_n-p\|^2 + \alpha_n(1-k)\left[\frac{\alpha_n M_1 - k\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)}{1-k}\|y_n-w_n\|+2\langle fp-p,x_{n+1}-p\rangle\right].$$

Claim 4. Show that $\{x_n\}$ converges strongly to some $p \in \daleth$.

Case 1: Assume that there exists $N \in \mathbb{N}$ such that $\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2$ for all $n \geq N$. Thus, $\lim_{n\to\infty}\|x_n-p\|^2$ exists. From Claim 2, we obtain that

$$\lim_{n\to\infty}\|y_n-w_n\|=0 \qquad \text{and} \qquad \lim_{n\to\infty}\|z_n-Tz_n\|=0. \tag{3.33}$$

Now,

$$\begin{aligned}\|x_n-w_n\| &=\gamma_n\|Sx_n-Sx_{n-1}\|\\ &\leq\gamma_n\|x_n-x_{n-1}\|\\ &\leq\alpha_n\cdot\frac{\gamma_n}{\alpha_n}\|x_n-x_{n-1}\|\longrightarrow 0 \text{ as } n\to\infty.\end{aligned} \tag{3.34}$$

Then, (3.33) and (3.34) imply that

$$\|x_n-y_n\|\longrightarrow 0 \tag{3.35}$$

and

$$\begin{aligned}\|z_n - y_n\| =& \lambda_n \|\mathscr{A} y_n - \mathscr{A} w_n\| \\ \leq& \frac{\lambda_n}{\lambda_{n+1}} \mu \|y_n - w_n\| \longrightarrow 0 \end{aligned} \tag{3.36}$$

as $n \longrightarrow \infty$. Combining (3.33) and (3.36), we have

$$\|w_n - z_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty \tag{3.37}$$

which further implies

$$\|x_n - z_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.38}$$

Observe that

$$\|t_n - z_n\| = \beta_n \|z_n - Tz_n\| \leq (1-\lambda)\|z_n - Tz_n\|. \tag{3.39}$$

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Indeed,

$$\begin{aligned}\|t_{n+1} - t_n\| =& \|(1-\beta_{n+1})(z_{n+1} - z_n) + \beta_{n+1}(Tz_{n+1} - Tz_n) + (\beta_{n+1} - \beta_n)(Tz_n - z_n)\| \\ \leq& (1-\beta_{n+1})\|z_{n+1} - z_n\| + \beta_{n+1}\|Tz_{n+1} - Tz_n\| + |\beta_{n+1} - \beta_n|\|Tz_n - z_n\| \end{aligned} \tag{3.40}$$

and

$$\begin{aligned}\|z_{n+1} - z_n\| \leq& \|y_{n+1} - y_n\| + \lambda_{n+1}\|\mathscr{A} y_{n+1} - \mathscr{A} w_{n+1}\| + \lambda_n \|\mathscr{A} y_n - \mathscr{A} w_n\| \\ \leq& \|y_{n+1} - y_n\| + \frac{\lambda_{n+1}}{\lambda_{n+2}}\mu\|y_{n+1} - w_{n+1}\| + \frac{\lambda_n}{\lambda_{n+1}}\mu\|y_n - w_n\| \\ \leq& \|y_{n+1} - y_n\| + \lambda_0 \mu L_1, \end{aligned} \tag{3.41}$$

where $L_1 = \sup_{n\in\mathbb{N}}\{\|y_{n+1} - w_{n+1}\| + \|y_n - w_n\|\}$. Since $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$, we find that there exists $n_3 \in \mathbb{N}$ such that

$$\frac{1}{2}\|z_n - Tz_n\| < \varepsilon \leq \|z_n - z_{n+1}\|, \quad \text{for all} \quad n, \geq n_3$$

which implies that

$$\|Tz_n - Tz_{n+1}\| \leq \|z_n - z_{n+1}\|, \quad \text{for all} \quad n \geq n_3.$$

It follows from (3.40) that

$$\|t_{n+1} - t_n\| \leq \|z_{n+1} - z_n\| + |\beta_{n+1} - \beta_n|\|Tz_n - z_n\|. \tag{3.42}$$

