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VOLTERRA INTEGRAL OPERATORS AND CARLESON EMBEDDING ON CAMPANATO SPACES

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Abstract. Let $1 \le p < q < \infty$. The boundedness and compactness of the embedding from Campanato spaces $\mathscr{L}_{p,\lambda}$ into tent spaces $\mathscr{T}_s^q(\mu)$ are investigated in this paper. Meanwhile, the boundedness and the essential norm of the Volterra operators T_g and I_g from $\mathscr{L}_{p,\lambda}$ to $F(q, q-2+\frac{q(1-\lambda)}{p}, s)$ are also studied.
Keywords. Campanato space; Volterra integral operator; Embedding; Carleson measure.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathscr{H}(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. As usual, $\mathscr{H}^\infty$ denotes the space of bounded analytic functions. Let $p \ge 1$ and $0 < \lambda < \infty$. We say that an $f \in H^p$ belongs to the analytic Campanato spaces $\mathscr{L}_{p,\lambda}$ if

$$\|f\|^p_{\mathscr{L}_{p,\lambda}} = \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^p \frac{|d\zeta|}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}, \; I \subseteq \partial\mathbb{D},$$

and the Hardy space H^p (see [1]) consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

If $p = 2$, then space $\mathscr{L}_{2,\lambda}$ is called the Morrey space, which was first studied by Wu and Xie [2] in the case of $\mathbb{D}$. We refer to [3, 4] for more study on the Morrey space $\mathscr{L}_{2,\lambda}$. If $p = 2$ and $\lambda = 1$, then $\mathscr{L}_{2,\lambda}$ is just the $BMOA$ space. For more information on $BMOA$ space, we refer to [5].

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Suppose that $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. The space $F(p,q,s)$ is defined by those $f \in \mathscr{H}(\mathbb{D})$ with

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. This space was first introduced by Zhao [6]. If $p = 2$ and $q = 0$, it gives Q_s spaces (see [7]). From [6], we see that $F(p, p-2, s)$ is equivalent to a Bloch space for all $s > 1$, where the Bloch space $\mathscr{B}$ is the class of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathscr{B}_0$ consists of all $f \in \mathscr{H}(\mathbb{D})$ for which

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

Let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I. The Carleson square based on I, denoted by $S(I)$, is defined by

$$S(I) := \left\{ z = re^{i\theta} \in \mathbb{D} : 1-|I| \le r < 1, e^{i\theta} \in I \right\}.$$

Let $0 < \alpha < \infty$, and let μ be a positive Borel measure on $\mathbb{D}$. Recall that μ is called an α-Carleson measure if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

If μ is an α-Carleson measure, then we set

$$\|\mu\|_\alpha = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha}.$$

If $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^\alpha} = 0$, then μ is called a vanishing α-Carleson measure.

Let μ be a positive Borel measure on $\mathbb{D}$. For $0 < q < \infty$ and $0 < s < \infty$, the tent space $\mathscr{T}_s^q(\mu)$ consists of all μ-measurable functions f such that

$$\|f\|_{\mathscr{T}_s^q(\mu)}^q = \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^q d\mu(z) < \infty.$$

For more information about some function spaces embedding into tent space $\mathscr{T}_s^q(\mu)$, we refer to [4, 8, 9, 10] and the references therein.

In this paper, we study the embedding of $\mathscr{L}_{p,\lambda}$ into $\mathscr{T}_s^q(\mu)$ when $1 \le p < q < \infty$. That is, we prove that the identity mapping $i : \mathscr{L}_{p,\lambda} \to \mathscr{T}_s^q(\mu)$ is bounded (resp.compactly) if and only if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}} < \infty \ \ (resp. \ \ \lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}} = 0),$$

when $0 < \lambda < 1$ and $\frac{q\lambda}{p} < s < \infty$. Furthermore, the boundedness, the compactness and the essential norm of operators T_g and I_g from $\mathscr{L}_{p,\lambda}$ to $F(q, q-2+\frac{q(1-\lambda)}{p}, s)$ are also investigated.

Throughout this paper, let f and g be two positive functions, and write $f \lesssim g$ if $f \le Cg$ holds, where C is a positive constant independent of f and g. If $f \lesssim g$ and $g \lesssim f$, then we say $f \asymp g$.

2. EMBEDDING $\mathscr{L}_{p,\lambda}$ INTO TENT SPACES $\mathscr{T}_s^q(\mu)$

In this section, we investigate the boundedness and compactness of the identity mapping $i:\mathscr{L}_{p,\lambda}\to\mathscr{T}_s^q(\mu)$. First, let us recall the following result, which can be found in [8].

