J. Nonlinear Var. Anal. 5 (2021), No. 5, pp. 795-802
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.5.11

A GENERALIZATION OF EULER'S PENTAGONAL NUMBER THEOREM WITH AN APPLICATION

HARI M. SRIVASTAVA[1,2,3,4,*], CHAO-PING CHEN[5]

[1] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*
[2] *Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[3] *Department of Mathematics and Informatics, Azerbaijan University, Baku AZ1007, Azerbaijan*
[4] *Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy*
[5] *School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, China*

Abstract. In this paper, we give a generalization of Euler's pentagonal number theorem. As an application, we derive a recursion formula for successively determining the coefficients a_k of $1/n^k$. We conclude this paper by presenting brief descriptions for some related results and other motivating developments.
Keywords. Asymptotic expansions; Euler-Mascheroni constant; Harmonic numbers; Jacobi's triple product identity; Recursion formulas.

1. Introduction

Euler's pentagonal number theorem states that (see, for example, [1, p. 11, Eq. (1.3.1)])

$$\begin{aligned}\prod_{k=1}^{\infty}(1-x^k) &= \sum_{m=-\infty}^{\infty}(-1)^m\, x^{m(3m-1)/2} \\ &= 1+\sum_{m=1}^{\infty}(-1)^m\left(x^{m(3m-1)/2}+x^{m(3m+1)/2}\right). \end{aligned} \tag{1.1}$$

Franklin [2] gave a wonderful combinatorial proof of this theorem.

The following special case of Jacobi's triple product identity (see [3]; see also [1, p. 21, Eq. (2.2.10); p. 23, Eq. (2.2.12)]

$$\begin{aligned}\prod_{k=1}^{\infty}(1-x^k)^3 &= \sum_{j=0}^{\infty}(-1)^j\, x^{x^j} = \prod_{j=1}^{\infty}\left(\frac{1-x^j}{1+x^j}\right) \\ &= \sum_{m=0}^{\infty}(-1)^m\,(2m+1)\,x^{m(m+1)/2}\end{aligned}$$

*Corresponding author.
E-mail addresses: harimsri@math.uvic.ca (H.M. Srivastava), chenchaoping@sohu.com (C.-P. Chen).
Received April 8, 2021; Accepted June 19, 2021

is widely known. Further explicit formulas for the powers of the Euler product given by

$$\prod_{k=1}^{\infty}(1-x^k)^s=\sum_{k=0}^{\infty} f_k(s)\, x^k \qquad (s\in\mathbb{R})$$

have been derived for several special values for the power s such as those which we recall below (see [4] and the references cited therein):

$$s=1,3,8,10,14,15,21,24,26,28,35,36,\cdots.$$

Our first aim in this paper is to give an explicit formula for determining the coefficients b_j such that

$$\prod_{k=1}^{\infty}(1+a_k\, x^k)=\sum_{j=0}^{\infty} b_j\, x^j, \tag{1.2}$$

where a_k ($k\in\mathbb{N}:=\{1,2,3,\cdots\}$) are given real numbers. We then consider the expansion for the powers of the following product:

$$\prod_{k=1}^{\infty}(1+a_k\, x^k).$$

The Euler-Mascheroni constant γ given by

$$\gamma=0.577215664\cdots$$

is defined as the limit of the following sequence:

$$D_n=H_n-\ln n \qquad (n\in\mathbb{N}), \tag{1.3}$$

where H_n given by

$$H_n=\sum_{k=1}^{n}\frac{1}{k} \qquad (n\in\mathbb{N})$$

denotes the nth harmonic number.

It is well known that (see [5, p. 258, Eq. (6.3.2)])

$$\psi(n+1)=-\gamma+H_n \qquad (n\in\mathbb{N}), \tag{1.4}$$

where $\psi(x)$ given by

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$$

is the psi (or digamma) function.

For any positive integer m, in 2018, You and Chen presented the following family of sequences (see, for details, [6, Theorem 1]):

$$\gamma_m(n)=H_n-\ln n-\sum_{k=1}^{m}\ln\left(1+\frac{a_k}{n^k}\right) \qquad (n\in\mathbb{N}), \tag{1.5}$$

which would converge to the Euler-Mascheroni constant γ, where

$$a_1=\frac{1}{2},\ a_2=\frac{1}{24},\ a_3=-\frac{1}{24},\ a_4=\frac{143}{5760},\ a_5=-\frac{1}{160},$$
$$a_6=-\frac{151}{290304},\ a_7=-\frac{1}{896},\ \cdots. \tag{1.6}$$

However, You and Chen [6] did not give the general formula for the coefficients a_k of $1/n^k$ in (1.5).

