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A SIMPLE STRONG CONVERGENT METHOD FOR SOLVING SPLIT COMMON FIXED POINT PROBLEMS

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Abstract. In this paper, we study the split common fixed point problem for demicontractive mappings in real Hilbert spaces. We propose an alternative regularization scheme with a self-adaptive step size for solving the problem and prove its strong convergence. Furthermore, we apply our main results to split feasibility problems, split variational inequality problems and signal processing. Several numerical examples show that our algorithm has some competitive advantage over some existing algorithms in the literature. Our results extend, complement, and generalize many recent related results in the literature.
Keywords. Fixed point problem; Demicontractive mappings; Split common problem; Hilbert spaces.

1. Introduction

In this paper, we focus on the following split common fixed point problem (in short, SCFPP). Let H_1 and H_2 be two real Hilbert spaces, and let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ two mappings with nonempty fixed point sets. Given a bounded and linear operator $A : H_1 \to H_2$, the split common fixed point problem is defined as follows:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in Fix(U) \text{ and } Ax^* \in Fix(T). \tag{1.1}$$

If U and T are taken as the metric projections onto some convex sets $C \subseteq H_1$ and $Q \subseteq H_2$, respectively, then the SCFP (1.1) translates to the split feasibility problem (in short, SFP) defined as follows:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in C \text{ and } Ax^* \in Q. \tag{1.2}$$

The SFP was introduced by Censor and Elfving [1] in finite dimensional Euclidean spaces for modeling inverse problems, which arise from phase retrievals and in medical image reconstruction [2]. Several iterative methods have been introduced for solving the SFP and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces (see, for example, [3, 4, 5, 6] and the references therein).

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In 2009, Censor and Segal [7] considered the case where U and T are directed mappings in finite dimensional spaces and then proposed the following algorithm:

$$x_{n+1} = U(x_n - \gamma A^t(I-T)Ax_n),\ n \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix A^tA (t stands for matrix transposition) and proved that the sequence generated converges weakly to a solution of SCFP (1.1).

In 2011, Moudafi [8] studied the SCFPs with the facts that U and T being quasi-nonexpansive mappings, and $I-U$ and $I-T$ are demiclosed at 0. He proposed the following algorithm and proved a weak convergence theorem:

$$\begin{cases} x_0 \in H_1, \\ u_n = x_n - \gamma\beta A^*(I-T)Ax_n, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n U(u_n), \end{cases} \tag{1.4}$$

where A^* is the adjoint of A, $\alpha_n \in (\delta, 1-\delta)$ for a small $\delta > 0$, $\beta > 0$ and $\gamma \in (0, \frac{1}{\lambda\beta})$ with λ being the spectral radius of A^*A. For recent and interesting extensions, modifications and improvements on the results by Moudafi [8], we refer to [9, 10] and the references therein.

In some cases, to estimate the norm of a bounded linear operator is difficult; see ([11], Theorem 2.3). Thus a challenge that may arises when one tries to compute the step size γ or implement algorithms (1.3), (1.4) and the algorithms introduced by Eslamian and Eslamian [9] and Shehu and Cholamjiak [10] is to determine $\|A\|$. Recently, some authors considered alternative ways of constructing variable step sizes. For instance, Cui and Wang [12] proposed the following Algorithm (1.5) for solving the SCFP (1.1) for the case that U and T are demicontractive mappings with constants $0 \le \kappa < 1$ and $0 \le \tau < 1$ such that $I-U$ and $I-T$ are demiclosed at 0, respectively, and they obtained a weak convergence result:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = U_\lambda(x_n - \rho_n A^*(I-T)Ax_n), n \ge 0, \end{cases} \tag{1.5}$$

where $U_\lambda = (1-\lambda)I + \lambda U$, $\lambda \in (0, 1-\kappa)$ and

$$\rho_n = \begin{cases} \frac{(1-\tau)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n) \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Another choice of the step size, introduced by Wang and Xu [13], is

$$\rho_n = \frac{\tau_n}{\|x_n - Ux_n + A^*(I-T)Ax_n\|},$$

where $\tau_n \subset (0, +\infty)$ is a sequence satisfying

$$\sum_{n=0}^{\infty} \tau_n = \infty \text{ and } \sum_{n=0}^{\infty} \tau_n^2 < \infty.$$

Next, let us focus on the fixed point problem of finding a point $x^* \in H_1$ such that $Tx^* = x^*$ for a given mapping $T : H_1 \to H_1$. In 2000, Moudafi [14] introduced the so-called viscosity approximation method, which is defined as follows:

Choose an arbitrary point $x_1 \in K$, where K is some subset in the domain of T, and define

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n,\ n \ge 1, \tag{1.7}$$

where $f : K \to K$ is a contraction and $\{\alpha_n\} \subset [0,1]$ satisfies the following control conditions:

(M1) $\lim_{n\to\infty} \alpha_n = 0$;
(M2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(M3) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

If $f(x) = u \in K$, $\forall x \in K$, then (1.7) becomes the Halpern algorithm [15]. Moudafi [14] proved the following result in Hilbert spaces:

Theorem 1.1. [14] *If $\{\alpha_n\}$ satisfies the conditions (M1) - (M3) as above, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a fixed point x^* of T, which also solves the following variational inequality*

$$\text{Find } x^* \in Fix(T) \text{ such that } \langle f(x^*) - x^*, x - x^* \rangle \leq 0, \quad x \in Fix(T).$$

Motivated by Moudafi [14] and Xu [16], Yang and He [17] proposed the following alternative regularization iterative scheme for finding a fixed points of nonexpansive mapping $T : K \to K$. Let $x_1 \in K$ be arbitrary and define

$$x_{n+1} = T(\alpha_n f(x_n) + (1-\alpha_n)x_n), \ n \geq 1, \tag{1.8}$$

where $f : K \to K$ is a contraction and $\{\alpha_n\} \subset [0,1]$ satisfies (M1) - (M3). It happens that the iterative schemes (1.7) and (1.8) converge to the same solution. For more details on alternative regularization methods and their convergence analysis, we refer to [17, 18, 19] and the references therein.

