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AN EXTRAGRADIENT METHOD FOR VECTOR EQUILIBRIUM PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. We consider the vector equilibrium problem in Hadamard manifolds, which extends the scalar equilibrium problem to vector valued bifunctions. We propose an extragradient method for solving this problem. Under suitable assumptions on the bifunction, we prove that the generated sequence converges to a solution of the problem. We also give some examples of Hadamard manifolds and vector equilibrium problems to which our main result can be applied. Finally, we present some numerical experiments.
Keywords. Extragradient method; Hadamard manifold; Linesearch; Vector equilibrium problem; Vector valued bifunction.

1. INTRODUCTION

Let M be a connected n-dimensional manifold. We always assume that M can be endowed with a Riemannian metric to become a Riemannian manifold. The tangent space at the point $x \in M$ is denoted by T_xM. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on T_xM with the associated norm $\|\cdot\|_x$, where the subscript x is sometimes omitted. The tangent bundle of M is denoted by $TM = \cup_{x \in M} T_xM$, which is naturally a manifold. Given a piecewise smooth curve $\gamma : [a,b] \to M$ joining x to y (i.e. $\gamma(a) = x$ and $\gamma(b) = y$), we can define the length of γ by $L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$. Then the Riemannian distance $d(x,y)$, which induces the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y. The set $K \subset M$ is said to be convex if it contains the geodesic segment γ whenever it contains the end points of γ, that is, $\gamma((1-t)a + tb)$ is in K whenever $\gamma(a) = x$ and $\gamma(b) = y$ are in K, and $t \in [0,1]$.

We assume that $K \subset M$ is a nonempty, closed and convex set, $C \subset \mathbb{R}^m$ is a closed, convex and pointed cone with nonempty interior (denoted as $\mathrm{int}(C)$), and $f : M \times M \to \mathbb{R}^m$ is a vector valued bifunction. The *vector equilibrium problem*, denoted as $\mathrm{VEP}(f,K)$, consists of finding $x^* \in K$ such that

$$f(x^*, y) \notin -\mathrm{int}(C) \tag{1.1}$$

for all $y \in K$. If x^* satisfies (1.1), then x^* is said to be a solution or equilibrium point for $\mathrm{VEP}(f,K)$. We denote the set of all equilibrium points of $\mathrm{VEP}(f,K)$ as $S(f,K)$. It is easy to see

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that when $m = 1$, then vector equilibrium problems become ordinary (i.e., scalar) equilibrium problems.

The associated dual vector equilibrium problem, denoted by DVEP(f,K), consists of finding $x^* \in K$ such that

$$f(y,x^*) \in -C, \quad \forall y \in K. \tag{1.2}$$

The set of solutions of DVEP(f,K) will be denoted as $DS(f,K)$.

Equilibrium problems with monotone and pseudo-monotone bifunctions have been studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (e.g. [1, 2, 3, 4, 5, 6]).

The equilibrium problem encompasses, among its particular cases, convex optimization problems, variational inequalities, saddle point problems, fixed point problems and other problems of interest in many applications (see, for example [5, 6, 7] and the references therein). The prototypical example of an equilibrium problem is a variational inequality problem, and as mentioned by Németh [8], there are numerous problems in many fields of mathematics and physics can be reformulated as variational inequalities or boundary value problems on manifolds, like e.g., convex optimization problems, PDEs boundary value problems, lubrication problems, the analysis of filtration of a liquid through a porous medium, the determination of a flow past a given profile, the deflection of a simply supported beam, etc. Therefore, the extension of concepts, results and techniques of equilibrium problems and variational inequalities from linear spaces to Riemannian manifolds is natural and has some important advantages. For example, optimization problems with nonconvex objective functions become convex optimization problems by introducing an appropriate Riemannian metric, and also constrained optimization problems can be considered as unconstrained optimization problems from the Riemannian geometry point of view. Therefore the study of convergence analysis of approximation methods for finding the solutions of equilibrium problems, optimization problems and variational inequalities over Hadamard manifolds is emerging as an interesting research topic; see, e.g., [7, 9, 10, 11, 12, 13, 14, 15] and the references therein. We will present some interesting examples of vector equilibrium problems in Hadamard manifolds in Section 4.

The prototypical example of vector equilibrium problems occurs when C is the nonnegative cone, i.e. $C = \mathbb{R}^m_+$. If we take $G : M \to \mathbb{R}^m$ and $f(x,y) = G(y) - G(x)$, then VEP(f,K) is equivalent to the problem of finding a Pareto minimizer of G on K, i.e. a point $x^* \in K$ such that there exists no $x \in K$ such that $G(x) \le G(x^*), G(x) \ne G(x^*)$ (here $G(x) \le G(x^*)$ means $G(x)_i \le G(x^*)_i$ for all $i \in \{1, \ldots m\}$).

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. If M is a Hadamard manifold, then M have the same topology and differential structure of the Euclidean space $\mathbb{R}^n$.

We will deal in this paper with the extragradient (or Korpelevich's) method for vector equilibrium problems on Hadamard manifolds, and thus we start with an introduction to its well known finite dimensional formulation when applied to variational inequalities, i.e., we assume that $M = \mathbb{R}^n$, $m = 1$ and $f(x,y) = \langle T(x), y - x\rangle$ with $T : \mathbb{R}^n \to \mathbb{R}^n$. We assume that T is *monotone*, i.e. $\langle T(x) - T(y), x - y\rangle \ge 0$ for all $x, y \in M$. In this setting, there are several iterative methods for solving VIP(T,K). One of the most useful ones is the *extragradient method* presented in [16], which generates a sequence $\{x^k\} \subset M$ according to:

$$y^k = P_K(x^k - \alpha_k T(x^k)), \tag{1.3}$$

$$x^{k+1} = P_K(x^k - \alpha_k T(y^k)), \tag{1.4}$$

where P_K denotes the orthogonal projection onto K and $\{\alpha_k\} \subset \mathbb{R}$ is a sequence of positive stepsizes.

It was proved in [16] that if T is monotone and Lipschitz continuous with constant L, and VIP(T,K) has solutions, then the sequence generated by (1.3)–(1.4) converges to a solution of VIP(T,K) provided that $\alpha_k = \alpha \in (0, 1/L)$.

In the absence of Lipschitz continuity of T, it is natural to search for an appropriate stepsize in an inner loop. This is achieved in the following procedure:

Take $\delta \in (0,1)$, $\hat{\beta}$, $\tilde{\beta}$ satisfying $0 < \hat{\beta} \leq \tilde{\beta}$, and a sequence $\{\beta_k\} \subseteq [\hat{\beta}, \tilde{\beta}]$. The method is initialized with any $x^0 \in K$ and the iterative step is as follows:

Given x^k, define

$$z^k := x^k - \beta_k T(x^k). \tag{1.5}$$

If $x^k = P_K(z^k)$ stop. Otherwise take

$$j(k) := \min\left\{ j \geq 0 : \left\langle T(2^{-j} P_K(z^k) + (1-2^{-j})x^k), x^k - P_K(z^k) \right\rangle \geq \frac{\delta}{\beta_k} \|x^k - P_K(z^k)\|^2 \right\}, \tag{1.6}$$

$$\alpha_k := 2^{-j(k)}, \tag{1.7}$$

$$y^k := \alpha_k P_K(z_k) + (1-\alpha_k)x^k, \tag{1.8}$$

$$H_k := \left\{ z \in \mathbb{R}^n : \langle T(y^k), z - y^k \rangle = 0 \right\}, \tag{1.9}$$

$$x^{k+1} := P_K\left(P_{H_k}(x^k)\right). \tag{1.10}$$

This method converges to a solution of VIP(T,K) under the only assumptions of monotonicity of T and existence of solutions; see [17].