Lemma 2.1. *Let μ be a positive Borel measure on $\mathbb{D}$. If $0<\alpha,t<\infty$, then μ is an α-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^t}{|1-\bar{a}z|^{\alpha+t}}d\mu(z)<\infty.$$

Moreover,

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^{\alpha}}\asymp\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^t}{|1-\bar{a}z|^{\alpha+t}}d\mu(z).$$

Lemma 2.2. [9] *Let $1\le p<\infty$ and $0<\lambda<1$. If $f\in\mathscr{L}_{p,\lambda}$, then*

$$|f(z)|\lesssim\frac{\|f\|_{\mathscr{L}_{p,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{p}}},\ z\in\mathbb{D}.$$

Lemma 2.3. [11] *For $0<r<1$, let $\chi_{\{z:|z|<r\}}$ be the characteristic function of the set $\{z:|z|<r\}$. If μ is a α-Carleson measure on $\mathbb{D}$, then μ is a vanishing α-Carleson measure if and only if $\|\mu-\mu_r\|_{\alpha}\to 0$ as $r\to 1^-$, where $d\mu_r=\chi_{\{z:|z|<r\}}d\mu$.*

Now, we are in a position to prove the main result in this section.

Theorem 2.1. *Let μ be a positive Borel measure on $\mathbb{D}$. Let $1\le p<q<\infty$, $0<\lambda<1$ and $\frac{q\lambda}{p}<s<\infty$. Then the identity mapping $i:\mathscr{L}_{p,\lambda}\to\mathscr{T}_s^q(\mu)$ is bounded if and only if*

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}}<\infty.$$

Proof. Assume that $i:\mathscr{L}_{p,\lambda}\to\mathscr{T}_s^q(\mu)$ is bounded. For $a\in\mathbb{D}$, set

$$f_a(z)=\frac{(1-|a|^2)^{1+\frac{\lambda-1}{p}}}{(1-\bar{a}z)},\ z\in\mathbb{D}.\tag{2.1}$$

By [9, Lemma 2.3], we have that $f_a\in\mathscr{L}_{p,\lambda}$ and $\sup_{a\in\mathbb{D}}\|f_a\|_{\mathscr{L}_{p,\lambda}}\lesssim 1$. Fixed an arc $I\subset\partial\mathbb{D}$. Let $e^{i\theta}$ be the center of I and $a=(1-|I|)e^{i\theta}$. Then

$$|1-\bar{a}z|\asymp 1-|a|=|I|,\ |f_a(z)|\asymp|I|^{\frac{\lambda-1}{p}},$$

whenever $z\in S(I)$. So

$$\frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}}\asymp\frac{1}{|I|^s}\int_{S(I)}|f_a(z)|^q d\mu(z)\le\|f_a\|^q_{\mathscr{T}_s^q(\mu)}<\infty.$$

Therefore, μ is a $(s+\frac{q(1-\lambda)}{p})$-Carleson measure.

Conversely, suppose that μ is a $(s+\frac{q(1-\lambda)}{p})$-Carleson measure. Fix $f\in\mathscr{L}_{p,\lambda}$. Let I be any arc on $\partial\mathbb{D}$ and $a=(1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the midpoint of I. From Lemma 2.2, we have

$$|f(a)|\lesssim\frac{\|f\|_{\mathscr{L}_{p,\lambda}}}{(1-|a|)^{\frac{1-\lambda}{p}}}=\frac{\|f\|_{\mathscr{L}_{p,\lambda}}}{|I|^{\frac{1-\lambda}{p}}}.$$

By using the triangle inequality, we deduce that

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^q d\mu(z)&\lesssim\frac{1}{|I|^s}\int_{S(I)}|f(a)|^q d\mu(z)+\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(a)|^q d\mu(z)\\&=I_1+I_2.\end{aligned}$$

It is obvious that

$$I_1\lesssim\frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}}\|f\|^q_{\mathscr{L}_{p,\lambda}}\lesssim\|f\|^q_{\mathscr{L}_{p,\lambda}}.$$

Since $s>\frac{q\lambda}{p}$, we obtain $s+\frac{q(1-\lambda)}{p}>1$. By the assumed condition and Theorem 7.4 in [12], we know that $i:A^p_{\frac{ps}{q}-1-\lambda}\to L^q(d\mu)$ is bounded. Note that

$$\mathscr{L}_{p,\lambda}\subseteq H^p\subseteq A^p_{\frac{ps}{q}-1-\lambda}.$$

Based on these facts, we turn to estimate I_2. The estimate will be divided into two cases.