Based on the above results, for solving split common fixed point problems, Huimin et al. [20] considered a split common fixed point problem with demicontractive mappings U and T. They proposed the following viscosity-type algorithm and proved its strong convergence:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda(x_n - \rho_n A^*(I-T)Ax_n), n \geq 0, \end{cases} \tag{1.9}$$

where f is a contraction, U_λ is as defined above, $\{\alpha_n\}$ is a sequence in $[0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and ρ_n is defined as in (1.6).

Recently, Jirakitpuwapat et al. [21] introduced the following viscosity-type algorithm for solving the SCFP (1.1) with demicontractive mappings:

$$\begin{cases} x_0 \in H_1, \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)U_{\lambda_n}(x_n - \rho_n A^*(I-T)Ax_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n f(y_n), n \geq 0 \end{cases} \tag{1.10}$$

where f is a contraction, $U_{\lambda_n} = (1-\lambda_n)I + \lambda_n U$ for $\lambda_n \in (0, 1-\kappa)$, ρ_n is as defined in (1.6) and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n < \infty$.

Inspired by the work of Wang and Xu [13], Wang, Fang and Kim [22] introduced the following viscosity-type algorithm and proved its strong convergence:

$$\begin{cases} x_0 \in H_1, \\ \text{If } \|x_n - Ux_n + A^*(I-T)Ax_n\| = 0, \text{ then STOP}; x_n \text{ solves (1.1)} \\ \text{Otherwise, } x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[x_n - \tilde{\rho}_n(x_n - Ux_n + A^*(I-T)Ax_n)], \end{cases} \tag{1.11}$$

where

$$\tilde{\rho}_n = \frac{\tau_n}{\|x_n - Ux_n + A^*(I-T)Ax_n\|},$$

U and T are demicontractive mappings, and f is a contraction with constant $\nu \in [0, \frac{1}{\sqrt{2}})$.

Following the above research going on this direction, we propose an alternative regularization scheme for solving the SCFP (1.1) with demicontractive mappings. We prove a strong convergence theorem in real Hilbert spaces. Mathematical applications and numerical examples are presented to illustrate the generality of our scheme and its numerical behaviour.

This paper is organised as follows. In Section 2, we give some useful definitions, notations and lemmas, which are needed for the algorithm analysis. In Section 3, our algorithm and its strong convergence are presented. In Section 4, we give some applications of our main results. In Section 5, we give numerical example to illustrate our algorithms and compare them with some existing algorithms in literature. We conclude this paper by Section 6, the last section.

2. Preliminaries

Let H be a real Hilbert space, and let K be a nonempty closed convex subset of H. We denote by '$x_n \rightharpoonup x$' and '$x_n \to x$', the weak and the strong convergence of $\{x_n\}$ to a point $x \in H$, respectively.

The following inequality is well-known and trivial.

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \forall x, y \in H. \tag{2.1}$$

Definition 2.1. A mapping $T : H \to H$ is said to be:

(i) Lipschitzian if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \le \beta\|x-y\|, \forall\, x, y \in H.$$

If $\beta \in [0,1)$ then T is said to be a contraction. If $\beta = 1$ then T is said to be nonexpansive;

(ii) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - x^*\| \le \|x - x^*\|, \forall\, x \in H, x^* \in Fix(T);$$

(iii) directed if

$$\|Tx - x^*\|^2 \le \|x - x^*\|^2 - \|x - Tx\|^2, \forall\, x \in H, x^* \in Fix(T);$$

(iv) κ-demicontractive if there exists $\kappa \in [0,1)$ such that

$$\|Tx - x^*\|^2 \le \|x - x^*\|^2 + \|x - Tx\|^2, \forall\, x \in H, x^* \in Fix(T),$$

or equivalently,

$$\langle x - Tx, x - x^*\rangle \ge \frac{1-\kappa}{2}\|x - Tx\|^2;$$

(v) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \beta\|x - y\|^2;$$

(vi) β-inverse strongly monotone if there exists $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \beta\|Tx - Ty\|^2.$$

Remark 2.1. It is obvious from the definitions above that if T is quasi-nonexpansive or directed, then T is demicontractive. It is known that if T is κ-demicontractive, then $Fix(T)$ is closed and convex.

Definition 2.2. A mapping $T : H \to H$ is said to be $I - T$ demiclosed at 0 if, for any sequence $\{x_n\}$ in H, $x_n \rightharpoonup x^*$ and $(I-T)x_n \to 0$ imply $Tx^* = x^*$.

The metric projection from H onto K, denoted by P_K, is the mapping that assigns each point $x \in H$ to its unique nearest point in K i.e.,

$$\|x - P_K x\| \leq \|x - y\|, \ \forall\, y \in K.$$

The metric projection is characterized by $P_K x \in K$ and

$$\langle x - P_K x, y - P_K x\rangle \leq 0, \ \forall y \in K.$$

Moreover, P_K is nonexpansive and $Fix(P_K) = K$. For more properties and details on P_K, we refer to, for example, [23].

Let $h : K \to K$ be a nonlinear operator. The variational inequality problem (VIP) is defined as follows:

$$\text{Find } x^* \in K \text{ such that } \langle h(x^*), x - x^*\rangle \geq 0\ , \forall\, x \in K.$$

Lemma 2.1. [24] *Let H be a real Hilbert space. Suppose that $h : H \to H$ is κ- Lipschitzian and β- strongly monotone over a closed convex subset $K \subset H$. Then, the following VIP*

$$\langle h(u^*), v - u^*\rangle \geq 0, \ \forall\, v \in K$$

has its unique solution $u^* \in K$.

Lemma 2.2. [12] *Assume that $T : H \to H$ is a κ-demicontractive mapping. Define $T_\lambda := (1-\lambda)I + \lambda T$ for any $\lambda \in (0, 1-\kappa)$. Then, for any $x \in H$ and $x^* \in Fix(T)$,*

$$\|T_\lambda x - x^*\|^2 \leq \|x - x^*\|^2 - \lambda(1-\kappa-\lambda)\|x - Tx\|^2.$$

Lemma 2.3. [12] *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $T : H_2 \to H_2$ be a κ-demicontractive operator. If $A^{-1}(Fix(T)) \neq \emptyset$, then*

$$\|x - \rho A^*(I-T)Ax - x^*\|^2 \leq \|x - x^*\|^2 - \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax\|^4}{\|A^*(I-T)Ax\|^2},$$

where $x \in H_1$, A^* is the adjoint of A, $Ax \neq T(Ax)$, $x^* \in A^{-1}(Fix(T))$ and

$$\rho := \frac{(1-\tau)\|(I-T)Ax\|^2}{2\|A^*(I-T)Ax\|^2}.$$

Lemma 2.4. [25] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation: $s_{n+1} \leq (1-t_n)s_n + t_n\rho_n$, $n \geq n_0$, where $\{t_n\} \subset (0,1)$ and $\{\rho_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$, and $\limsup_{n\to\infty} \rho_n \leq 0$. Then $s_n \to 0$ as $n \to \infty$.*

Lemma 2.5. [26] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Let $\{\tau(n)\}_{n\geq n_0}$ be a sequence of integers defined by*

$$\tau(n) := \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau_n\}_{n\geq n_0}$ is a non-decreasing sequence such that $\lim_{n\to\infty} \tau_n = \infty$, and, for all $n \geq n_0$, the following two estimates hold: $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, and $\Gamma_n \leq \Gamma_{\tau(n)+1}$.