The above backtracking procedure for determining the right α_k is sometimes called an Armijo-type search (see [18]). It has been analyzed for VIP(T,K) in [17] and [19]. Other variants of Korpelevich's method can be found in [20, 21, 22, 23, 24, 25, 26]. An extragradient method for scalar equilibrium problems in Hadamard spaces has been studied in [21] and also an extragradient method for vector equilibrium problems in Banach spaces has been introduced in [23]. In this paper, we extend the method in [23] to vector equilibrium problems in Hadamard manifolds.

The paper is organized as follows. In Section 2, we introduce some preliminary material related to properties of Hadamard manifolds and vector optimization. In Section 3, we present our extragradient method for solving vector equilibrium problems and prove convergence of the generated sequence to a solution of the problem. In Section 4, we give some examples of vector equilibrium problems in Hadamard manifolds to which our main result can be applied. Finally, we present some numerical experiments.

2. Preliminaries

In this section, we give some definitions and preliminary material related to properties of Hadamard manifolds and vector optimization.

Definition 2.1. Let (X,d) be a complete metric space and $C \subset X$ be nonempty. A sequence $\{x^k\} \subset X$ is called Fejér convergent to C if

$$d(z,x^{k+1}) \leq d(z,x^k)$$

for all $z \in C$ and for all k.

Lemma 2.1. (Lemma 4.6 of [27]) *Let (X,d) be a complete metric space and let $C \subset X$ be a nonempty set. Let $\{x^k\} \subset X$ be Fejér convergent to C and suppose that any cluster point of $\{x^k\}$ belongs to C.Then $\{x^k\}$ converges to a point of C.*

If M is a Hadamard manifold, then for all $p \in M$, the exponential mapping is a diffeomorphism. Denoting the inverse of the exponential mapping by $\exp_p^{-1} : M \to T_pM$, we have $d(p,q) = \|exp_p^{-1}q\|$.

Proposition 2.1. (Proposition 2.1 of [28]) *Let M be a Hadamard manifold and $p \in M$. Then we know that $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p,q \in M$, there exists a unique minimal geodesic $\gamma_{p,q} = \exp_p t \exp_p^{-1} q$ for all $t \in [0,1]$ joining p to q.*

Proposition 2.2. (Comparison theorem for triangles, Proposition 2.2 in [27]) *Let $\gamma_i : [0,1] \to M$ be the geodesic joining p_i to p_{i+1} and $\Delta(p_1,p_2,p_3)$ be the geodesic triangle. Take $l_i := L(\gamma_i)$, $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$, where $i \equiv 1,2,3(mod\ 3)$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi,$$

$$l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \leq l_{i-1}^2.$$

Since

$$\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i,p_{i+1})d(p_{i+1},p_{i+2})\cos\alpha_{i+1},$$

we have

$$d^2(p_i,p_{i+1}) + d^2(p_{i+1},p_{i+2}) - d^2(p_i,p_{i+2}) \leq 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \tag{2.1}$$

Definition 2.2. The subdifferential of a function $h : M \to \mathbb{R}$ is the set-valued mapping $\partial h : M \to 2^{TM}$ defined by

$$\partial h(x) = \Big\{ u \in T_xM : \langle \exp_x^{-1} y, u \rangle \leq h(y) - h(x),\ \forall y \in M \Big\}, \qquad \forall x \in M.$$

Proposition 2.3. (Proposition 3.3 of [29]) *Let M be a Hadamard manifold and $z \in M$. Then the map $\rho_z(x) = \frac{1}{2}d^2(z,x)$ is strictly convex and its gradient at x is $\triangledown\rho_z(x) = -\exp_x^{-1} z$.*

Throughout this paper K will denote a nonempty, closed and convex set in a Riemannian manifold. For any $p \in M$ and $K \subset M$, there exists a unique $q \in K$ such that $d(p,q) \leq d(p,z)$ for all $z \in K$; this unique point is called the projection of p onto the convex set K and is denoted as $P_K(p)$. The next result gives a characterization of the projection P_K.

Proposition 2.4. [30] *For any $p \in M$, there exists a unique projection $P_K(p)$. Furthermore, the following inequality holds:*

$$\langle \exp_{P_K(p)}^{-1} z, \exp_{P_K(p)}^{-1} p \rangle \leq 0, \qquad \forall z \in K.$$

We continue by establishing some standard notation. The norm in $\mathbb{R}^m$ will be denoted as $\|\cdot\|$, while the inner product in $\mathbb{R}^m$ will be denoted as $\langle\cdot,\cdot\rangle$. The dual cone C^+ of C is defined as $C^+ = \{z \in \mathbb{R}^m : \langle y,z\rangle \geq 0,\ \forall y \in C\}$. We define the partial order $\preceq$ in $\mathbb{R}^m$, induced by the cone C, as

$$y \preceq y' \Longleftrightarrow y' - y \in C,$$

with its associate relation $\prec$, by

$$y \prec y' \Longleftrightarrow y' - y \in \mathrm{int}(C).$$

We define $\bar{\mathbb{R}}^m = \mathbb{R}^m \cup \{-\infty, +\infty\}$, where a neighbourhood of $+\infty$ is defined as a set $N \subset \bar{\mathbb{R}}^m$ containing $r + C \cup \{+\infty\}$ for some $r \in \mathbb{R}^m$ and its opposite $-N$ is a neighbourhood of $-\infty$. The binary relations $\preceq$ and $\prec$ defined above are extended to $\bar{\mathbb{R}}^m$ by

$$-\infty \prec y \prec +\infty, \qquad -\infty \preceq y \preceq +\infty.$$

for all $y \in \mathbb{R}^m$.

Note that the embedding $\mathbb{R}^m \subset \bar{\mathbb{R}}^m$ is continuous and dense. We extend by continuity every $z \in C^+ \setminus \{0\}$ to $\bar{\mathbb{R}}^m$, by putting $\langle \pm\infty, z\rangle = \pm\infty$. Given a set $T \subset \bar{\mathbb{R}}^m$, we denote its topological closure in the topological space $\bar{\mathbb{R}}^m$ by $\bar{T}$. To a given set $T \subset \bar{\mathbb{R}}^m$, we associate the following set:

$$\inf\nolimits_w^C(T) = \{y \in \bar{T} \mid \nexists z \in T : z \prec y\}.$$

Given $S \subset M$ and $G : S \to \mathbb{R}^m \cup \{+\infty\}$, the point $a \in M$ is called *weakly efficient* if $a \in S$ and $G(a) \in \inf_w^C(G(S))$. We denote as $\mathrm{argmin}_w^C\{G(x) | x \in S\}$ the set of weakly efficient points. We observe that

$$\mathrm{argmin}_w^C\{G(x) | x \in S\} = S \cap G^{-1}(\inf\nolimits_w^C(G(S))).$$

A map $G : M \to \mathbb{R}^m \cup \{+\infty\}$ is called C-convex whenever for any piecewise smooth curve $\gamma : [a,b] \to M$ joining x to y, we have

$$G(\gamma(t)) \preceq tG(x) + (1-t)G(y), \quad \forall x,y \in M \quad \text{and} \quad \forall t \in [0,1].$$

Definition 2.3. A vector valued function $G : M \to \mathbb{R}^m \cup \{+\infty\}$ is called positively lower semicontinuous, if for every $z \in C^+$ the extended scalar function $x \mapsto \langle G(x), z\rangle$ is lower semicontinuous. Also we say that G is positively upper semicontinuous whenever $-G$ is positively lower semicontinuous.