By using (1.2), we derive here a recursion formula for successively determining the coefficients a_k of $1/n^k$ in (1.5), which is the second aim of this paper. Some computations in this paper were performed using the Maple software.

2. Generalization of Euler's Pentagonal Number Theorem

Using the Maple software, we find that

$$\begin{aligned}\prod_{k=1}^{\infty}(1+a_kx^k) = 1+a_1x+a_2x^2+(a_3+a_1a_2)x^3+(a_4+a_1a_3)x^4 \\ +(a_5+a_1a_4+a_2a_3)x^5+(a_6+a_1a_5+a_2a_4+a_1a_2a_3)x^6 \\ +(a_7+a_1a_6+a_2a_5+a_3a_4+a_1a_2a_4)x^7+\cdots. \end{aligned} \tag{2.1}$$

Even though as many coefficients as we please in the right-hand side of (2.1) can be obtained by using the Maple software, here we aim at giving a formula for determining these coefficients.

For our later use, we introduce the following set of partitions of an integer $n \in \mathbb{N}$:

$$\begin{aligned}\mathscr{A}_n := \Big\{ (k_1, k_2, \cdots, k_n) \in \mathbb{N}_0^n : k_1+k_2+\cdots+k_n = n \\ \text{and} \quad k_j \text{ satisfies } k_1<k_2<\cdots<k_n \text{ when } k_j \neq 0\ (j=1,2,\cdots,n)\Big\}. \end{aligned} \tag{2.2}$$

Upon setting

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0,1,2,\cdots\},$$

the coefficients b_j ($j \in \mathbb{N}_0$) in (1.2) can be calculated by the following explicit formula:

$$b_0 = 1 \qquad \text{and} \qquad b_j = \sum_{(k_1,k_2,\cdots,k_j)\in\mathscr{A}_j} a_{k_1}a_{k_2}\cdots a_{k_j}, \tag{2.3}$$

where the $\mathscr{A}_j$ is given by (2.2). We stipulate that $a_0 = 1$.

Here we give explicit numerical values of some first terms of b_j by using the partition set (2.2) and the formula (2.3). Obviously, we have

$$b_0 = 1 \qquad \text{and} \qquad b_1 = \sum_{(k_1 \in mathcalA_1)} a_{k_1} = a_1.$$

For $k_1+k_2 = 2$ (k_j satisfies $k_1<k_2$ when $k_j \neq 0$), the partition set $\mathscr{A}_2$ in (2.2) is seen to have 1 element:

$$\mathscr{A}_2 = \{(0,2)\}.$$

From (2.3), we have

$$b_2 = \sum_{(k_1,k_2)\in\mathscr{A}_2} a_{k_1}\, a_{k_2} = a_0\, a_2 = a_2.$$

For $k_1+k_2+k_3 = 3$ (k_j satisfies $k_1<k_2<k_3$ when $k_j \neq 0$), as above, the partition set $\mathscr{A}_3$ in (2.2) contains 2 elements:

$$\mathscr{A}_3 = \{(0,0,3),\, (0,1,2)\}.$$

We then find from (2.3) that

$$b_3 = \sum_{(k_1,k_2,k_3)\in A_3} a_{k_1}a_{k_2}a_{k_j} = a_0a_0a_3+a_0a_1a_2 = a_3+a_1a_2.$$

Likewise, the partition sets $\mathscr{A}_4$, $\mathscr{A}_5$, $\mathscr{A}_6$ and $\mathscr{A}_7$ have 2, 3, 4 and 5 elements, respectively, and so we get

$$\begin{aligned}
&\mathscr{A}_4 = \{(0,0,0,4), (0,0,1,3)\}, \mathscr{A}_5 = \{(0,0,0,0,5), (0,0,0,1,4), (0,0,0,2,3)\},\\
&\mathscr{A}_6 = \{(0,0,0,0,0,6), (0,0,0,0,1,5), (0,0,0,0,2,4), (0,0,0,1,2,3)\},\\
&\mathscr{A}_7 = \{(0,0,0,0,0,0,7), (0,0,0,0,0,1,6), (0,0,0,0,0,2,5), (0,0,0,0,0,3,4),\\
&\qquad (0,0,0,0,1,2,4)\},
\end{aligned}$$

which yields

$$b_4 = a_4 + a_1 a_3, \quad b_5 = a_5 + a_1 a_4 + a_2 a_3, \quad b_6 = a_6 + a_1 a_5 + a_2 a_4 + a_1 a_2 a_3,$$
$$\text{and} \qquad b_7 = a_7 + a_1 a_6 + a_2 a_5 + a_3 a_4 + a_1 a_2 a_4.$$

We note that the values of b_j (for $j = 1,2,3,4,5,6,7$) above are equal to the coefficients appearing in (2.1).