Note that

$$\begin{aligned}\|y_{n+1} - y_n\| \leq& \|w_{n+1} - \lambda_{n+1}\mathscr{A} w_{n+1} - w_n + \lambda_n \mathscr{A} w_n\| \\ \leq& \|w_{n+1} - w_n\| + \lambda_{n+1}\|\mathscr{A} w_{n+1}\| + \lambda_n\|\mathscr{A} w_n\| \\ \leq& \|w_{n+1} - w_n\|, + \lambda_0 L_2 \end{aligned}$$

where $L_2 = \sup_{n\in\mathbb{N}}\{\|\mathscr{A} w_{n+1}\| + \|\mathscr{A} w_n\|\}$. As

$$\begin{aligned}\|w_{n+1} - w_n\| \leq& \|x_{n+1} - x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| + \gamma_n\|x_n - x_{n-1}\| \\ \leq& (1+\gamma_{n+1})\|x_{n+1} - x_n\| + \alpha_n \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\|, \end{aligned}$$

we arrive at

$$\|z_{n+1}-z_n\| \le (1+\gamma_{n+1})\|x_{n+1}-x_n\| + \alpha_n \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| + \lambda_0 L_3, \tag{3.43}$$

where $L_3 > 0$ is some real number. Note that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| \le& \|\alpha_{n+1} f z_{n+1} + (1-\alpha_{n+1})t_{n+1} - \alpha_n f z_n - (1-\alpha_n)t_n\| \\ \le& \|\alpha_{n+1}(f z_{n+1} - f z_n) + (1-\alpha_{n+1})(t_{n+1}-t_n) + (\alpha_{n+1}-\alpha_n)(f z_n - t_n)\| \\ \le& \alpha_{n+1}\|f z_{n+1} - f z_n\| + (1-\alpha_{n+1})\|t_{n+1}-t_n\| + |\alpha_{n+1}-\alpha_n|\|f z_n - t_n\| \\ \le& \alpha_{n+1} k\|z_{n+1}-z_n\| + (1-\alpha_{n+1})\|t_{n+1}-t_n\| + |\alpha_{n+1}-\alpha_n|\|f z_n - t_n\|.\end{aligned}$$

From (3.42), we obtain

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\| \\ &\le \alpha_{n+1} k\|z_{n+1}-z_n\| + (1-\alpha_{n+1})\|z_{n+1}-z_n\| + (1-\alpha_{n+1})|\beta_{n+1}-\beta_n|\|T z_n - z_n\| \\ &\quad + |\alpha_{n+1}-\alpha_n|\|f z_n - t_n\| \\ &\le (1-(1-k)\alpha_{n+1})\|z_{n+1}-z_n\| + |\beta_{n+1}-\beta_n|\|T z_n - z_n\| + |\alpha_{n+1}-\alpha_n|\|f z_n - t_n\|.\end{aligned}$$

It follows from (3.43) that

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\| \\ &\le (1-(1-k)\alpha_{n+1})(1+\gamma_{n+1})\|x_{n+1}-x_n\| + \alpha_n \cdot \frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\quad + |\beta_{n+1}-\beta_n|\|T z_n - z_n\| + |\alpha_{n+1}-\alpha_n|\|f z_n - t_n\| + \lambda_0 L_3 \\ &\le (1-(1-k)\alpha_{n+1})\|x_{n+1}-x_n\| + (1-k)\alpha_{n+1}\left(\frac{1}{1-k}\cdot\frac{\gamma_{n+1}}{\alpha_{n+1}}\|x_{n+1}-x_n\|\right. \\ &\quad \left. + \frac{1}{1-k}\cdot\frac{\alpha_n}{\alpha_{n+1}}\cdot\frac{\gamma_n}{\alpha_n}\|x_n - x_{n-1}\|\right) + |\beta_{n+1}-\beta_n|\|T z_n - z_n\| + \left|1 - \frac{\alpha_n}{\alpha_{n+1}}\right| L_4 + \lambda_0 L_3,\end{aligned}$$

where $L_4 = \sup_{n\in\mathbb{N}}\|f z_n - t_n\|$. Now, all the assumptions of Lemma 2.3 are satisfied. Hence,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.44}$$