Now we recall an essential theorem from [31], which is needed in the next sections.

Theorem 2.1. *If $S \subset M$ is a convex set and $G : S \to \mathbb{R}^m \cup \{+\infty\}$ is a C-convex proper map, then*

$$\mathrm{argmin}_w^C\{G(x) \mid x \in S\} = \bigcup_{z \in C^+ \setminus \{0\}} \mathrm{argmin}\{\langle G(x), z\rangle \mid x \in S\}.$$

For convergence of the scalar extragradient method for equilibrium problems in $\mathbb{R}^n$, some monotonicity-like assumptions on the bifunction f are needed. We define next two suitable properties of this kind in the context of vector equilibrium problems in Hadamard manifolds. The bifunction f is said to be *C-pseudomonotone* whenever $f(x,y) \in \mathbb{R}^m \setminus (-C)$ with $x,y \in M$, it holds that $f(y,x) \in -C \setminus \{0\}$, and f is weakly *C-pseudomonotone* whenever $f(x,y) \notin -\mathrm{int}(C)$ with $x,y \in M$, it holds that $f(y,x) \in -C$.

We introduce some assumptions on a vector valued bifunction $f : M \times M \to \mathbb{R}^m$, which will be needed in our convergence analysis.

B1: $f(x,x)=0$ for all $x\in M$,
B2: $f(\cdot,\cdot):M\times M\to\mathbb{R}^m$ is uniformly continuous on bounded sets,
B3: $f(x,\cdot):M\to\mathbb{R}^m$ is C-convex for all $x\in M$.
B4: f is weakly C-pseudomonotone.

In connection with B4, we mention that for the case of scalar valued bifunctions, the usual monotonicity-like assumption is *pseudo-monotonicity of* f, meaning that whenever $f(x,y)\geq 0$ with $x,y\in M$, it holds that $f(y,x)\leq 0$. Clearly, B4 is a suitable extension of pseudo-monotonicity.

We recall now some properties of the solution set of dual vector equilibrium problems.

Proposition 2.5. *Assume that $f:M\times M\to\mathbb{R}^m$ satisfies B1, that $f(\cdot,y)$ is positively upper semicontinuous for all $y\in M$ and that $f(x,\cdot)$ is C-convex for all $x\in M$. Then $DS(f,K)\subseteq S(f,K)$.*

Proof. Take $x^*\in DS(f,K)$ and define $\gamma(t)=\exp_{x^*}t\exp_{x^*}^{-1}y$ with $t\in(0,1)$ and $y\in K$. Take any $c\in C^+\setminus\{0\}$. B1 and C-convexity of $f(\gamma(t),\cdot)$ imply that

$$0=\langle f(\gamma(t),\gamma(t)),c\rangle\leq(1-t)\langle f(\gamma(t),x^*),c\rangle+t\langle f(\gamma(t),y),c\rangle. \tag{2.2}$$

Since $\langle f(\gamma(t),x^*),c\rangle\leq 0$, (2.2) implies that

$$\langle f(\gamma(t),y),c\rangle\geq 0. \tag{2.3}$$

Since $\langle f(\cdot,y),c\rangle\geq 0$ is upper semicontinuous, taking limsup with $t\to 0$ in (2.3) gives

$$\langle f(x^*,y),c\rangle\geq 0.$$

Hence $f(x^*,y)\notin-\text{int}(C)$. Since $y\in K$ is arbitrary, $DS(f,K)\subseteq S(f,K)$. □

Corollary 2.1. *Under B1–B3, $DS(f,K)\subseteq S(f,K)$. If B4 holds, then $S(f,K)\subseteq DS(f,K)$.*

Proof. Elementary. □

Proposition 2.6. *If $f(x,\cdot)$ is C-convex and positively lower semicontinuous for all $x\in M$, then $DS(f,K)$ is closed and convex.*

Proof. Take $\bar{x},x^*\in DS(f,K)$ and define $\gamma(t)=\exp_{x^*}t\exp_{x^*}^{-1}\bar{x}$ with $t\in(0,1)$. Take any $c\in C^+\setminus\{0\}$. By C-convexity of $f(x,\cdot)$, we have

$$\langle f(x,\gamma(t)),c\rangle\leq(1-t)\langle f(x,x^*),c\rangle+t\langle f(x,\bar{x}),c\rangle\leq 0, \tag{2.4}$$

for all $x\in K$. Since $c\in C^+\setminus\{0\}$ is arbitrary, it follows that $\gamma(t)\in DS(f,K)$, i.e. $DS(f,K)$ is convex. Closedness of $DS(f,K)$ follows from positive lower semicontinuity of $f(x,\cdot)$ for all $x\in M$. □

3. Extragradient Method with Linesearch and Convergence Analysis

Let M be a Hadamard manifold and $K\subset M$ be nonempty, closed and convex. We take a closed, convex and pointed cone $C\subset\mathbb{R}^m$ with nonempty interior and a bifunction $f:M\times M\to\mathbb{R}^m$ which satisfies B1-B4 and such that $S(f,K)\neq\emptyset$. We consider the vector equilibrium problem VEP(f,K) as defined in (1.1), and propose the following **Extragradient Method with Linesearch (EML)** for solving this problem.

Take $\delta\in(0,1)$, $\hat{\beta}$, $\tilde{\beta}$ satisfying $0<\hat{\beta}\leq\tilde{\beta}$, a sequence $\{\beta_k\}\subseteq[\hat{\beta},\tilde{\beta}]$, and a sequence $\{e^k\}\subset\text{int}(C)$ such that $e^k\to e\in\text{int}(C)$.

1. Initialization:

$$x^0 \in K. \tag{3.1}$$

2. Iterative step: Given x^k, define

$$z^k \in \operatorname{argmin}_w^C \left\{ f(x^k, y) + \frac{1}{2\beta_k} d^2(x^k, y) e^k : y \in K \right\}. \tag{3.2}$$

If $x^k = z^k$ stop. Otherwise, let

$$\ell(k) = \min \left\{ \ell \geq 0 : -\beta_k f(y^\ell, x^k) + \beta_k f(y^\ell, z^k) + \frac{\delta}{2} d^2(z^k, x^k) e^k \notin \operatorname{int}(C) \right\}, \tag{3.3}$$

where

$$y^\ell = \exp_{x^k}(2^{-\ell} \exp_{x^k}^{-1} z^k). \tag{3.4}$$

We take

$$\alpha_k := 2^{-\ell(k)}, \tag{3.5}$$

$$y^k := \exp_{x^k}(\alpha_k \exp_{x^k}^{-1} z^k), \tag{3.6}$$

$$w^k = P_{H_k}(x^k), \tag{3.7}$$

where

$$H_k = \left\{ y \in M : f(y^k, y) \in -C \right\}.$$

Finally we define

$$x^{k+1} = P_K(w^k). \tag{3.8}$$

We start the analysis of the algorithm with some elementary properties of EML.