Case 1: $\frac{ps}{q}-\lambda\geq 1$.

$$\begin{aligned}I_2&\asymp\int_{S(I)}\frac{|f(z)-f(a)|^q}{|1-\bar{a}z|^s}d\mu(z)\\&\asymp(1-|a|^2)^{\frac{(3-\lambda)q}{p}}\int_{S(I)}\frac{|f(z)-f(a)|^q}{|1-\bar{a}z|^{\frac{(3-\lambda)q}{p}+s}}d\mu(z)\\&\lesssim(1-|a|^2)^{\frac{(3-\lambda)q}{p}}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^q}{\left|(1-\bar{a}z)^{\frac{3-\lambda}{p}+\frac{s}{q}}\right|^q}d\mu(z)\\&\lesssim(1-|a|^2)^{\frac{(3-\lambda)q}{p}-\frac{2q}{p}}\left(\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^{3-\lambda+\frac{ps}{q}}}(1-|z|^2)^{\frac{ps}{q}-1-\lambda}dA(z)\right)^{q/p}\\&\lesssim(1-|a|^2)^{\frac{(3-\lambda)q}{p}-\frac{2q}{p}}\left(\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^4}dA(z)\right)^{q/p}\\&=\left((1-|a|^2)^{3-\lambda-2}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^4}dA(z)\right)^{q/p}\\&=\left((1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|f\circ\varphi_a(w)-f(a)|^p dA(w)\right)^{q/p}\\&\lesssim\left((1-|a|^2)^{1-\lambda}\int_{\partial\mathbb{D}}|f\circ\varphi_a(\zeta)-f(a)|^p d\zeta\right)^{q/p}\\&\leq\|f\|^q_{\mathscr{L}_{p,\lambda}}<\infty.\end{aligned}$$

The last second inequality is from [13, Theorem 1].

Case 2: $0 < \frac{ps}{q} - \lambda < 1$.

$$\begin{aligned}
I_2 &\asymp (1-|a|^2)^{-s} \int_{S(I)} |f(z)-f(a)|^q d\mu(z) \\
&\asymp (1-|a|^2)^{\frac{4q}{p}-s} \int_{S(I)} \frac{|f(z)-f(a)|^q}{|1-\bar{a}z|^{\frac{4q}{p}}} d\mu(z) \\
&\lesssim (1-|a|^2)^{\frac{4q}{p}-s} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^q}{|1-\bar{a}z|^{\frac{4q}{p}}} d\mu(z) \\
&\lesssim (1-|a|^2)^{\frac{4q}{p}-\frac{2q}{p}-s} \left(\int_{\mathbb{D}} \frac{|f(z)-f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} (1-|z|^2)^{\frac{ps}{q}-\lambda-1} dA(z) \right)^{q/p} \\
&= \left((1-|a|^2)^{2-\frac{ps}{q}} \int_{\mathbb{D}} |f\circ\varphi_a(w)-f(a)|^p (1-|\varphi_a(w)|^2)^{\frac{ps}{q}-\lambda-1} dA(w) \right)^{q/p} \\
&\lesssim \left((1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f\circ\varphi_a(w)-f(a)|^p (1-|w|^2)^{\frac{ps}{q}-\lambda-1} dA(w) \right)^{q/p} \\
&\lesssim \left((1-|a|^2)^{1-\lambda} \int_{\partial\mathbb{D}} |f\circ\varphi_a(\zeta)-f(a)|^p d\zeta \right)^{q/p} \\
&\lesssim \|f\|^q_{\mathscr{L}_{p,\lambda}} < \infty.
\end{aligned}$$

Combining the estimates I_1 and I_2, we conclude that the identity mapping $i : \mathscr{L}_{p,\lambda} \to \mathscr{T}_s^q(\mu)$ is bounded. □

Recall that the identity mapping $i : \mathscr{L}_{p,\lambda} \to \mathscr{T}_s^q(\mu)$ is compact if

$$\lim_{n\to\infty} \frac{1}{|I|^s} \int_{S(I)} |f_n(z)|^q d\mu(z) = 0$$

whenever $I \subseteq \partial\mathbb{D}$ and $\{f_n\}$ is a bounded sequence in $\mathscr{L}_{p,\lambda}$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

Theorem 2.2. *Let μ be a positive Borel measure on $\mathbb{D}$. Let $1 \le p < q < \infty$, $0 < \lambda < 1$ and $\frac{q\lambda}{p} < s < \infty$. Then the identity mapping $i : \mathscr{L}_{p,\lambda} \to \mathscr{T}_s^q(\mu)$ is compact if and only if*