3. Main Results

In this section, we present our new relaxed viscosity scheme for solving the split common fixed point problem (1.1) with demicontractive mappings. For simplicity, we let

$$\Omega := \{x^* \in H_1 : x^* \in Fix(U) \text{ and } Ax^* \in Fix(T)\}.$$

For an arbitrary starting point $x_0 \in H_1$ and a contraction $f : H_1 \to H_1$ with constant $\nu \in [0, \frac{1}{\sqrt{2}})$, we define the following iterative sequence

$$x_{n+1} = U_n(\alpha_n f(x_n) + (1-\alpha_n)(x_n - \rho_n A^*(I-T)Ax_n)), n \geq 1, \tag{3.1}$$

where $U_n = (1-\lambda_n)I + \lambda_n U$ for some sequence $\{\lambda_n\}$.

For the convergence of the method, we assume that the following conditions hold.

(A1) the mappings $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are κ and μ demicontractive, respectively;
(A2) $\Omega \neq \emptyset$;
(A3) $Fix(U)$ and $Fix(T)$ are nonempty and $I-U$ and $I-T$ are demiclosed at 0;
(A4) the sequence $\{\lambda_n\}$ satisfy the following conditions:
 (i) $0 < a \leq \lambda_n \leq b < 1-\kappa$;
 (ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
 (iii) $\rho_n := \frac{(1-\tau)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, 0 \leq \tau < 1$.

Lemma 3.1. *Assume that conditions (A1)-(A4) hold. Then the sequence $\{x_n\}$ generated by (3.1) is bounded.*

Proof. Let $z \in \Omega$. We have

$$U_n z = (1-\lambda_n)z + \lambda_n Uz = z.$$

Set $w_n = x_n - \rho_n A^*(I-T)Ax_n$ and $z_n = \alpha_n f(x_n) + (1-\alpha_n)w_n$. It then follows from Lemma 2.2 and Lemma 2.3 that

$$\|U_n z_n - z\| \leq \|z_n - z\| \tag{3.2}$$

and

$$\|w_n - z\| \leq \|x_n - z\|. \tag{3.3}$$

From (3), (3.2) and (3.3), we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|U_n z_n - z\| \\
&\leq \|z_n - z\| \\
&\leq \alpha_n \|f(x_n) - z\| + (1-\alpha_n)\|w_n - z\| \\
&\leq \alpha_n(\|f(x_n) - f(z)\| + \|f(z) - z\|) + (1-\alpha_n)\|w_n - z\| \\
&\leq \alpha_n \nu \|x_n - z\| + \alpha_n \|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&= (1-\alpha_n(1-\nu))\|x_n - z\| + \frac{\alpha_n(1-\nu)\|f(z) - z\|}{1-\nu} \\
&\leq \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1-\nu}\right\} \\
&\vdots \\
&\leq \max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1-\nu}\right\}.
\end{aligned}$$

The last inequality shows that $\{\|x_{n+1}-z\|\}$ is bounded. Consequently, $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. $\square$

Theorem 3.1. *Assume that conditions (A1)-(A4) hold. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $z \in \Omega$, where z is the unique solution of the variational inequality problem: Find $z \in \Omega$ such that*

$$\langle (I-f)z, x-z\rangle \geq 0, \forall\, x \in \Omega. \tag{3.4}$$

Proof. Since f is a contraction mapping with constant ν, we see that, for any $x, y \in H_1$,

$$\begin{aligned} \|(I-f)x-(I-f)y\| &= \|(x-y)+(fy-fx)\| \\ &\leq \|x-y\|+\|fy-fx\| \\ &\leq (1+\nu)\|x-y\| \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \langle (I-f)x-(I-f)y, x-y\rangle &= \langle x-y, x-y\rangle - \langle fx-fy, x-y\rangle \\ &\geq \|x-y\|^2 - \nu\|x-y\|^2 \\ &= (1-\nu)\|x-y\|^2. \end{aligned} \tag{3.6}$$

Thus, we can infer from (3.5) and (3.6) that $(I-f)$ is $(1+\nu)$-Lipschitzian and $(1-\nu)$-strongly monotone, respectively. From Lemma 2.1, we have that there exists a unique $z \in \Omega$ such that (3.4) is satisfied.

We prove below that $\{x_n\}$ converges strongly to z. Setting $w_n = x_n - \rho_n A^*(I-T)Ax_n$ and $z_n = \alpha_n f(x_n) + (1-\alpha_n)w_n$, we obtain

$$\begin{aligned} \langle f(x_n)-z, w_n-z\rangle &= \langle f(x_n)-f(z), w_n-z\rangle + \langle f(z)-z, w_n-z\rangle \\ &\leq \|f(x_n)-f(z)\|\|w_n-z\| + \langle f(z)-z, w_n-z\rangle \\ &\leq \frac{1}{2}(\|f(x_n)-f(z)\|^2+\|w_n-z\|^2) + \langle f(z)-z, w_n-z\rangle \\ &\leq \frac{1}{2}\nu^2\|x_n-z\|^2 + \frac{1}{2}\|w_n-z\|^2 + \langle f(z)-z, w_n-z\rangle \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|f(x_n)-z\|^2 &\leq (\|f(x_n)-f(z)\|+\|f(z)-z\|)^2 \\ &\leq (\nu\|x_n-z\|+\|f(z)-z\|)^2 \\ &\leq 2\nu^2\|x_n-z\|^2 + 2\|f(z)-z\|^2. \end{aligned} \tag{3.8}$$