Proposition 3.1. *The sequence $\{z^k\}$ is well defined.*

Proof. Take any $c \in C^+ \setminus \{0\}$. Since $e^k \in \operatorname{int}(C)$, it follows from the definition of C^+ that $\langle e^k, c \rangle > 0$. Define $\psi : M \to \mathbb{R} \cup \{+\infty\}$ as

$$\psi(y) = \langle f(x^k, y), c \rangle + \frac{1}{2\beta_k} d^2(x^k, y) \langle e^k, c \rangle. \tag{3.9}$$

It is easy to see that ψ is proper, convex and lower semicontinuous. By Lemma 4.2 of [29], the subdifferential $\partial \psi$ has some zero, which is a minimizer of ψ. In view of Theorem 2.1, such minimizer satisfies (3.2) and can be taken as z^k. □

Proposition 3.2. *Assume that f satisfies B1–B3. Take $x \in K$, $\beta \in \mathbb{R}^+$ and $e \in \operatorname{int}(C)$ such that $\|e\| = 1$. If*

$$z \in \operatorname{argmin}_w^C \left\{ f(x, y) + \frac{1}{2\beta} d^2(x, y) e : y \in K \right\} \tag{3.10}$$

then there exists $c \in C^+ \setminus \{0\}$ such that

$$\langle \exp_z^{-1} y, \exp_z^{-1} x \rangle \langle e, c \rangle \leq \beta [\langle f(x, y), c \rangle - \langle f(x, z), c \rangle], \qquad \forall y \in K.$$

Proof. Let $N_K(z)$ be the normal cone of K at $z \in K$, i.e., $N_K(z) = \{v \in T_zM : \langle \exp_z^{-1} y, v\rangle \leq 0, \forall y \in K\}$. Since z solves the vector optimization problem in (3.10), in view of Theorem 2.1 there exists $c \in C^+ \setminus \{0\}$ such that z satisfies the first order optimality condition, given by

$$0 \in \partial \left\{ \langle f(x,\cdot), c\rangle + \frac{1}{2\beta} d^2(x,y)\langle e, c\rangle \right\}(z) + N_K(z).$$

Therefore, in view of Proposition 2.3, there exist $w \in \partial\langle f(x,\cdot),c\rangle(z)$ and $\bar{w} \in N_K(z)$ such that

$$0 = w - \frac{\langle e,c\rangle}{\beta} \exp_z^{-1} x + \bar{w}.$$

Now since $\bar{w} \in N_K(z)$, we have $\langle \exp_z^{-1} y, -w + \frac{\langle e,c\rangle}{\beta} \exp_z^{-1} x\rangle \leq 0$, so that, using the fact that $w \in \partial\langle f(x,\cdot),c\rangle(z)$,

$$\frac{\langle e,c\rangle}{\beta} \langle \exp_z^{-1} y, \exp_z^{-1} x\rangle \leq \langle \exp_z^{-1} y, w\rangle \leq \langle f(x,y),c\rangle - \langle f(x,z),c\rangle. \tag{3.11}$$

□

Corollary 3.1. *Assume that $\{x^k\}$ and $\{z^k\}$ are the sequences generated by EML. Then there exists $\{c^k\} \subset C^+ \setminus \{0\}$ such that*

$$\langle \exp_{z^k}^{-1} y, \exp_{z^k}^{-1} x^k\rangle \langle e^k, c^k\rangle \leq \beta_k \left[\langle f(x^k,y),c^k\rangle - \langle f(x^k,z^k),c^k\rangle \right] \qquad \forall y \in K.$$

Proof. Follows from Proposition 3.2 and (3.2). □

Proposition 3.3. *If Algorithm EML stops at the k-th iteration, then x^k is a solution of VEP(f,K).*

Proof. If $x^k = z^k$, then Corollary 3.1 implies that $\langle f(x^k,y),c^k\rangle \geq 0$ for all $y \in K$. Since $c^k \in C^+ \setminus \{0\}$, $f(x^k,y) \notin -\text{int}(C)$ for all $y \in K$. □

Proposition 3.4. *The following statements hold for Algorithm EML.*
i) $\ell(k)$ is well defined, (i.e. the Armijo-type search for α_k is finite), and consequently the same holds for the sequence $\{x^k\}$.
ii) $x^k \in K$ for all $k \geq 0$.
iii) If the algorithm does not stop at iteration k, then $f(y^k,x^k) \notin -C$.

Proof. i) We proceed inductively, i.e. we assume that x^k is well defined, and proceed to establish that the same holds for x^{k+1}. Note that z^k is well defined by Proposition 3.1. It suffices to check that $\ell(k)$ is well defined. Assume by contradiction that

$$-\beta_k f(y^\ell,x^k) + \beta_k f(y^\ell,z^k) + \frac{\delta}{2} d^2(z^k,x^k)e^k \in \text{int}(C) \tag{3.12}$$

for all ℓ. Since $c^k \in C^+ \setminus \{0\}$, we have

$$\beta_k[\langle f(y^\ell,x^k),c^k\rangle - \langle f(y^\ell,z^k),c^k\rangle] < \frac{\delta}{2} d^2(z^k,x^k)\langle e^k,c^k\rangle \tag{3.13}$$

for all ℓ. Note that the sequence $\{y^\ell\}$ is strongly convergent to x^k. In view of B2, taking limits in (3.13) as $\ell \to +\infty$,

$$\beta_k[\langle f(x^k,x^k),c^k\rangle - \langle f(x^k,z^k),c^k\rangle] \leq \frac{\delta}{2} d^2(z^k,x^k)\langle e^k,c^k\rangle. \tag{3.14}$$

Since $x^k \in K$ by (3.8), we apply Corollary 3.1 with $y = x^k$ in (3.14), obtaining

$$\langle \exp_{z^k}^{-1} x^k, \exp_{z^k}^{-1} x^k \rangle \leq \frac{\delta}{2} d^2(z^k, x^k). \tag{3.15}$$

Thus by (2.1) and (3.15), we get

$$d^2(z^k, x^k) + d^2(x^k, z^k) \leq \delta d^2(z^k, x^k). \tag{3.16}$$

Since $\delta \in (0,1)$, this contradiction shows that $\ell(k)$ is well defined.
ii) It follows from (3.1) and (3.8).
iii) Assume that $f(y^k, x^k) \in -C$. Using B1, B3 and (3.6), we have

$$0 = f(y^k, y^k) \preceq \alpha_k f(y^k, z^k) + (1-\alpha_k) f(y^k, x^k).$$

Since $-(1-\alpha_k) f(y^k, x^k)$ and $\alpha_k f(y^k, z^k) + (1-\alpha_k) f(y^k, x^k)$ belong C, and C is a convex cone, we conclude that $f(y^k, z^k) \in C$. Therefore

$$-\beta_k f(y^k, x^k) + \beta_k f(y^k, z^k) + \frac{\delta}{2} d^2(z^k, x^k) e^k \in \operatorname{int}(C), \tag{3.17}$$

which contradicts (3.3)–(3.6). Note that the inclusion in (3.17) is due to the fact that $x^k \neq z^k$ and $e^k \in \operatorname{int}(C)$. □

We continue the analysis of the convergence properties of EML. Recall that $S(f,K)$ denotes the solution set of VEP(f,K).

Proposition 3.5. *Assume that $S(f,K) \neq \emptyset$ and the bifunction f satisfies B1-B4. Let $\{x^k\}$, $\{y^k\}$, $\{z^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm EML. If the algorithm does not have finite termination, then*
i) the sequence $\{d(x^*, x^k)\}$ is nonincreasing (and henceforth convergent) for any $x^* \in S(f,K)$,
ii) the sequence $\{x^k\}$ is bounded,
iii) $\lim_{k\to+\infty} d(w^k, x^k) = 0$,
iv) the sequence $\{z^k\}$ is bounded,
v) all cluster points of $\{f(y^k, x^k)\}$ belong to $-C$.