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}} = 0.$$

Proof. Assume that $i : \mathscr{L}_{p,\lambda} \to \mathscr{T}_s^q(\mu)$ is compact. Give a sequence of arcs $\{I_n\}$ with $\lim_{n\to\infty} |I_n| = 0$. Denote the center of I_n by $e^{i\theta_n}$ and $a_n = (1-|I_n|)e^{i\theta_n}$. Set

$$f_n(z) = \frac{(1-|a_n|^2)^{1+\frac{\lambda-1}{p}}}{(1-\bar{a}_n z)}, \; z \in \mathbb{D}. \tag{2.2}$$

It is clear that $\{f_n\}$ is bounded in $\mathscr{L}_{p,\lambda}$ and $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$. Then $\lim_{n\to\infty} \|f_n\|_{\mathscr{T}_s^q(\mu)} = 0$. Since

$$|f_n(z)| \asymp (1-|a_n|)^{\frac{\lambda-1}{p}} = |I_n|^{\frac{\lambda-1}{p}}, \; z \in S(I_n),$$

we obtain

$$\frac{\mu(S(I_n))}{|I_n|^{s+\frac{q(1-\lambda)}{p}}} \asymp \frac{1}{|I_n|^s}\int_{S(I_n)} |f_n(z)|^q d\mu(z) \le \|f_n\|^q_{\mathcal{T}_s^q(\mu)} \to 0,\ n\to\infty.$$

Then μ is a vanishing $(s+\frac{q(1-\lambda)}{p})$-Carleson measure by the arbitrariness of $\{I_n\}$.

Conversely, suppose that μ is a vanishing $(s+\frac{q(1-\lambda)}{p})$-Carleson measure. Then μ is also a $(s+\frac{q(1-\lambda)}{p})$-Carleson measure and $\lim_{r\to 1^-}\|\mu-\mu_r\|_{s+\frac{q(1-\lambda)}{p}} = 0$ by Lemma 2.3. It follows from the boundedness above, the mapping $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_s^q(\mu)$ is bounded. Let $\{f_n\}$ be a bounded sequence in $\mathcal{L}_{p,\lambda}$ such that $\{f_n\}$ converges to zero uniformly on each compact subset of $\mathbb{D}$. We have

$$\begin{aligned}
\frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu(z) &\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu_r(z) + \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d(\mu-\mu_r)(z)\\
&\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu_r(z) + \|\mu-\mu_r\|_{s+\frac{q(1-\lambda)}{p}}\|f_n\|^q_{\mathcal{L}_{p,\lambda}}\\
&\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu_r(z) + \|\mu-\mu_r\|_{s+\frac{q(1-\lambda)}{p}}\\
&\to 0,
\end{aligned}$$

as $r\to 1^-$ and $n\to\infty$. Therefore, $\lim_{n\to\infty}\|f_n\|_{\mathcal{T}_s^q(\mu)} = 0$. This shows that $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_s^q(\mu)$ is compact. □

3. Boundedness of T_g, I_g and M_g

Let $f, g \in \mathcal{H}(\mathbb{D})$. The Volterra integral operator T_g and the integral operator I_g are defined by

$$T_g f(z) = \int_0^z g'(w) f(w)dw,\ I_g f(z) = \int_0^z g(w) f'(w)dw,\ z\in\mathbb{D},$$

respectively.

There are many interesting results associated with T_g. For example, it was showed that T_g is bounded on Hardy spaces if and only if $g \in BMOA$ (see, e.g., [14, 15]). In [16], Aleman and Siskakis showed that T_g is bounded on the Bergman space A^p if and only if $g\in\mathcal{B}$. In [17], Siskakis and Zhao proved that $T_g : BMOA \to BMOA$ is bounded if and only if $g \in BMOA_{\log}$. Pau and Peláez [18] and Xiao [10] studied T_g acting on Q_p, respectively. For more information on operator T_g, see [3, 8, 11, 14, 16, 17, 19, 20, 21, 22, 23, 24] and the references therein.

In this section, via the embedding theorem (Theorem 2.2), we provide the characterizations for the boundedness of Volterra integral operator T_g and its companion operator I_g from $\mathcal{L}_{p,\lambda}$ to $F(q,q-2+\frac{q(1-\lambda)}{p},s)$. Moreover, we study the multiplication operator M_g from $\mathcal{L}_{p,\lambda}$ to $F(q,q-2+\frac{q(1-\lambda)}{p},s)$.