By Lemma 2.1 below and Eq. (1.2), we now present a generalization of Euler's pentagonal number theorem given by Theorem 2.1.

Lemma 2.1. (see [7]) *Let g be a function with a formal power series given by*

$$g(x) = \sum_{n=0}^{\infty} b_n\, x^n \qquad (b_0 \neq 0).$$

Then, for all $s \in \mathbb{R}$, it is asserted that

$$[g(x)]^s = \sum_{n=0}^{\infty} P_n(s)\, x^n,$$

where

$$P_0(s) = b_0^s \qquad \textit{and} \qquad P_n(s) = \frac{1}{nb_0} \sum_{k=1}^{n} [k(1+s) - n] b_k\, P_{n-k}(s). \tag{2.4}$$

Theorem 2.1. *Let $s \neq 0$. Then*

$$\prod_{k=1}^{\infty} (1 + a_k\, x^k)^s = \left(\sum_{j=0}^{\infty} b_j\, x^j \right)^s = \sum_{n=0}^{\infty} P_n(s)\, x^n, \tag{2.5}$$

where

$$P_0(s) = 1 \qquad \textit{and} \qquad P_n(s) = \frac{1}{n} \sum_{k=1}^{n} [k(1+s) - n] b_k\, P_{n-k}(s) \tag{2.6}$$

and the coefficients b_k can be calculated in (2.3), that is,

$$\begin{aligned}
\prod_{k=1}^{\infty} (1 + a_k x^k)^s &= 1 + s a_1\, x + \frac{s(2a_2 + s a_1^2 - a_1^2)}{2}\, x^2 \\
&\quad + \frac{s(6a_3 + s^2 a_1^3 - 3 s a_1^3 + 2a_1^3 + 6 s a_1 a_2)}{6}\, x^3 + \cdots.
\end{aligned}$$

Remark 2.1. The Ramanujan function $\tau(n)$ is defined by the following expansion:

$$x \prod_{k=1}^{\infty} (1 - x^k)^{24} = \sum_{n=1}^{\infty} \tau(n)\, x^n. \tag{2.7}$$

We write (2.5) as follows:

$$x\prod_{k=1}^{\infty}(1+a_k\,x^k)^s=\sum_{n=0}^{\infty}P_n(s)\,x^{n+1}=\sum_{n=1}^{\infty}\tau(n,s)\,x^n, \tag{2.8}$$

where

$$\tau(n,s)=P_{n-1}(s). \tag{2.9}$$

The formula (2.8) is a generalization of (2.7).

3. An Application of the Product Formula (1.2)

As an application of the product formula (1.2), we derive a recursion formula for successively determining the coefficients a_k of $1/n^k$ in (1.5).

Theorem 3.1. *The coefficients a_k of $1/n^k$ in (1.5) can be derived from the following recursion formula*:

$$\sum_{(k_1,k_2,\cdots,k_j)\in\mathscr{A}_j}a_{k_1}\,a_{k_2}\cdots a_{k_j}=b_j \qquad (j\in\mathbb{N}), \tag{3.1}$$

where

$$b_j=\sum_{k_1+2k_2+\cdots+jk_j=j}\frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!\,k_2!\cdots k_j!}\left(\frac{B_1}{1}\right)^{k_1}\left(\frac{B_2}{2}\right)^{k_2}\cdots\left(\frac{B_j}{j}\right)^{k_j} \tag{3.2}$$

and B_j $(j\in\mathbb{N}_0)$ are the Bernoulli numbers, and the sum is taken over all nonnegative integers k_j satisfying the following equation:

$$k_1+2k_2+\cdots+jk_j=j.$$

Proof. In view of (1.4) and (1.5), we can set

$$\frac{e^{\psi(x+1)}}{x}\sim\prod_{k=1}^{\infty}\left(1+\frac{a_k}{x^k}\right) \qquad (x\to\infty), \tag{3.3}$$

where a_k $(k\in\mathbb{N})$ are real numbers to be determined. It follows from the known result [8, Theorem 2.1] that, as $x\to\infty$,

$$\begin{aligned}\frac{e^{\psi(x+1)}}{x}&\sim 1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}\\&=1+\frac{1}{2x}+\frac{1}{24x^2}-\frac{1}{48x^3}+\frac{23}{5760x^4}+\frac{17}{3840x^5}-\frac{10099}{2903040x^6}-\frac{2501}{1161216x^7}\\&\quad+\frac{795697}{199065600x^8}+\frac{870041}{398131200x^9}-\frac{2727899759}{367873228800x^{10}}-\frac{318246113}{81749606400x^{11}}+\cdots,\end{aligned} \tag{3.4}$$