Proof. i) Take $x^* \in S(f,K)$. By B4, $x^* \in H_k$ for all k. By Proposition 3.4(iii), $x^k \notin H_k$. Also, we have $w^k = P_{H_k}(x^k)$. Using Proposition 2.4,

$$\langle \exp_{w^k}^{-1} x^*, \exp_{w^k}^{-1} x^k \rangle \leq 0. \tag{3.18}$$

Therefore (2.1) shows that

$$d^2(w^k, x^k) + d^2(x^*, w^k) - d^2(x^*, x^k) \leq 0. \tag{3.19}$$

Now, since $x^{k+1} = P_K(w^k)$, again Proposition 2.4 implies that

$$\langle \exp_{x^{k+1}}^{-1} x^*, \exp_{x^{k+1}}^{-1} w^k \rangle \leq 0 \tag{3.20}$$

and consequently by (2.1), we have

$$d^2(x^{k+1}, w^k) + d^2(x^*, x^{k+1}) - d^2(x^*, w^k) \leq 0. \tag{3.21}$$

In view of (3.19) and (3.21), we have

$$d^2(x^*, x^{k+1}) \leq d^2(x^*, w^k) - d^2(x^{k+1}, w^k) \leq d^2(x^*, w^k) \leq d^2(x^*, w^k) + d^2(w^k, x^k) \leq d^2(x^*, x^k). \tag{3.22}$$

ii) In view of (i), $\lim_{k\to+\infty} d^2(x^*,x^k)$ exists, so that $\{x^k\}$ is bounded.
iii) It follows from (i) and (3.22).
iv) Since $x^k \in K$ by (3.8), we conclude from (3.2) and Theorem 2.1 that there exists $c^k \in C^+ \setminus \{0\}$ such that

$$\beta_k \langle f(x^k,z^k),c^k\rangle + \frac{1}{2}d^2(x^k,z^k)\langle e^k,c^k\rangle \le \beta_k\langle f(x^k,x^k),c^k\rangle + \frac{1}{2}d^2(x^k,x^k)\langle e^k,c^k\rangle = 0, \tag{3.23}$$

using B1 in the equality. From (3.23), we get

$$d^2(x^k,z^k)\langle e^k,c^k\rangle \le -2\beta_k\langle f(x^k,z^k),c^k\rangle. \tag{3.24}$$

Take now $u_*^k \in \partial\langle f(x^k,\cdot),c^k\rangle(x^k)$ and define $u^k = \langle e^k,c^k\rangle^{-1}u_*^k$. By definition of $\partial\langle f(x^k,\cdot),c^k\rangle$ at x^k, we have

$$\langle \exp_{x^k}^{-1} y,u^k\rangle\langle e^k,c^k\rangle \le \langle f(x^k,y),c^k\rangle - \langle f(x^k,x^k),c^k\rangle = \langle f(x^k,y),c^k\rangle. \tag{3.25}$$

Define $B_1(x^k) = \{x \in M : d(x,x^k) \le 1\}$. Since f is bounded on bounded sets and $\{x^k\}$ is bounded by item (ii), there is $N > 0$ such that $\left\|f(x^k,y)\right\| < N$ for all k and for all $y \in B_1(x^k)$. Now without loss of generality, we can assume $\|c^k\| = 1$ for all k. Then we have

$$\|u^k\|\langle e^k,c^k\rangle = \sup_{y\in B_1(x^k)} \langle \exp_{x^k}^{-1} y,u^k\rangle\langle e^k,c^k\rangle \le \sup_{y\in B_1(x^k)} \langle f(x^k,y),c^k\rangle \le N. \tag{3.26}$$

Since $e^k \to e \in \mathrm{int}(C)$, therefore $\{u^k\}$ is bounded. Now from (3.25), we have

$$\langle \exp_{x^k}^{-1} z^k,u^k\rangle\langle e^k,c^k\rangle \le \langle f(x^k,z^k),c^k\rangle. \tag{3.27}$$

Combining (3.24) and (3.27), we get, after dividing by $\langle e^k,c^k\rangle$,

$$d^2(x^k,z^k) \le -2\beta_k\langle \exp_{x^k}^{-1} z^k,u^k\rangle \le 2\tilde{\beta}\left\|u^k\right\| d(x^k,z^k). \tag{3.28}$$

Now since $\{x^k\}$ is bounded by item (ii), we obtain that $\{z^k\}$ is bounded.

v) Note that $\{x^k\}$, $\{y^k\}$, and $\{w^k\}$ are bounded by items (ii), (iv), (3.6) and (3.22). Since $f(\cdot,\cdot)$ is uniformly continuous on bounded sets by B2, and $\lim_{k\to+\infty} d(w^k,x^k) = 0$ by item (iii), we conclude that

$$\lim_{k\to+\infty}\left\|f(y^k,x^k) - f(y^k,w^k)\right\| = 0. \tag{3.29}$$

Note that if the algorithm does not stop at iteration k, then $f(y^k,x^k) \notin -C$, by Proposition 3.4(iii). Also, since $w^k \in H_k$, we have $f(y^k,w^k) \in -C$ for all k. Since C is closed and convex, these two facts, together with (3.29), easily imply that each limit point of $\{f(y^k,x^k)\}$ belongs to $-C$. □

Proposition 3.6. *Assume that $S(f,K) \neq \emptyset$ and the bifunction f satisfies B1-B4. Let $\{x^k\}$ and $\{z^k\}$ be the sequences generated by Algorithm EML. If $\{x^{k_i}\}$ is a subsequence of $\{x^k\}$ satisfying*

$$\lim_{i\to+\infty} d(z^{k_i},x^{k_i}) = 0, \tag{3.30}$$

then each cluster point of $\{x^{k_i}\}$ solves $VEP(f,K)$.

Proof. Note that $\{x^{k_i}\}$ and $\{z^{k_i}\}$ are bounded, and

$$\lim_{i\to+\infty} d(z^{k_i},x^{k_i}) = \lim_{i\to+\infty} \left\|\exp^{-1}_{z^{k_i}} x^{k_i}\right\| = 0.$$

Therefore, by B2, we get

$$\lim_{i\to+\infty} f(x^{k_i},z^{k_i}) = 0. \tag{3.31}$$

In the sequel, without loss of generality, we can assume that $x^{k_i}\to\bar{x}$. Take now any $y\in K$. By Corollary 3.1, we have

$$\langle \exp^{-1}_{z^{k_i}} y, \exp^{-1}_{z^{k_i}} x^{k_i}\rangle\langle e^{k_i},c^{k_i}\rangle \le \beta_{k_i}\left[\langle f(x^{k_i},y),c^{k_i}\rangle - \langle f(x^{k_i},z^{k_i}),c^{k_i}\rangle\right],$$

which implies that

$$-\left\|\exp^{-1}_{z^{k_i}} y\right\|\left\|\exp^{-1}_{z^{k_i}} x^{k_i}\right\|\langle e^{k_i},c^{k_i}\rangle \le \beta_{k_i}\left[\langle f(x^{k_i},y),c^{k_i}\rangle - \langle f(x^{k_i},z^{k_i}),c^{k_i}\rangle\right]. \tag{3.32}$$