Theorem 3.1. *Let $g\in\mathcal{H}(\mathbb{D})$, $1\le p<q<\infty$, $0<\lambda<1$ and $\frac{q\lambda}{p}<s<\infty$. Then $T_g : \mathcal{L}_{p,\lambda} \to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is bounded if and only if $g\in\mathcal{B}$. Furthermore, $\|T_g\|\asymp\|g\|_{\mathcal{B}}$.*

Proof. Let $g \in \mathscr{B}$. Using the equivalent norm of Bloch function, we obtain

$$\begin{aligned}
\|g\|_{\mathscr{B}}^q &\asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^{s+\frac{q(1-\lambda)}{p}} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-2+s+\frac{q(1-\lambda)}{p}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{s+\frac{q(1-\lambda)}{p}} dA(z) \\
&\asymp \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{s+\frac{q(1-\lambda)}{p}}} \int_{S(I)} |g'(z)|^q (1-|z|^2)^{q-2+s+\frac{q(1-\lambda)}{p}} dA(z).
\end{aligned}$$

This means that

$$d\mu_g(z) = |g'(z)|^p (1-|z|^2)^{q-2+s+\frac{q(1-\lambda)}{p}} dA(z)$$

is a $(s+\frac{q(1-\lambda)}{p})$-Carleson measure. From Theorem 2.2, we find that $i : \mathscr{L}_{p,\lambda} \to \mathscr{T}_s^q(\mu_g)$ is bounded. Let $f \in \mathscr{L}_{p,\lambda}$. We deduce that

$$\begin{aligned}
\|T_g f\|^q_{F(q,q-2+\frac{q(1-\lambda)}{p},s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}} (1-|\varphi_a(z)|^2)^s dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-2+s+\frac{q(1-\lambda)}{p}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^s dA(z) \\
&\asymp \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^q d\mu_g(z) \\
&= \|f\|^q_{\mathscr{T}_s^q(\mu_g)} \\
&\lesssim \|\mu_g\|_{s+\frac{q(1-\lambda)}{p}} \|f\|^q_{\mathscr{L}_{p,\lambda}} \\
&\asymp \|g\|^q_{\mathscr{B}} \|f\|^q_{\mathscr{L}_{p,\lambda}}.
\end{aligned}$$

That is, $T_g : \mathscr{L}_{p,\lambda} \to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is bounded and $\|T_g\| \lesssim \|g\|_{\mathscr{B}}$.

Suppose that $T_g : \mathscr{L}_{p,\lambda} \to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is bounded. Let $a \in \mathbb{D}$ and f_a be as in (2.1). Then $f_a \in \mathscr{L}_{p,\lambda}$ and $\|f_a\|_{\mathscr{L}_{p,\lambda}} \lesssim 1$. It follows that

$$\|T_g f_a\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)} \le \|T_g\|\, \|f_a\|_{\mathscr{L}_{p,\lambda}} \lesssim \|T_g\|.$$

By Lemma 4.12 of [12], we have

$$\begin{aligned}
&\|T_g f_a\|^q_{F(q,q-2+\frac{q(1-\lambda)}{p},s)} \\
&\ge \int_{\mathbb{D}} |g'(z)|^q \frac{(1-|a|^2)^{q+\frac{q(\lambda-1)}{p}}}{|1-\bar{a}z|^q} (1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}} (1-|\varphi_a(z)|^2)^s dA(z) \\
&\ge \int_{\mathbb{D}(a,r)} |g'(z)|^q \frac{(1-|a|^2)^{q+\frac{q(\lambda-1)}{p}}}{|1-\bar{a}z|^q} (1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}} (1-|\varphi_a(z)|^2)^s dA(z) \\
&\gtrsim |g'(a)|^q (1-|a|^2)^q.
\end{aligned}$$

Here $\mathbb{D}(a,r)=\{z\in\mathbb{D}:\beta(a,z)<r\}$ denotes the Bergman metric disk centered at a with radius r (see [12]). It follows that

$$|g'(a)|(1-|a|^2)\lesssim \|T_gf_a\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\lesssim \|T_g\|.$$

Thus, $g\in\mathscr{B}$ and $\|g\|_{\mathscr{B}}\lesssim\|T_g\|$. □

Theorem 3.2. *Suppose that $2\le p<q<\infty$, $0<\lambda<1$ and $\lambda<s<2$. If $g\in\mathscr{H}(\mathbb{D})$, then $I_g:\mathscr{L}_{p,\lambda}\to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is bounded if and only if $g\in\mathscr{H}^\infty$. Furthermore, $\|I_g\|\asymp\|g\|_{\mathscr{H}^\infty}$.*