Using (2.1), (3.7), (3.8) and Lemma 2.3, we obtain

$$\begin{aligned}
&\|z_n - z\|^2 \\
&= \alpha_n^2 \|f(x_n) - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - z, w_n - z\rangle + (1-\alpha_n)^2\|w_n - z\|^2 \\
&\leq 2\alpha_n^2 \nu^2 \|x_n - z\|^2 + 2\alpha_n^2\|f(z) - z\|^2 + \alpha_n(1-\alpha_n)\nu^2\|x_n - z\|^2 \\
&(1-\alpha_n)\|w_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(z) - z, w_n - z\rangle \\
&\leq 2\alpha_n^2 \nu^2 \|x_n - z\|^2 + 2\alpha_n^2\|f(z) - z\|^2 + \alpha_n(1-\alpha_n)\nu^2\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(z) - z, w_n - z\rangle - \frac{(1-\alpha_n)(1-\tau)^2\|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^2} \\
&\leq (1-\alpha_n(1-(1+\alpha_n)\nu^2))\|x_n - z\|^2 + \alpha_n(2\alpha_n\|f(z) - z\|^2 + 2(1-\alpha_n)\langle f(z) - z, w_n - z\rangle) \\
&\quad - \frac{(1-\alpha_n)(1-\tau)^2\|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^2}.
\end{aligned} \tag{3.9}$$

From Lemma 2.2 and (3.9), we obtain

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|U_n z_n - z\| \\
&\leq \|z_n - z\|^2 - \lambda_n(1-\kappa-\lambda_n)\|z_n - Uz_n\|^2 \\
&\leq (1-\alpha_n(1-(1+\alpha_n)\nu^2))\|x_n - z\|^2 + \alpha_n(2\alpha_n\|f(z) - z\|^2 + 2(1-\alpha_n)\langle f(z) - z, w_n - z\rangle) \\
&\quad - \frac{(1-\alpha_n)(1-\tau)^2\|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^2} - \lambda_n(1-\kappa-\lambda_n)\|z_n - Uz_n\|^2.
\end{aligned}$$

Setting $\delta_n = \alpha_n(1-(1+\alpha_n)\nu^2)$ and $d_n = \frac{2\alpha_n\|f(z)-z\|^2 + 2(1-\alpha_n)\langle f(z)-z, w_n - z\rangle}{1-(1+\alpha_n)\nu^2}$ in (3.10), we conclude

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\delta_n)\|x_n - z\|^2 + \delta_n d_n - \frac{(1-\alpha_n)(1-\tau)^2\|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^2} \\
&\quad -\lambda_n(1-\kappa-\lambda_n)\|z_n - Uz_n\|^2 \qquad (3.10) \\
&\leq (1-\delta_n)\|x_n - z\|^2 + \delta_n d_n. \qquad (3.11)
\end{aligned}$$

We further divide the proof into two cases:
<u>Case 1</u>: Suppose that there exists some $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}$ is monotone nonincreasing for $n > n_0$. From (3.10), we have

$$\begin{aligned}
\frac{(1-\alpha_n)(1-\tau)^2\|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^2} &+ \lambda_n(1-\kappa-\lambda_n)\|z_n - Uz_n\|^2 \\
&\leq (1-\delta_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \delta_n d_n. \qquad (3.12)
\end{aligned}$$

Note that $\delta_n \to 0$ as $n \to \infty$. Taking the limit of (3.12) as $n \to \infty$, we obtain

$$\frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|} \to 0 \text{ as } n \to \infty \tag{3.13}$$

and

$$\|z_n - Uz_n\| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

From (3.13), we obtain

$$\|(I-T)Ax_n\| \leq \frac{\|A^*\|\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|} \to 0 \text{ as } n\to\infty. \tag{3.15}$$

Consequently, we derive that

$$\begin{aligned} \|z_n-x_n\| &\leq \alpha_n\|f(x_n)-x_n\|+(1-\alpha_n)\rho_n\|A^*(I-T)Ax_n\| \\ &\to 0 \text{ as } n\to\infty, \end{aligned} \tag{3.16}$$

and

$$\|w_n-x_n\| = \rho_n\|A^*(I-T)Ax_n\| \to 0 \text{ as } n\to\infty. \tag{3.17}$$

We next prove

$$\|x_{n+1}-x_n\| \to 0 \text{ as } n\to\infty.$$

Indeed, using (3.14) and (3.16), we obtain

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|(1-\lambda_n)z_n+\lambda_n Uz_n-x_n\| \\ &\leq \|z_n-x_n\|+\lambda_n\|z_n-Uz_n\| \\ &\to 0 \text{ as } n\to\infty. \end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k\to\infty$. From the linearity, it follows that $Ax_{n_k} \rightharpoonup A\bar{x}$ as $n\to\infty$. Therefore from (3.15) and the fact that $(I-T)$ is demiclosed at 0, one has $A\bar{x}\in Fix(T)$. From (3.16), we can conclude that $z_{n_k} \rightharpoonup \bar{x}$. Since $(I-U)$ is demiclosed at 0, then it follows from (3.14) that $\bar{x}\in Fix(U)$. Therefore $\bar{x}\in\Omega$. We conclude by showing that $x_n\to z$ as $n\to\infty$. In the first instance, since $\bar{x}\in\Omega$, we have from (3.4) that

$$\langle (I-f)z, \bar{x}-z\rangle \geq 0. \tag{3.18}$$

Moreover, from the choices of $\{\alpha_n\}$ and ν, one can verify that $\sum_{n=1}^{\infty}\delta_n = \sum_{n=1}^{\infty}\alpha_n(1-(1+\alpha_n)\nu^2) = \infty$ and $\delta_n \to 0$ as $n\to\infty$. Using (3.17) and (3.18), we derive that

$$\begin{aligned} \limsup_{n\to\infty}\langle f(z)-z, w_n-z\rangle &= \lim_{k\to\infty}\langle f(z)-z, w_{n_k}-z\rangle \\ &= \lim_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle \\ &= \langle f(z)-z, \bar{x}-z\rangle \\ &\leq 0. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} d_n \leq 0. \tag{3.19}$$