Since $c^{k_i}\in C^+\setminus\{0\}$, without loss of generality, we can assume that $\left\|c^k\right\|=1$ and $c^{k_i}\to c^*$. Taking the limit when $i\to\infty$ from (3.32) and using (3.30) and (3.31), we get

$$0\le \limsup_{i\to\infty}\langle f(x^{k_i},y),c^{k_i}\rangle. \tag{3.33}$$

Now since $x^{k_i}\to\bar{x}$ and $c^{k_i}\to c^*$, we get from B2 that

$$\langle f(\bar{x},y),c^*\rangle \ge 0.$$

Since $c^*\in C^+\setminus\{0\}$, we have $f(\bar{x},y)\notin -\text{int}(C)$. Now since $y\in K$ is arbitrary, $f(\bar{x},y)\notin -\text{int}(C)$ for all $y\in K$. □

Proposition 3.7. *Assume that $S(f,K)\neq\emptyset$ and the bifunction f satisfies B1-B4. If a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_k\}$, as defined in (3.5), converges to 0, then each cluster point of $\{x^{k_i}\}$ solves VEP(f,K).*

Proof. For proving the result, we will use Proposition 3.6. Thus, we must show that

$$\lim_{i\to+\infty} d(z^{k_i},x^{k_i}) = 0.$$

For the sake of contradiction, and without loss of generality, let us assume that

$$\text{liminf}_{i\to+\infty} d^2(z^{k_i},x^{k_i}) \ge \eta > 0. \tag{3.34}$$

Define

$$\hat{y}^i := \exp_{x^{k_i}}(2\alpha_{k_i}\exp^{-1}_{x^{k_i}} z^{k_i}), \tag{3.35}$$

or equivalently

$$\exp^{-1}_{x^{k_i}}\hat{y}^i = 2\alpha_{k_i}\exp^{-1}_{x^{k_i}} z^{k_i}. \tag{3.36}$$

Note that, since $\lim_{i\to+\infty}\alpha_{k_i}=0$, $\ell(k_i)>1$ for large enough i. Also, in view of (3.35), we have that $\hat{y}^i = y^{\ell(k_i)-1}$ in the inner loop of the linesearch for determining α_{k_i}, i.e., in (3.4). Since $\ell(k_i)$ is the first integer for which the exclusion in (3.3) holds, such exclusion does not hold for $\ell(k_i)-1$. i.e., we have

$$-\beta_{k_i} f(\hat{y}^i,x^{k_i}) + \beta_{k_i} f(\hat{y}^i,z^{k_i}) + \frac{\delta}{2} d^2(z^{k_i},x^{k_i})e^{k_i} \in \text{int}(C) \tag{3.37}$$

for large enough i. On the other hand, since $\lim_{i\to+\infty}\alpha_{k_i}=0$ by hypothesis, and $\{\exp^{-1}_{x^{k_i}} z^{k_i}\}$ is bounded by Proposition 3.5(ii) and (iv), it follows from (3.36) that

$$\lim_{i\to+\infty} d(x^{k_i},\hat{y}^i)=\lim_{i\to+\infty}\left\|\exp^{-1}_{x^{k_i}}\hat{y}^i\right\|=0. \tag{3.38}$$

Since $f(\cdot,\cdot)$ is uniformly continuous on bounded sets by B2, (3.37) and (3.38) imply that

$$-\beta_{k_i}f(x^{k_i},x^{k_i})+\beta_{k_i}f(x^{k_i},z^{k_i})+\frac{\delta}{2}d^2(z^{k_i},x^{k_i})e^{k_i}\in C \tag{3.39}$$

for large enough i. Since δ belongs to $(0,1)$, it follows from (3.39) that

$$\beta_{k_i}f(x^{k_i},z^{k_i})+\frac{1}{2}d^2(z^{k_i},x^{k_i})e^{k_i}\in \operatorname{int}(C). \tag{3.40}$$

Take now $y=x^{k_i}$ in Corollary 3.1. Then by (2.1), we have

$$\begin{aligned}\frac{1}{2}\left[d^2(z^{k_i},x^{k_i})+d^2(x^{k_i},z^{k_i})\right]\langle e^{k_i},c^{k_i}\rangle &\le \langle \exp^{-1}_{z^{k_i}}x^{k_i},\exp^{-1}_{z^{k_i}}x^{k_i}\rangle\langle e^{k_i},c^{k_i}\rangle\\ &\le \beta_{k_i}\left[\langle f(x^{k_i},x^{k_i}),c^{k_i}\rangle-\langle f(x^{k_i},z^{k_i}),c^{k_i}\rangle\right],\end{aligned} \tag{3.41}$$

which implies that

$$\frac{1}{2}\left[d^2(z^{k_i},x^{k_i})+d^2(x^{k_i},z^{k_i})\right]\langle e^{k_i},c^{k_i}\rangle+\beta_{k_i}\langle f(x^{k_i},z^{k_i}),c^{k_i}\rangle\le 0. \tag{3.42}$$

Since $c^{k_i}\in C^+\setminus\{0\}$, we have

$$\frac{1}{2}d^2(z^{k_i},x^{k_i})e^{k_i}+\frac{1}{2}d^2(x^{k_i},z^{k_i})e^{k_i}+\beta_{k_i}f(x^{k_i},z^{k_i})\notin \operatorname{int}(C). \tag{3.43}$$

Note that $d^2(x^{k_i},z^{k_i})>0$. Hence, (3.43) contradicts (3.40), establishing the result.

□

Proposition 3.8. *Assume that $S(f,K)\neq\emptyset$ and the bifunction f satisfies B1-B4. Then each cluster point of the sequence $\{x^k\}$ generated by Algorithm EML solves VEP(f,K).*

Proof. First assume that there exists a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_k\}$ which converges to 0. In this case, by Proposition 3.7, we obtain that each cluster point of $\{x^{k_i}\}$ solves VEP(f,K). Now assume that $\{\alpha_{k_i}\}$ is any subsequence of $\{\alpha_k\}$ bounded away from zero (say $\alpha_{k_i}\ge\bar{\alpha}>0$). It follows from (3.3) and (3.6) that

$$-\beta_{k_i}f(y^{k_i},x^{k_i})+\beta_{k_i}f(y^{k_i},z^{k_i})+\frac{\delta}{2}d^2(z^{k_i},x^{k_i})e^{k_i}\notin \operatorname{int}(C). \tag{3.44}$$

Note that, since $\alpha_{k_i}\le 1$ by (3.5), we get, in view of B1 and B3,

$$0=f(y^{k_i},y^{k_i})\preceq \alpha_{k_i}f(y^{k_i},z^{k_i})+(1-\alpha_{k_i})f(y^{k_i},x^{k_i})\in C. \tag{3.45}$$

Hence we have

$$-\beta_{k_i}f(y^{k_i},z^{k_i})+\frac{-\beta_{k_i}(1-\alpha_{k_i})}{\alpha_{k_i}}f(y^{k_i},x^{k_i})\in -C. \tag{3.46}$$

Summing up (3.44) and (3.46), we have

$$\frac{-\beta_{k_i}}{\alpha_{k_i}}f(y^{k_i},x^{k_i})+\frac{\delta}{2}d^2(z^{k_i},x^{k_i})e^{k_i}\notin \operatorname{int}(C). \tag{3.47}$$

Note that each cluster point of $\{f(y^{k_i},x^{k_i})\}$ belongs to $-C$ by Proposition 3.5(v). Since C is a closed and convex cone, taking limits in (3.47) with $i \to +\infty$, we obtain

$$\lim_{i\to+\infty} d^2(z^{k_i},x^{k_i}) = 0.$$

We are within the assumptions of Proposition 3.6, and thus we conclude that each cluster point of $\{x^{k_i}\}$ solves VEP(f,K). It follows that each cluster point of every subsequence of $\{x^k\}$ solves VEP(f,K), and hence the same holds for the whole sequence $\{x^k\}$. □

Now we prove our main result.