Proof. Suppose that $g\in\mathscr{H}^\infty$ and $f\in\mathscr{L}_{p,\lambda}$. For any $a\in\mathbb{D}$, we have

$$|f'(z)|^q\lesssim\frac{\|f\|^q_{\mathscr{L}_{p,\lambda}}}{(1-|z|^2)^{q+\frac{q(1-\lambda)}{p}}}.$$

From [25, Theorem 1], we have

$$\|\cdot\|_{F(p,p-1-\lambda,\lambda)}\lesssim\|\cdot\|_{\mathscr{L}_{p,\lambda}}.$$

Thus,

$$\begin{aligned}\|I_gf\|^q_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}&\le\|g\|^q_{\mathscr{H}^\infty}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^q(1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}}(1-|\varphi_a(z)|^2)^s dA(z)\\&\le\|g\|^q_{\mathscr{H}^\infty}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-2+1-\lambda}(1-|\varphi_a(z)|^2)^\lambda dA(z)\\&\le\|g\|^q_{\mathscr{H}^\infty}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\varphi_a(z)|^2)^\lambda dA(z)\\&\lesssim\|g\|^q_{\mathscr{H}^\infty}\|f\|^p_{\mathscr{L}_{p,\lambda}}.\end{aligned}$$

Conversely, Let $a\in\mathbb{D}$ and

$$F_a=\frac{(1-|a|^2)^{1+\frac{\lambda-1}{p}}}{\bar{a}(1-\bar{a}z)}\in\mathscr{L}_{p,\lambda}.$$

It is easily to see that $\sup_{a\in\mathbb{D}}\|F_a\|_{\mathscr{L}_{p,\lambda}}\lesssim 1$ and hence

$$\|I_gF_a\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\le\|I_g\|\ \|F_a\|_{\mathscr{L}_{p,\lambda}}\lesssim\|I_g\|.$$

Furthermore, Lemma 4.12 of [12] gives

$$\begin{aligned}&\|I_gF_a\|^q_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\\&\gtrsim\int_{\mathbb{D}}|g(z)|^q\frac{(1-|a|^2)^{q+\frac{q(\lambda-1)}{p}}}{|1-\bar{a}z|^{2q}}(1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}}(1-|\varphi_a(z)|^2)^s dA(z)\\&\gtrsim\int_{\mathbb{D}(a,r)}|g(z)|^q\frac{(1-|a|^2)^{q+\frac{q(\lambda-1)}{p}}}{|1-\bar{a}z|^{2q}}(1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}}(1-|\varphi_a(z)|^2)^s dA(z)\\&\gtrsim|g(a)|^q.\end{aligned}$$

Therefore, $|g(a)|\lesssim\|I_g\|$. By the choice of a, we deduce that $g\in\mathscr{H}^\infty$ and $\|g\|_{\mathscr{H}^\infty}\lesssim\|I_g\|$. □

Recall that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are analytic function spaces. Denote by $M(X,Y)$ the set of multipliers from X to Y, that is,

$$M(X,Y) = \{g \in \mathscr{H}(\mathbb{D}) : fg \in Y, \ \forall f \in X\}.$$

The multiplication operator M_g is defined by

$$M_g f(z) = g(z) f(z), \ f \in \mathscr{H}(\mathbb{D}), \ z \in \mathbb{D}.$$

The operators T_g and I_g are closely related to M_g due to

$$T_g f + I_g f = M_g f - f(0) g(0).$$

Using Theorems 3.1 and 3.2, we characterize the multipliers from $\mathscr{L}_{p,\lambda}$ to $F(q, q-2+\frac{q(1-\lambda)}{p}, s)$.

Theorem 3.3. *Suppose that* $2 \le p < q < \infty$, $0 < \lambda < 1$ *and* $\frac{q\lambda}{p} < s < 2$. *Then* $M(\mathscr{L}_{p,\lambda}, F(q, q-2+\frac{q(1-\lambda)}{p}, s)) = \mathscr{H}^\infty$.

Proof. Let $g \in \mathscr{H}^\infty$. It follows from Theorems 3.1 and 3.2 that

$$T_g : \mathscr{L}_{p,\lambda} \to F(q, q-2+\frac{q(1-\lambda)}{p}, s) \ \text{ and } \ I_g : \mathscr{L}_{p,\lambda} \to F(q, q-2+\frac{q(1-\lambda)}{p}, s)$$

are bounded. So $M_g : \mathscr{L}_{p,\lambda} \to F(q, q-2+\frac{q(1-\lambda)}{p}, s)$ is bounded.