Therefore, applying Lemma 2.4 to (3.11) and using (3.19), we obtain that $\lim_{n\to\infty}\|x_n-z\| = 0$.
<u>Case 2</u>: Suppose that $\{\|x_n-z\|\}$ is not monotonically decreasing. Define the sequence $\tau:\mathbb{N}\to\mathbb{N}$ for all $n\geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k\in\mathbb{N} \mid k\leq n, \phi(p,x_k)\leq\phi(p,x_{k+1})\}.$$

Observe that $\{\tau(n)\}$ is non-decreasing with $\lim_{n\to\infty}\tau(n)=\infty$ and

$$0\leq \|x_{\tau(n)}-z\| \leq \|x_{\tau(n)+1}-z\| \text{ for all } n\geq n_0. \tag{3.20}$$

Following similar argument as in Case 1, we see that

$$\lim_{n\to\infty}\|z_{\tau(n)}-Uz_{\tau(n)}\|=0\ ,\ \lim_{n\to\infty}\|Ax_{\tau(n)}-TAx_{\tau(n)}\|=0,$$
$$\lim_{n\to\infty}\|z_{\tau(n)}-x_{\tau(n)}\|=0\ ,\ \lim_{n\to\infty}\|w_{\tau(n)}-x_{\tau(n)}\|=0,$$
$$\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle f(z)-z,w_{\tau(n)}-z\rangle\leq 0. \tag{3.21}$$

It also follows from (3.11) and (3.20) that

$$\begin{aligned}
0 &\leq (1-\delta_{\tau(n)})\|x_{\tau(n)}-z\|-\|x_{\tau(n)+1}-z\|+\delta_{\tau(n)}\langle f(z)-z,w_{\tau(n)}-z\rangle\\
0 &\leq (1-\delta_{\tau(n)})\|x_{\tau(n)+1}-z\|-\|x_{\tau(n)+1}-z\|+\delta_{\tau(n)}\langle f(z)-z,w_{\tau(n)}-z\rangle\\
&= -\delta_{\tau(n)}\|x_{\tau(n)+1}-z\|+\delta_{\tau(n)}\langle f(z)-z,w_{\tau(n)}-z\rangle.
\end{aligned}$$

Therefore,

$$\|x_{\tau(n)+1}-z\|\leq\langle f(z)-z,w_{\tau(n)}-z\rangle.$$

By taking the limit and recalling (3.21), we obtain that $\lim_{n\to\infty}\|x_{\tau(n)+1}-z\|=0$. Consequently, $\lim_{n\to\infty}\|x_{\tau(n)}-z\|=0$. From Lemma 2.5, we then obtain

$$0\leq\|x_n-z\|\leq\max\{\|x_n-z\|,\|x_{\tau(n)}-z\|\}\leq\|x_{\tau(n)+1}-z\|.$$

Thus $\lim_{n\to\infty}\|x_n-z\|=0$ and therefore, $\{x_n\}$ converges strongly to z.

In both cases, we proved that the sequence $\{x_n\}$ converges strongly to $z\in\Omega$, where z is the unique solution of the variational inequality problem (3.4). This completes the proof. □

Observe that our scheme is quite general and if one chooses f as a constant function, that is, $f(x)=u$, then we obtain the following scheme, which is a special case of scheme

$$x_{n+1}=U_n(\alpha_n u+(1-\alpha_n)(x_n-\rho_n A^*(I-T)Ax_n)),n\geq 1.$$

4. Applications

In this section, we present two mathematical applications of the split common fixed point problem: a split feasibility problem and a split variational inequality problem.

4.1. **The split feasibility problem.** Recall the SFP (1.2): let C and Q be nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively. Let $A:H_1\to H_2$ be a bounded linear operator. The SFP is formulated as:

$$\text{Find } x^*\in H_1 \text{ such that } x^*\in C \text{ and } Ax^*\in Q.$$

Note that the SFP (1.2) can be formulated as

$$\text{Find } x^*\in H_1 \text{ such that } x^*\in Fix(P_C) \text{ and } Ax^*\in Fix(P_Q),$$

where P_C and P_Q denote the metric projection onto C and Q, respectively. Denote the solution set of the SFP by $SFP(C,Q)$. Since P_C and P_Q are nonexpansive with nonempty fixed point sets, they are 0-demicontractive. We denote the solution set of the SFP (1.2) by $SFP(C,Q)$ and assume $SFP(C,Q)\neq\emptyset$. We have the following result.

Theorem 4.1. *Consider the SCFP (1.1) with $U = P_C$ and $T = P_Q$ and assume that conditions (A1) and (A4) hold. Then the sequence $\{x_n\}$ generated by (3) converges strongly to $z \in SFP(C,Q)$, where z is the unique solution of the variational inequality problem: Find $z \in SFP(C,Q)$ such that $\langle (I-f)z, x-z\rangle \geq 0$, $x \in SFP(C,Q)$.*

4.2. **The split variational inequality problem.** Let C and Q be nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively. Consider the variational inequality problem (VIP):

$$\text{Find } x^* \in C \text{ such that } \langle h_1(x^*), x - x^*\rangle \geq 0,$$

where $h_1 : H_1 \to H_1$ is α-inverse strongly monotone mapping. We denote the solution set of the VIP by $Sol(C,h_1)$. For any $\lambda \in (0,2\alpha)$, it is known that $P_C(I-\lambda h_1)$ is nonexpansive and $x^* \in Sol(C,h_1)$ if and only if $x^* \in Fix(P_C(I-\lambda h_1))$; see [27]. Several iterative methods have been introduced for solving the VIP in Hilbert and Banach spaces, (see, for example, [28, 29, 30, 31, 32, 33] and the references therein). Let $A : H_1 \to H_2$ be a bounded linear operator and let $h_2 : H_2 \to H_2$ be a β-inverse strongly monotone mapping. We consider the following split variational inequality problem (SVIP):

$$\begin{cases} \text{find } x^* \in C \text{ such that } \langle h_1(x^*), x - x^*\rangle \geq 0, \forall x \in C \\ y^* = Ax^* \in Q \text{ such that } \langle h_2(y^*), y - y^*\rangle \geq 0, \forall y \in Q. \end{cases} \tag{4.1}$$

It follows that, for $\lambda \in (0,2\beta)$, $P_Q(1-\lambda h_2)$ is nonexpansive. We denote the solution set of the SVIP (4.1) by $SVIP(h_1,h_2)$ and assume it is nonempty. By applying Theorem 3.1, we obtain the following result for solving SVIP (4.1) immediately.