Theorem 3.1. *Assume that $S(f,K) \neq \emptyset$ and the bifunction f satisfies B1-B4. Let $\{x^k\}$ be the sequence generated by Algorithm EML. Then the sequence $\{x^k\}$ converges to a solution of VEP(f,K).*

Proof. i) Proposition 3.8 shows that each cluster point of $\{x^k\}$ solves VEP(f,K), so that all cluster points of $\{x^k\}$ belong to $S(f,K)$. By (3.22), the sequence $\{x^k\}$ is Fejér convergent to $S(f,K)$. Therefore Lemma 2.1 implies that $\{x^k\}$ converges to a point of $S(f,K)$. □

4. Examples and Applications

In this section, we first give some examples of vector equilibrium problems in several Hadamard manifolds to which our main theorem can be applied for finding a solution. We also present some numerical experiments. We start by recalling hyperbolic spaces.

The hyperbolic space $\mathbb{H}^n$:

We equip $\mathbb{R}^{n+1}$ with the inner product

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1},$$

for $x=(x_1,x_2,\ldots,x_{n+1})$ and $y=(y_1,y_2,\ldots,y_{n+1})$. Define

$$\mathbb{H}^n := \Big\{x=(x_1,x_2,\ldots,x_{n+1})\in\mathbb{R}^{n+1} : \langle x,x\rangle = -1,\ \ x_{n+1}>0\Big\}.$$

Then $\langle\cdot,\cdot\rangle$ induces the Riemannian metric d on the tangent spaces $T_p\mathbb{H}^n \subset T_p\mathbb{R}^{n+1}$ as

$$d(x,y) = \operatorname{arccosh}(-\langle x,y\rangle), \qquad \forall x,y\in\mathbb{H}^n \tag{4.1}$$

for $p\in\mathbb{H}^n$. Note that a normalized geodesic $\gamma_x : [0,1]\to\mathbb{H}^n$ starting from $\gamma_x(0)=x$, will have the equation

$$\gamma_x(t) = (\cosh t)x + (\sinh t)v,$$

where $v=\gamma_x'(0)\in T_x\mathbb{H}^n$ is the tangent unit vector of γ in the starting point. Also, we get from (4.1)

$$\exp_x^{-1} y = \operatorname{arccosh}(-\langle x,y\rangle)\frac{y+\langle x,y\rangle x}{\sqrt{\langle x,y\rangle^2-1}}, \qquad \forall x,y\in\mathbb{H}^n. \tag{4.2}$$

It is well known that $(\mathbb{H}^n,d)$ is a Hadamard manifold with sectional curvature -1 at every point (see [32]).

Now, we give an example of vector equilibrium problems in the hyperbolic space $\mathbb{H}^n$.

Example 4.1. Let $M = \mathbb{H}^n$ be the hyperbolic space and $C = \{z \in \mathbb{R}^3 : z_i \geq 0, i = 1,2,3\}$. We define the vector valued bifunction $f : M \times M \to \mathbb{R}^3$ as

$$f(x,y) = \langle x - T(x), y - x \rangle \varepsilon,$$

where ε belongs to the positive orthant $\mathbb{R}^3_{++} = \{z \in \mathbb{R}^3 : z_i > 0, i = 1,2,3\}$ and T is a map from M into itself defined by $T(x) = (-x_1, -x_2, \ldots, -x_n, x_{n+1})$. If x is an equilibrium point of VEP(f,M), then

$$\langle x - T(x), y - x \rangle \varepsilon \notin -\text{int}(C) \tag{4.3}$$

for all $y \in M$. Taking $y = T(x)$, we get

$$\langle x - T(x), T(x) - x \rangle \varepsilon \notin -\text{int}(C). \tag{4.4}$$

Therefore there exists $c^* \in C^+ \setminus \{0\}$ such that

$$\langle x - T(x), T(x) - x \rangle \langle \varepsilon, c^* \rangle \geq 0. \tag{4.5}$$

Since $\varepsilon \in \mathbb{R}^3_{++}$, $\langle \varepsilon, c^* \rangle > 0$. Therefore (4.5) implies that

$$\langle x - T(x), T(x) - x \rangle \geq 0. \tag{4.6}$$

Now (4.6) shows that $-4x_1^2 - 4x_2^2 - \cdots - 4x_n^2 \geq 0$. Hence, we get $x_1 = x_2 = \cdots = x_n = 0$. On the other hand, we have $\langle x, x \rangle = -1$. We conclude that $x_{n+1} = 1$. Hence $x = (0,0,\ldots,0,1)$ is the solution of VEP(f,M).

We continue with other examples of vector equilibrium problems in Hadamard manifolds to which our main result can be applied, some of which were adapted from [13].

Example 4.2. Let $M = \mathbb{R} \times \mathbb{H}^1$ be a Hadamard manifold and

$$K = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{H}^1 : x_2^2 - x_3^2 = -1,\ 0 \leq x_1 \leq 1,\ x_2 \geq 0,\ x_3 \geq 1 \right\}.$$

K is a nonempty, closed and convex subset of M. We define the vector valued bifunction $f : K \times K \to \mathbb{R}^3$ as

$$f(x,y) = ((2 - x_1)(y_2^2 + y_3^2 - x_2^2 - x_3^2), x_1^2 - x_1 y_1, (3 - 2x_1)(y_2^2 - x_2^2)).$$

Now suppose that $C = \{x \in \mathbb{R}^3 : x_i \geq 0, i = 1,2,3\}$. It is easy to see that $x^* = (1,0,1)$ is an equilibrium point of VEP(f,K). Note that $f(x,x) = 0$ for all $x \in K$, f is weakly C-pseudomonotone and C-convex with respect to the second variable and $f(\cdot,\cdot) : K \times K \to \mathbb{R}^3$ is uniformly continuous on bounded sets. If $\{x^k\}$ is the sequence generated by Algorithm EML, then $\{x^k\}$ converges to a solution of VEP(f,K) by Theorem 3.1.

The positive orthant with another metric:

We endow the positive orthant $\mathbb{R}^n_{++} = \{z \in \mathbb{R}^n : z_i > 0, i = 1, \cdots, n\}$ with the so-called affine metric defined by $G : \mathbb{R}^n_{++} \to S^n_{++}$,

$$G(x) = \text{diag}\left(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \cdots, \frac{1}{x_n^2} \right).$$

In other words, for any $x \in M := \mathbb{R}^n_{++}$ and $u, v \in T_x M$, we have $\langle u, v \rangle_x = \langle G(x)u, v \rangle = \sum_{i=1}^n \frac{u_i v_i}{x_i^2}$. The geodesic joining $x \in M$ to $y \in M$ is the curve $\gamma : [0,1] \to M$ defined by

$$\gamma(t) = \langle x_1^{1-t} y_1^t, x_2^{1-t} y_2^t, \cdots, x_n^{1-t} y_n^t \rangle.$$

Also, we can obtain that the distance between x and y is $d(x,y)=\left(\sum_{i=1}^{n}(\ln\frac{x_i}{y_i})^2\right)^{\frac{1}{2}}$. It is well known that (M,d) is a Hadamard manifold such that the tangent space at a point x is $\mathbb{R}^n$ (see also [33]).