As $\{x_n\}$ is bounded, we have that there exists $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle fp - p, x_n - p\rangle = \limsup_{i\to\infty}\langle fp - p, x_{n_i} - p\rangle. \tag{3.45}$$

Let $x_{n_i} \rightharpoonup x^* \in E$. Then, (3.45) becomes

$$\limsup_{n\to\infty}\langle fp - p, x_n - p\rangle = \limsup_{i\to\infty}\langle fp - p, x^* - p\rangle. \tag{3.46}$$

By use of (3.34), we obtain that $w_{n_i} \rightharpoonup x^*$. By use of (3.33) and Lemma 2.2, we have $x^* \in VC(I,\mathscr{A})$. Again, from (3.38), we obtain $z_{n_i} \rightharpoonup x^*$. It follows from Lemma 2.1 and (3.33) that $x^* \in F(T)$. Therefore, $x^* \in \daleth$ and

$$\langle fp - p, x^* - p\rangle \le 0. \tag{3.47}$$

On the other hand, (3.44), (3.46) and (3.47) imply that

$$\limsup_{n\to\infty}\langle fp - p, x_{n+1} - p\rangle \le \limsup_{n\to\infty}\langle fp - p, x_n - p\rangle \le 0. \tag{3.48}$$

It follows from Claim 3, (3.48) and Lemma 2.3 that $x_n \to p$.

Case 2: There exists a subsequence $\{\|x_{n_i} - p\|\}$ of $\{\|x_n - p\|\}$ such that $\{\|x_{n_i} - p\|\} < \{\|x_{n_i+1} - p\|\}$ for all $i \in \mathbb{N}$. From Lemma 2.4, there exists a nondecreasing sequence $\{t_i\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} t_i = \infty$ and the following hold, for all $i \in \mathbb{N}$,

$$\begin{cases} \|x_{t_i} - p\|^2 \leq & \|x_{t_i+1} - p\|^2, \\ \|x_i - p\|^2 \leq & \|x_{t_i} - p\|^2. \end{cases} \tag{3.49}$$

From Claim 2, we have

$$\begin{aligned} &(1-(1-k)\alpha_{t_i})\left(1-\mu^2\frac{\lambda_{t_i}^2}{\lambda_{t_i+1}^2}\right)\|y_{t_i} - w_{t_i}\|^2 + \beta_{t_i}(1-\beta_{t_i})(1-\alpha_{t_i})\|z_{t_i} - Tz_{t_i}\|^2 \\ &\leq \|x_{t_i} - p\|^2 - \|x_{t_i+1} - p\|^2 + \alpha_{t_i} M_4 \\ &\leq \alpha_{t_i} M_4 \longrightarrow 0 \quad \text{as } n \to \infty \end{aligned}$$

and $\lim_{i\to\infty}\|y_{t_i} - w_{t_i}\| = \lim_{i\to\infty}\|z_{t_i} - Tz_{t_i}\| = 0$. As in Case 1, $\lim_{i\to\infty}\|x_{t_i} - x_{t_i+1}\| = 0$ and $\limsup_{i\to\infty}\langle fp - p, x_{t_i+1} - p\rangle \leq 0$. Therefore, from Claim 3 and (3.49), we have

$$\begin{aligned} \|x_i - p\|^2 \leq &(1-(1-k)\alpha_{t_i})\|x_{t_i} - p\|^2 + \alpha_{t_i}(1-k)\left[\frac{\alpha_{t_i} M_1 - k\left(1-\mu^2\frac{\lambda_{t_i}^2}{\lambda_{t_i+1}^2}\right)}{1-k}\|y_{t_i} - w_{t_i}\|\right. \\ &\left. + 2\langle fp - p, x_{t_i+1} - p\rangle\right]. \end{aligned}$$

Consequently, $\limsup_{i\to\infty}\|x_i - p\| \leq 0$, that is, x_i converges to p. This completes the proof. □

4. Computational experiments

This section provides numerical examples for testing the suggested algorithms. All the codes were written in Matlab (R2015a) and run on laptop with Intel(R) Core(TM) i7-75000U CPU 2.70GHz RAM 2.90 GHz.