Theorem 4.2. *Consider the SCFP (1.1) with $U = P_{C_n}(I-\lambda h_1)$ and $T = P_Q(1-\lambda h_2)$ and assume that h_1 and h_2 are α and β inverse strongly monotone mapping, respectively. Moreover assume that conditions (A1) and (A4) hold. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $z \in SVIP(h_1,h_2)$, where z is the unique solution of the following variational inequality problem: $\langle (I-f)z, x-z\rangle \geq 0$, $x \in SVIP(h,f)$.*

5. Numerical Examples

In this section, we give some numerical examples to illustrate the convergence of our algorithm and also to compare it with other existing algorithms with the similar features.

Example 5.1. Let $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R} \times \mathbb{R}$ equipped with the Euclidean norm. Define the mappings $U : H_1 \to H_1$ by

$$Ux = \begin{cases} 0, & x \in (-\infty, 0), \\ \frac{x}{3}, & x \in [0, \frac{1}{2}], \\ \frac{2}{3}x\sin(\frac{\pi}{3}x), & x \in (\frac{1}{2}, 1], \\ \frac{\sqrt{3}}{3}, & x \in (1, \infty), \end{cases}$$

and $Ty = (-2y_1, -3y_2), \forall\, y = (y_1,y_2) \in H_2$. Then U is 0-demicontractive and T is $\frac{1}{2}$-demicontractive. In addition $I-U$ and $I-T$ are demiclosed at 0. Let $A : H_1 \to H_2$ be defined by $Ax = (\frac{x}{3}, \frac{x}{5})$, $\forall\, x \in H_1$ with $A^* : H_2 \to H_1$ defined by $A^*(y_1,y_2) = \frac{y_1}{3} + \frac{y_2}{5}$, $\forall\, (y_1,y_2) \in H_2$. In this case, $\Omega = \{0\} \neq \emptyset$. Let $f : H_1 \to H_1$ be defined by $f(x) = \frac{1}{4}x$, $\forall x \in H_1$. Then f is a contraction mapping with constant $\frac{1}{4}$. We choose $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{n^2+3}, \lambda_n = \lambda = 0.6, \tau = 0.4$ and $\tau_n = \frac{1}{n^{0.9}}$.

We make different choices of the initial value x_1 as follows:
Case Ia: $x_1 = 20$;
Case Ib: $x_1 = -17$;
Case Ic: $x_1 = 53$;
Case Id: $x_1 = 0.5$.

Using MATLAB 2015(b), we compare the performance of our Algorithm (3.1) (Ade) with Algorithm (1.9) (Huimin), Algorithm (1.5) (Cui), Algorithm (1.10) (Jirak) and Algorithm (1.11) (Wang). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-7}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 1 and Table 1, respectively.

TABLE 1. Numerical results.

		Ade	Huimin	Cui	Jirak	Wang
Case Ia	CPU time (sec)	0.0066	0.0056	0.0082	0.0072	0.0089
	No of Iter.	11	15	20	13	15
Case Ib	CPU time (sec)	0.0064	0.0065	0.0070	0.0049	0.0064
	No. of Iter.	6	9	10	8	9
Case Ic	CPU time (sec)	0.0062	0.0072	0.0081	0.0066	0.0067
	No of Iter.	12	16	21	14	16
Case Id	CPU time (sec)	0.0064	0.0065	0.0086	0.0078	0.0087
	No of Iter.	10	12	17	11	12

Example 5.2. Let $H_1 = H_2 = (\ell_2(\mathbb{R}), \|\cdot\|_2)$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}, \forall\, x \in \ell_2(\mathbb{R})$. We define $U : H_1 \to H_1$ and $T : H_2 \to H_2$ by $Ux = -3x$ and $Tx = -5x$, $x \in \ell_2(\mathbb{R})$, respectively. Then U is $\frac{1}{2}$-demicontractive and T is $\frac{2}{3}$-demicontractive. In addition, $I - U$ and $I - T$ are demiclosed at 0 (see [34]). For any $x = (x_1, x_2, x_3, \cdots) \in \ell_2(\mathbb{R})$, define $A : H_1 \to H_2$ by $Ax = (0, x_1, x_2, \cdots)$. One can easily verify that A is a bounded linear operator with $\|A\| = 1$ and $A^*x = (x_2, x_3, x_4, \cdots)$. We choose $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{n^2+3}, \lambda_n = \lambda = 0.3, \tau = 0.4$ and $\tau_n = \frac{1}{n^{0.9}}$.

We make different choices of the initial value x_1 as follows:
Case IIa: $x_1 = (8, 4, 2, \cdots)$;
Case IIb: $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$;
Case IIc: $x_1 = (-100, 10, -1, \cdots)$;
Case IId: $x_1 = (12, -4, \frac{4}{3}, \cdots)$.

Using MATLAB 2015(b), we compare the performance of our Algorithm (3) (Ade) with Algorithm (1.9) (Huimin), Algorithm (1.5) (Cui), Algorithm (1.10) (Jirak) and Algorithm (1.11) (Wang). The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-7}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 2 and Table 2, respectively.