Example 4.3. Suppose that (M,d) is the Hadamard manifold defined as above. Consider the vector valued bifunction $f: M\times M\to\mathbb{R}^3$ defined by

$$f(x,y)=\sum_{i=1}^{m}\left\langle \alpha_i(\prod_{j=1}^{n}y_j^{a_{ij}}-\prod_{j=1}^{n}x_j^{a_{ij}}),\beta_i(\prod_{j=1}^{n}y_j^{b_{ij}}-\prod_{j=1}^{n}x_j^{b_{ij}}),\gamma_i(\prod_{j=1}^{n}y_j^{c_{ij}}-\prod_{j=1}^{n}x_j^{c_{ij}})\right\rangle,$$

where $\alpha_i,\beta_i,\gamma_i\in\mathbb{R}_{++}$ and $a_{ij},b_{ij},c_{ij}\in\mathbb{R}$ for any i,j. Note that each component of f is a nonconvex function with respect to the second variable, but it is geodesic convex on M with respect to the metric d. Let $K=\{z\in\mathbb{R}^n: z_i\geq 1, i=1,\cdots,n\}\subset M$. It is obvious that K is a nonempty, closed and convex set. Now take $C=\{z\in\mathbb{R}^3: z_i\geq 0, i=1,2,3\}$ which is a closed, convex and pointed cone with nonempty interior. Then it is easy to see that $S(f,K)\neq\emptyset$, $f(x,x)=0$ for all $x\in M$, and f is C-convex with respect to the second variable and weakly C-pseudomonotone. Now, if $f(\cdot,\cdot): M\times M\to\mathbb{R}^3$ is uniformly continuous on bounded sets, then we can use Algorithm EML to approximate a solution of VEP(f,K), and if $\{x^k\}$ is the sequence generated by Algorithm EML, then the sequence $\{x^k\}$ converges to a solution of the problem.

The Nash equilibrium problem:

Suppose that $I=\{1,2,\cdots,n\}$ is a finite index set which denotes the set of players. Let M_i be a Hadamard manifold where $i\in I$, and the strategy set K_i is subset of M_i for the i-th player. Note that $M:=M_1\times M_2\times\cdots\times M_n$ is a Hadamard manifold (see [27]), and the set $K:=K_1\times K_2\times\cdots\times K_n$ is a subset of the Hadamard manifold M. Let $\varphi_i: K\to\mathbb{R}$ be a payoff function which shows the loss of each player where $i\in I$. Also, φ_i depends on the strategies of all the player for any $i\in I$. The Nash equilibrium problem corresponding to $\{\varphi_i\}_{i\in I}$ and $\{K_i\}_{i\in I}$ is to find $x=(x_1,x_2,\cdots,x_n)\in K$ such that

$$\varphi_i(x)\leq\varphi_i(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n),$$

for all $i\in I$ and all $y_i\in K_i$. The point x is a solution of the problem and is called a Nash equilibrium point. The above inequality implies that each Nash equilibrium point corresponds to an optimal amount for minimizing the loss. Now, we define $f: K\times K\to\mathbb{R}$ as

$$f(x,y)=\sum_{i=1}^{n}(\varphi_i(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)-\varphi_i(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)),$$

where $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)$. So, f is a bifunction and its corresponding equilibrium problem is to find $x\in K$ such that

$$f(x,y)\geq 0,\qquad \text{for all } y\in K.$$

It is easy to see that x is a Nash equilibrium point if and only if it is an equilibrium point of f.

Now, for any $i\in I$, we extend the payoff function $\varphi_i: K\to\mathbb{R}$ to a finite family of functions $\varphi_{ij}: K\to\mathbb{R}$ showing the loss of each player in m areas separately (for example, losses in the areas of finance, energy, time, human resources and etc) where $1 \leq j \leq m$. Consider

$$f_j(x,y) = \sum_{i=1}^{n} (\varphi_{ij}(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n) - \varphi_{ij}(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)),$$

for all $1 \leq j \leq m$ and $x = (x_1,x_2,\cdots,x_n)$ and $y = (y_1,y_2,\cdots,y_n)$. We define $f : M \times M \to \mathbb{R}^m$ as

$$f(x,y) = \langle f_1(x,y), f_2(x,y),\cdots, f_m(x,y)\rangle. \tag{4.7}$$

Therefore, our problem has been formulated as a vector equilibrium problem on the Hadamard manifold M, and the solution of the problem is the vector which minimizes the losses of the nm payoff functions corresponding to the m areas of the problem.

Example 4.4. Consider the bifunction $f : M \times M \to \mathbb{R}^m$ as defined in (4.7) and assume that the cost function φ_{ij} is convex and uniformly continuous on bounded sets for all i, j. Let $K \subset M$ be nonempty, closed and convex, and $C = \{z \in \mathbb{R}^m : z_i \geq 0, i = 1,\cdots,m\}$ which is a closed, convex and pointed cone with nonempty interior. Then it is obvious that the assumptions B1-B4 are satisfied. Now, if $S(f,K) \neq \emptyset$ and $\{x^k\}$ is the sequence generated by Algorithm EML, then Theorem (3.1) implies that $\{x^k\}$ converges to a solution of VEP(f,K).

We end this paper by performing some numerical experiments.

Example 4.5. Consider $(\mathbb{R}^2,d)$ where the metric d is defined by

$$d((x_1,x_2),(y_1,y_2)) = \Big((x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2\Big)^{\frac{1}{2}},$$

for all $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$. Note that $(\mathbb{R}^2,d)$ is a Hadamard manifold with the geodesic

$$\gamma(t) = \Big((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\Big),$$

where $t \in [0,1]$ (see [33]). Let M be the Hadamard manifold $(\mathbb{R}^2,d)$, and define $f : M \times M \to \mathbb{R}$ by

$$f(x,y) = a\Big((y_2 - y_1^2)^2 - (x_2 - x_1^2)^2\Big) + b\Big((1-y_1)^2 - (1-x_1)^2\Big), \tag{4.8}$$

where $a,b \in \mathbb{R}_+$. In order to implement our algorithm (EML) in Section 3, we suppose that $K = \Big\{x = (x_1,x_2) \in M : x_1 \geq 0\Big\}$ and $C = [0,+\infty)$. It is easy to see that K is nonempty, closed and convex, f satisfies B1–B4 and $S(f,K) \neq \emptyset$; indeed, it is easy to check that for all $(a,b) \in \mathbb{R}^2_{++}$ the unique solution is $x^* = (1,1)$. We take $\delta = \frac{1}{100}$, $\beta_k \equiv 1$ and $e^k \equiv 1$. If $\{x^k\}$ is the sequence generated by EML in Section 3, then Theorem 3.1 ensures that $\{x^k\}$ converges to the solution of VEP(f,K). We performed some numerical experiments for this example. We chose randomly 100 random pairs $(a,b) \in [0,100] \times [0,100]$ and five starting points. Our stopping criterion is $d(x^{k-1},x^k) < \varepsilon$, and we take $\varepsilon = 10^{-8}$

The numerical results are displayed in the following table, where the starting points, the average number of iterations and the average CPU times have been reported.

Also, all tests for the 100 problems corresponding to each starting point were successful, meaning that the sequence $\{x^k\}$ converges to $(1,1)$, which is the solution of VEP(f,K). All problems were solved by the Optimization Toolbox in Matlab R2020a on a Laptop Intel(R) Core(TM) i7- 8665U CPU @ 1.90GHz RAM 8.00 GB.

Starting point: x^0	Average number of iterations	Average CPU time (Sec)
(3, -2)	16.67	1.2321
(7, -11)	17.89	1.3496
(19, 4)	18.38	1.5412
(73, 98)	19.48	2.0146
(0, -18)	15.10	1.1295

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