Conversely, let $f \in F(q, q-2+\frac{q(1-\lambda)}{p}, s)$ and $a \in \mathbb{D}$. It follows that

$$|f'(a)| \lesssim \frac{\|f\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}}{(1-|a|^2)^{1+\frac{1-\lambda}{p}}}.$$

Since a is arbitrary, we get

$$|f(a)| \lesssim \frac{\|f\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}}{(1-|a|^2)^{\frac{1-\lambda}{p}}}.$$

For any $a \in \mathbb{D}$, let f_a be defined as in (2.1). Then $\{f_a\}$ is bounded in $\mathscr{L}_{p,\lambda}$. It follows that $M_g f_a \in F(q, q-2+\frac{q(1-\lambda)}{p}, s)$ and then

$$|M_g f_a(z)| \lesssim \frac{\|M_g f_a\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \lesssim \frac{\|M_g\| \ \|f_a\|_{\mathscr{L}_{p,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{1-\lambda}{p}}}.$$

As a consequence,

$$\left| \frac{1-|a|^2}{(1-\bar{a}z)^{1+\frac{1-\lambda}{p}}} g(z) \right| \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{1-\lambda}{p}}}.$$

Using the arbitrariness of $z, a \in \mathbb{D}$, and letting $a = z$, we conclude that $g \in \mathscr{H}^\infty$ and $\|g\|_{\mathscr{H}^\infty} \lesssim \|M_g\|$. □

4. Essential norm of T_g and I_g

Firstly, let us recall some definitions. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of $T : X \to Y$, denoted by $\|T\|_e$, is defined by

$$\|T\|_e = \inf_K \{\|T - K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

It is clear that $T : X \to Y$ is compact if and only if $\|T\|_e = 0$. Let Ω be a closed subspace of X. Given $f \in X$, the distance from f to Ω, denoted by $\mathrm{dist}_X(f, \Omega)$, is defined by

$$\mathrm{dist}_X(f, \Omega) = \inf_{g\in\Omega} \|f - g\|_X.$$

Lemma 4.1. [26, 27] *If $f \in \mathscr{B}$, then*

$$\limsup_{|z|\to 1^-}(1 - |z|^2)|f'(z)| \asymp \mathrm{dist}_{\mathscr{B}}(f, \mathscr{B}_0) \asymp \limsup_{r\to 1^-} \|f - f_r\|_{\mathscr{B}},$$

where $f_r(z) = f(rz)$, $0 < r < 1, z \in \mathbb{D}$.

We need the following lemma.

Lemma 4.2. *Let $1 \le p < q < \infty$, $0 < \lambda < 1$ and $\frac{q\lambda}{p} < s < \infty$. If $0 < r < 1$ and $g \in \mathscr{B}$, then $T_{g_r} : \mathscr{L}_{p,\lambda} \to F(q, q - 2 + \frac{q(1-\lambda)}{p}, s)$ is compact.*

Proof. Given $\{f_n\} \subset \mathscr{L}_{p,\lambda}$ such that $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ and $\sup_n \|f_n\|_{\mathscr{L}_{p,\lambda}} \le 1$, for each $a \in \mathbb{D}$, we have

$$\begin{aligned}
&\|T_{g_r} f_n\|^q_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^q |g_r'(z)|^q (1 - |z|^2)^{q-2+\frac{q(1-\lambda)}{p}} (1 - |\varphi_a(z)|^2)^s dA(z)\\
&\lesssim \frac{\|g\|^q_{\mathscr{B}}}{(1 - r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^q (1 - |z|^2)^{q-2+\frac{q(1-\lambda)}{p}} (1 - |\varphi_a(z)|^2)^s dA(z)\\
&\lesssim \frac{\|g\|^q_{\mathscr{B}} \|f_n\|^q_{\mathscr{L}_{p,\lambda}}}{(1 - r^2)^q} \int_{\mathbb{D}} (1 - |z|^2)^{q-2} (1 - |\varphi_a(z)|^2)^s dA(z)\\
&\lesssim \frac{\|g\|^q_{\mathscr{B}} \|f_n\|^q_{\mathscr{L}_{p,\lambda}}}{(1 - r^2)^q}.
\end{aligned}$$

By the dominated convergence theorem, we get the desire result. The proof is complete. □

Theorem 4.1. *Let $1 \le p < q < \infty$, $0 < \lambda < 1$ and $\frac{q\lambda}{p} < s < \infty$. If $g \in \mathscr{H}(\mathbb{D})$ and $T_g : \mathscr{L}_{p,\lambda} \to F(q, q - 2 + \frac{q(1-\lambda)}{p}, s)$ is bounded, then*

$$\|T_g\|_e \asymp \limsup_{|z|\to 1^-} (1 - |z|^2)|g'(z)| \asymp \mathrm{dist}_{\mathscr{B}}(g, \mathscr{B}_0).$$