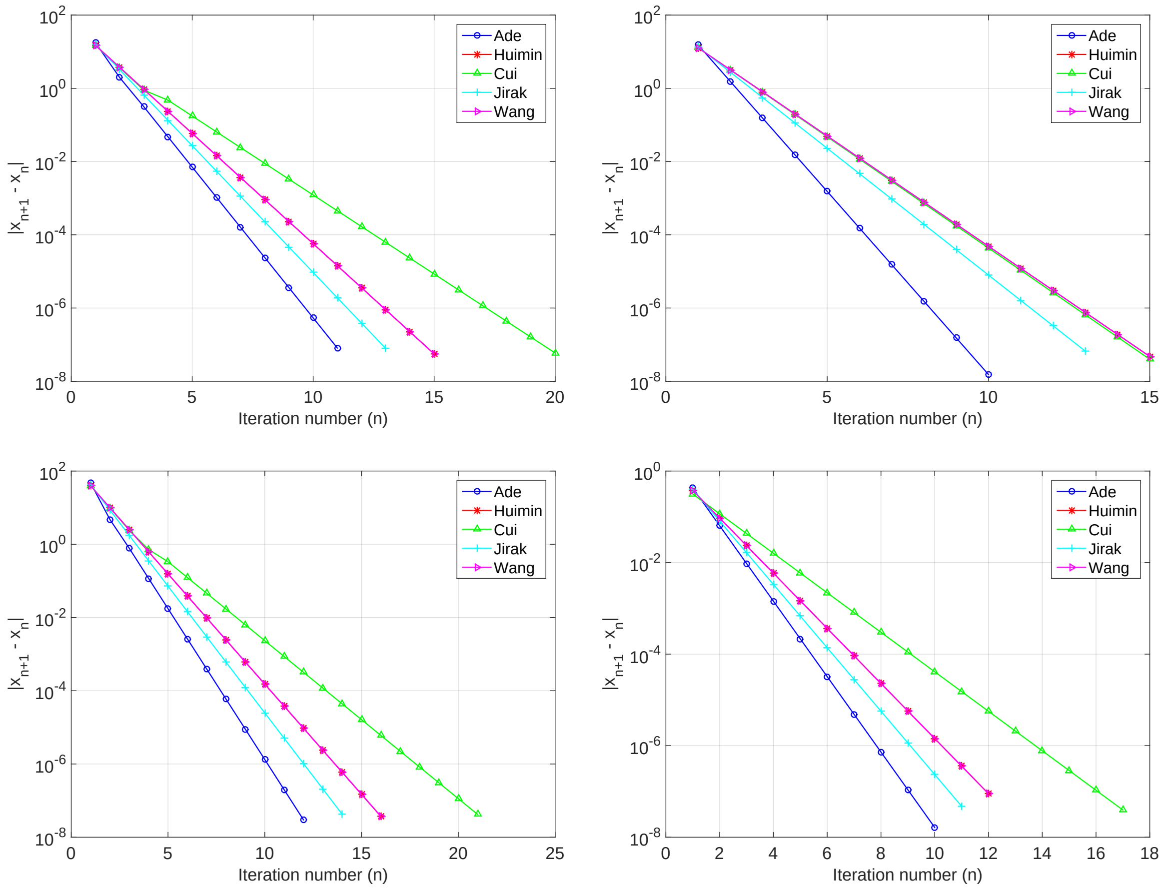

FIGURE 1. Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

TABLE 2. Numerical results.

		Ade	Huimin	Cui	Jirak	Wang
Case IIa	CPU time (sec)	0.0036	0.0059	0.0059	0.0051	0.0077
	No of Iter.	6	10	11	9	10
Case IIb	CPU time (sec)	0.0046	0.0054	0.0074	0.0061	0.0063
	No. of Iter.	6	9	10	8	9
Case IIc	CPU time (sec)	0.0036	0.0071	0.00787	0.0051	0.0086
	No of Iter.	7	11	13	10	11
Case IId	CPU time (sec)	0.0040	0.0072	0.0067	0.0051	0.0079
	No of Iter.	7	10	12	9	10

Example 5.3. (See [21, 35]) In this example, we consider the following linear model used in signal processing:

$$y = Ax + w, \tag{5.1}$$

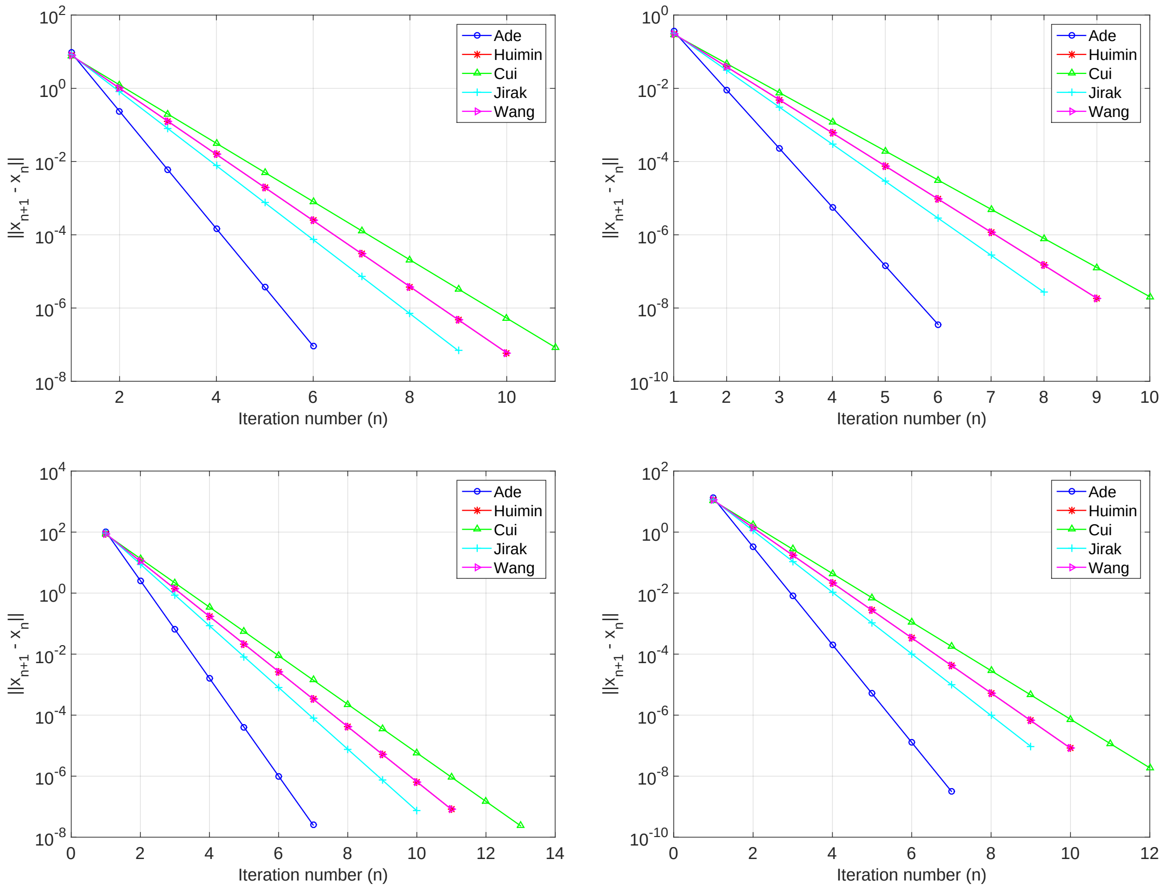

FIGURE 2. Top left: Case IIa; Top right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId.