Example 4.1. Let $E = \mathbb{R}$ and $\mathscr{A} : E \to E$ be given by $\mathscr{A}x = 2x$. Note that $\mathscr{A}$ is a monotone and Lipschitz continuous operator with $L = 2$. Define $T : E \to E$ by

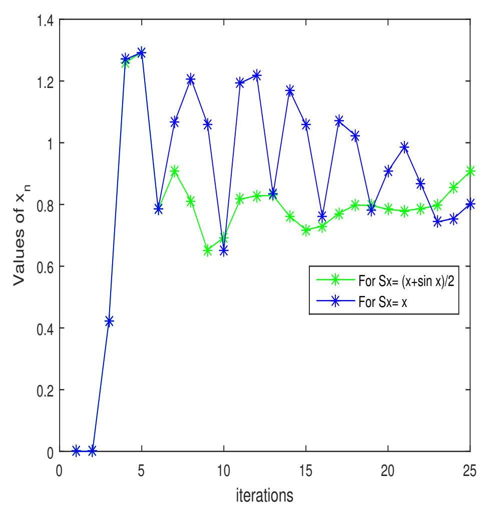
$$Tx = \begin{cases} |1-x|, & x \leq \frac{1}{8} \\ \frac{x+7}{8}, & x > \frac{1}{8}. \end{cases}$$

Note that T satisfies Condition (C) but is not nonexpansive. Also, $fx = \frac{3x}{4}$ and $Sx = \frac{x+\sin x}{2}$. The feasible set C is $[-1,2]$. We conduct two tests: $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-4}$. We also experiment with the parameter λ_0. We test for: $\lambda_0 = 0.6$ and $\lambda_0 = 0.2$. We also compare the computational efficiency of the proposed algorithm with Algorithm 3.1 [24] and ISA-SEGM [23]. Choose $x(0) = 0$, $x(1) = 0$, $\mu = 0.5$ and $\alpha_n = \frac{1}{n}$. We choose $\gamma_n = 5$ for the proposed algorithm and $\gamma_n = 0.7$ for ISA-SEGM [23]. In order to terminate the algorithm, we use the condition $\|x_{n+1} - x*\| < \varepsilon$ where $x*$ is the solution of the problem.

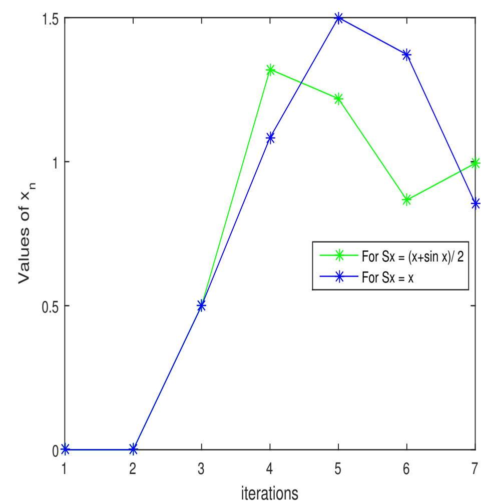
ε	Algo.1	Algo.[24]	ISA-SEGM [23]
	For $\lambda_0 = 0.6$		
10^{-2}	0.045754	0.050431	0.056131
10^{-4}	0.045754	0.050795	0.074446
	For $\lambda_0 = 0.2$		
10^{-2}	0.052763	0.067685	0.058734
10^{-4}	0.054501	0.089375	0.087988

TABLE 1. The comparison of elapsed CPU times

We can see that the proposed algorithm takes lesser computation time than the previously studied comparable algorithms. The following figures show the convergence comparison of the Algorithm 1 for different choices of the nonexpansive mapping S and parameter α_n.