Proof. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty} |a_n| = 1$. For each n, let f_n be defined as in (2.2). Then $\{f_n\}$ is bounded in $\mathscr{L}_{p,\lambda}$. Furthermore, $\{f_n\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. Given a compact operator $K:\mathscr{L}_{p,\lambda}\to F(q,q-2+\frac{q(1-\lambda)}{p},s)$, we have from [28] that $\lim_{n\to\infty}\|Kf_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}=0$. So

$$\begin{aligned}\|T_g-K\| &\gtrsim \limsup_{n\to\infty}\|(T_g-K)f_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\\ &\gtrsim \limsup_{n\to\infty}\left(\|T_gf_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}-\|Kf_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\right)\\ &= \limsup_{n\to\infty}\|T_gf_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\\ &\geq \limsup_{n\to\infty}\left(\int_{\mathbb{D}}|f_n(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}}(1-|\varphi_{a_n}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\ &\gtrsim \limsup_{n\to\infty}(1-|a_n|^2)|g'(a_n)|,\end{aligned}$$

which implies

$$\|T_g\|_e\gtrsim \limsup_{n\to\infty}(1-|a_n|^2)|g'(a_n)|.$$

It follows from Lemma 4.1 and the arbitrariness of $\{a_n\}$ that

$$\|T_g\|_e\gtrsim \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|\asymp \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

On the other hand, by Lemma 4.2, we have that $T_{g_r}:\mathscr{L}_{p,\lambda}\to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is compact. Then

$$\|T_g\|_e\leq\|T_g-T_{g_r}\|=\|T_{g-g_r}\|\asymp\|g-g_r\|_{\mathscr{B}}.$$

Using Lemma 4.1 again, we have

$$\|T_g\|_e\lesssim \limsup_{r\to 1^-}\|g-g_r\|_{\mathscr{B}}\asymp \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

The proof is complete. □

Using Theorem 4.1, we easily deduce the following corollary.

Corollary 4.1. *Let $1\leq p<q<\infty$, $0<\lambda<1$ and $\frac{q\lambda}{p}<s<\infty$. If $g\in\mathscr{H}(\mathbb{D})$, then $T_g:\mathscr{L}_{p,\lambda}\to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is compact if and only if $g\in\mathscr{B}_0$.*

Theorem 4.2. *Suppose that $2\leq p<q<\infty$, $0<\lambda<1$ and $\lambda<s<2$. If $g\in\mathscr{H}(\mathbb{D})$ and $I_g:\mathscr{L}_{p,\lambda}\to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is bounded, then $\|I_g\|_e\asymp\|g\|_{\mathscr{H}^\infty}$.*

Proof. Let $\{a_n\}$, $\{f_n\}$ and K be defined as in the proof of Theorem 4.1. Since $K:\mathscr{L}_{p,\lambda}\to F(q,q-2+\frac{q(1-\lambda)}{p},s)$ is compact, we get

$$\lim_{n\to\infty}\|Kf_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}=0.$$

Hence,

$$\begin{aligned}\|I_g - K\| &\gtrsim \limsup_{n\to\infty} \|(I_g - K)f_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\\ &\gtrsim \limsup_{n\to\infty}\left(\|I_g f_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)} - \|K f_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}\right)\\ &= \limsup_{n\to\infty}\|I_g f_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}.\end{aligned}$$

Therefore,

$$\|I_g\|_e \gtrsim \limsup_{n\to\infty}\|I_g f_n\|_{F(q,q-2+\frac{q(1-\lambda)}{p},s)}.$$

Similar argument as in the proof of Theorem 3.2 shows that

$$\|I_g f_n\|^q_{F(q,q-2+\frac{q(1-\lambda)}{p},s)} \gtrsim |g(a_n)|^q,$$

which implies that $\|I_g\|_e \gtrsim \|g\|_{\mathscr{H}^\infty}$.

On the other hand, Theorem 3.2 gives

$$\|I_g\|_e = \inf_K \|I_g - K\| \leq \|I_g\| \lesssim \|g\|_{\mathscr{H}^\infty}.$$

The proof is complete. □

Corollary 4.2. *Suppose that* $2 \leq p < q < \infty$, $0 < \lambda < 1$ *and* $\lambda < s < 2$. *If* $g \in \mathscr{H}(\mathbb{D})$, *then* $I_g : \mathscr{L}_{p,\lambda} \to F(q, q-2+\frac{q(1-\lambda)}{p}, s)$ *is compact if and only if* $g = 0$.

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