where $x \in \mathbb{R}^N$ is the signal to recover, $w \in \mathbb{R}^k$ is the noisy term, $y \in \mathbb{R}^k$ is the noisy observation and $A : \mathbb{R}^N \to \mathbb{R}^k$ is a bounded linear operator. One way of solving Problem (5.1) is to express it as the *least absolute selection and shrinkage operator* (LASSO) problem:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|y - Ax\|^2 + \lambda \|x\|_1, \tag{5.2}$$

where $\lambda > 0$ is a constant that is related to the noise term w and $\|\cdot\|_1$ is the ℓ_1 norm. An equivalent formulation of Problem (5.2) is the following convex constraint minimization problem:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|y - Ax\|^2 \text{ such that } \|x\|_1 \leq t, \tag{5.3}$$

for some nonnegative integer t. Thus letting $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$, Problem (5.3) becomes the SFP (1.2). Therefore, we can reformulate Problem (5.3) as the following SCFP:

$$\text{Find } x \in Fix(P_C) \text{ such that } Ax \in Fix(P_Q). \tag{5.4}$$

By making $U = P_C$ and $T = P_Q$ and using Theorem (3.1), we obtain that the sequence $\{x_n\}$ generated by (3.1) converges strongly to a solution of problem (5.3).

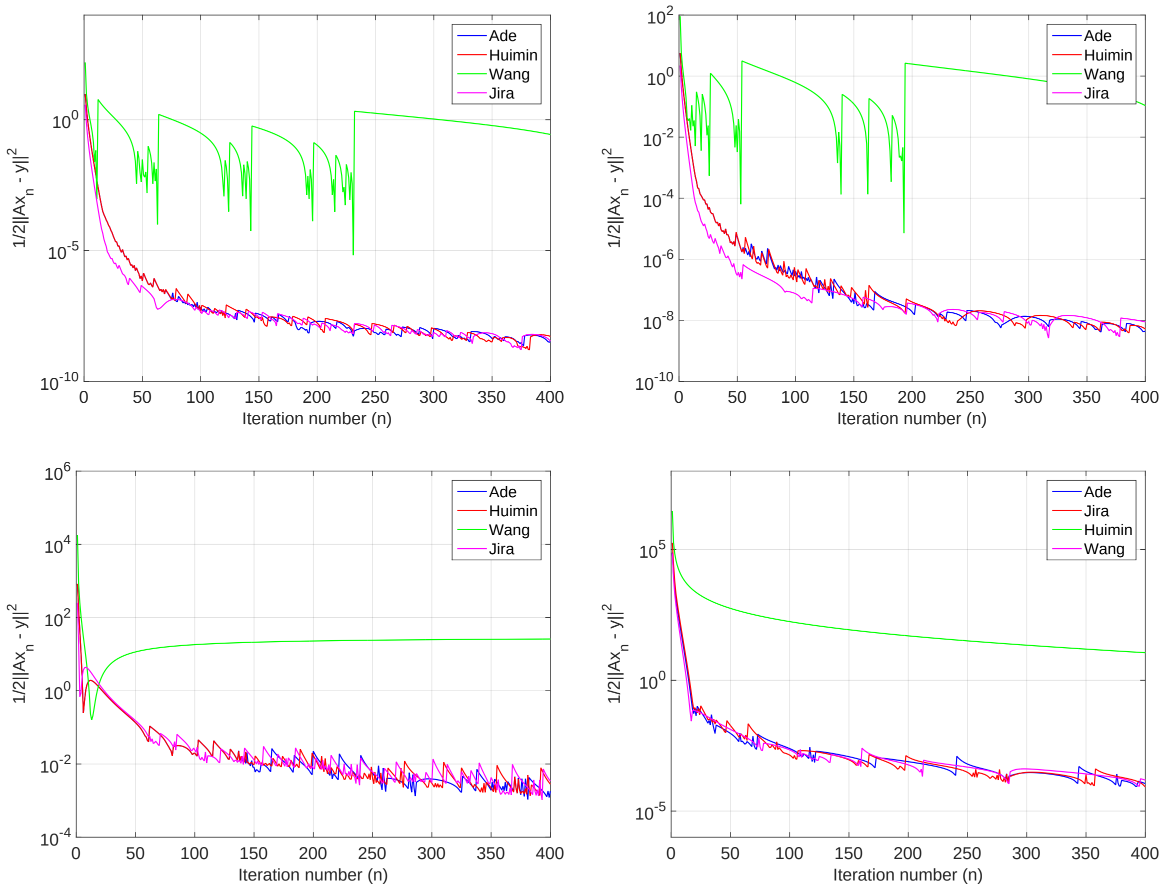

FIGURE 3. Top left: N=20, t=5, k=10; Top right: N=t=20, k=10; Bottom left: N=100, t=50, k=60; Bottom right: N=1000, t=15, k=30.

We next illustrate Problem (5.3). Let A be a $(k \times N)$ randomly generated matrix with entries in $[0,1]$. Let $y = Ax^*$, where $x^* \in C$. Let $f(x) = \frac{x}{4}, \forall\, x \in \mathbb{R}^N$, $\lambda_n = 0.4$, $\tau = 0.6$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n^2+1}$ and $\tau_n = \frac{1}{n^{0.9}}$. For different choices of N, k and t, we compare our Algorithm (3.1) (Ade) with Algorithms (1.9) (Huimin), (1.10) (Jira) and (1.11) (Wang) using Matlab 2015(b). We use 400 maximum number of iteration as the stopping criterion for all algorithms and plot the graph of $\frac{1}{2}\|Ax_n - y\|$ against the number of iterations in each case. The numerical result is given in Figure 3

Remark 5.1. We can infer from the numerical results that our algorithm has the least number of iteration and computation time.

6. THE CONCLUSION

We studied the split common fixed point problem with demicontractive mappings and proposed a new algorithm, which can be seen as alternative regularization version of the Algorithm (1.9) presented in [20]. A strong convergence theorem is established under mild and standard assumptions in real Hilbert spaces. We illustrated the performance of our algorithm by comparing and testing with some related results in the literature. Our algorithm is simple/flexible, and generalizes several existing results.

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