(A) The comparison of convergence for $\alpha_n = \frac{1}{\sqrt{n+1}}$



(B) The comparison of convergence for $\alpha_n = \frac{1}{n}$

FIGURE 1. The comparison of convergence for $Sx = x$ and $Sx = \sin x$

Example 4.2. Let $E = \mathbb{R}$ and $\mathscr{A} : E \to E$ be given by $\mathscr{A}x = x + \sin x$. Define $T : E \to E$ by

$$Tx = \begin{cases} \frac{x}{2}, & x \neq 1 \\ \frac{3}{5}, & x = 1. \end{cases}$$

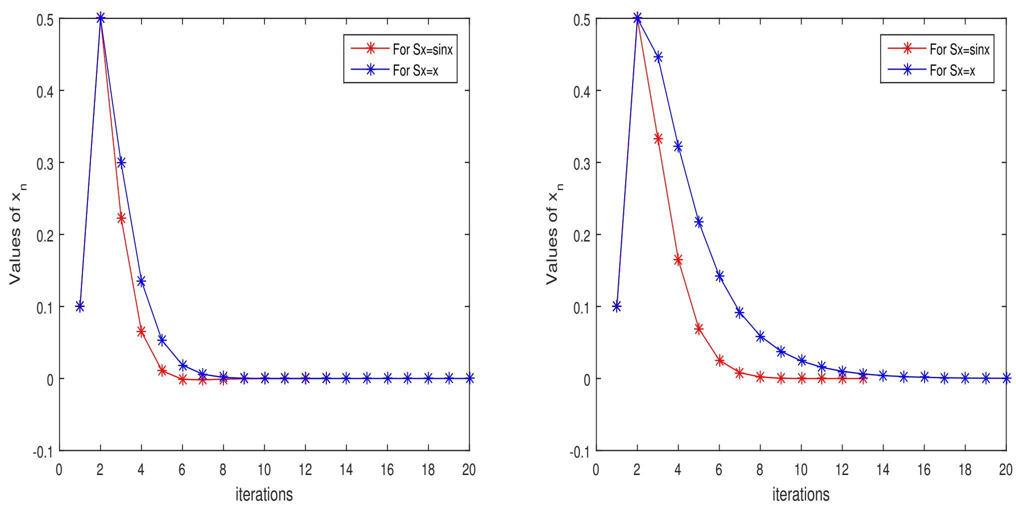
Note that T satisfies Condition (C). Choose $Sx = \sin x$, which is clearly nonexpansive. The feasible set C is $[-1, 2]$. Observe that $\mathscr{A}$ is a monotone and Lipschitz continuous operator with $L = 2$.

We conduct two tests: $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-10}$. We also experiment with the parameter λ_0. We test for: $\lambda_0 = 0.7$ and $\lambda_0 = 0.3$. Choose $x(0) = 0.1$, $x(1) = 0.5$, $fx = \frac{x}{4}$, $\gamma = 0.25$, $\mu = 0.5$ and $\alpha_n = \beta_n = 0.50$. In order to terminate the algorithm, we use the condition $\|x_{n+1} - x*\| < \varepsilon$,

ε	Algo.2	Algo.[24]	Algo. [25]
	For $\lambda_0 = 0.7$		
10^{-5}	0.000769	0.000788	0.001194 ($\lambda = 0.4$)
10^{-10}	0.000770	0.000898	0.001278 ($\lambda = 0.4$)
	For $\lambda_0 = 0.3$		
10^{-5}	0.000754	0.000793	0.001239 ($\lambda = 0.3$)
10^{-10}	0.000792	0.000821	0.001305 ($\lambda = 0.3$)

TABLE 2. The comparison of elapsed CPU times

where x^* is the solution of the problem. Since λ is fixed and determined by L in Algo. [25] we test $\lambda = 0.4$ and $\lambda = 0.3$. The table clearly illustrates that the proposed algorithm performs better than the previously known algorithms. The following figures show the convergence comparison of the Algorithm 2 for different choices of the nonexpansive mapping S and parameters α_n and β_n.



(A) The comparison of convergence for $\alpha_n = \frac{1}{n}$ and $\beta_n = 0.5$

(B) Comparison of convergence for $\alpha_n = 0.1$ and $\beta_n = 0.4$

FIGURE 2. The comparison of convergence for $Sx = \sin x$ and $Sx = x$

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