with the coefficients b_j $(j\in\mathbb{N})$ given by

$$b_j=\sum_{k_1+2k_2+\cdots+jk_j=j}\frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!\,k_2!\cdots k_j!}\left(\frac{B_1}{1}\right)^{k_1}\left(\frac{B_2}{2}\right)^{k_2}\cdots\left(\frac{B_j}{j}\right)^{k_j},$$

where $B_j \ (j \in \mathbb{N}_0)$ are the Bernoulli numbers and the sum is taken over all nonnegative integers k_j satisfying the following equation:

$$k_1 + 2k_2 + \cdots + jk_j = j.$$

Finally, by applying (1.2), we find that

$$\frac{e^{\psi(x+1)}}{x} \sim 1 + \sum_{j=1}^{\infty} \frac{b_j}{x^j} = \prod_{k=1}^{\infty} \left(1 + \frac{a_k}{x^k}\right), \tag{3.5}$$

where

$$b_j = \sum_{(k_1,k_2,\cdots,k_j)\in\mathscr{A}_j} a_{k_1}\, a_{k_2} \cdots a_{k_j} \qquad (j \in \mathbb{N}),$$

which completes the proof of Theorem 3.1. □

Remark 3.1. The asymptotic expansion of the function given by

$$x \mapsto \frac{e^{p\psi(x+t)}}{x^p}$$

was presented in the earlier works [9] and [10].

Lastly, we give explicit numerical values of the first few coefficients a_k by using the formula (3.1). This demonstrates the ease with which the coefficients a_k in (1.5) can be determined. Indeed, from (3.4), we observe that

$$b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{24}, \quad b_3 = -\frac{1}{48}, \quad b_4 = \frac{23}{5760}, \quad b_5 = \frac{17}{3840},$$
$$b_6 = -\frac{10099}{2903040} \qquad \text{and} \qquad b_7 = -\frac{2501}{1161216}.$$

Moreover, we fing from (3.1) that

$$\begin{aligned}
&a_1 = b_1 = \frac{1}{2},\\
&a_2 = b_2 = \frac{1}{24},\\
&b_3 = a_3 + a_1a_2 \implies a_3 = b_3 - a_1a_2 = -\frac{1}{24},\\
&b_4 = a_4 + a_1a_3 \implies a_4 = b_4 - a_1a_3 = \frac{143}{5760},\\
&b_5 = a_5 + a_1a_4 + a_2a_3 \implies a_5 = b_5 - a_1a_4 - a_2a_3 = -\frac{1}{160},\\
&b_6 = a_6 + a_1a_5 + a_2a_4 + a_1a_2a_3\\
&\qquad \implies a_6 = b_6 - a_1a_5 - a_2a_4 - a_1a_2a_3 = -\frac{151}{290304},\\
&b_7 = a_7 + a_1a_6 + a_2a_5 + a_3a_4 + a_1a_2a_4\\
&\qquad \implies a_7 = b_7 - a_1a_6 - a_2a_5 - a_3a_4 - a_1a_2a_4 = -\frac{1}{896}.
\end{aligned}$$

We note that the values of the coefficients $a_k \ (k = 1,2,3,4,5,6,7)$ above are equal to the coefficients appearing in (1.5).

4. Concluding Remarks and Observations

Our present investigation is motivated by an earlier work by You and Chen [6] who studied, for any positive integer m, a family $\gamma_m(n)$ of sequences given by

$$\gamma_m(n) = H_n - \ln n - \sum_{k=1}^{m} \ln\left(1 + \frac{a_k}{n^k}\right) \qquad (n = 1, 2, 3, \cdots),$$

which would converge to the Euler-Mascheroni constant γ, where

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{24}, \quad a_3 = -\frac{1}{24}, \quad a_4 = \frac{143}{5760}, \quad a_5 = -\frac{1}{160},$$
$$a_6 = -\frac{151}{290304}, \quad a_7 = -\frac{1}{896}, \cdots.$$

The fact that You and Chen [6] did not give the general formula for computing the coefficients a_k of $1/n^k$ led us to give a generalization of Euler's pentagonal number theorem. As an application of our generalization of Euler's pentagonal number theorem, we successfully presented recursion formula for successively determining the coefficients a_k of $1/n^k$. Moreover, with a view to encouraging further researches on the subject of our study in this paper, we included a couple of citations of related recent works (see, for example, [11] and [12]).

The list of additional references, which we included in this paper, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we dealt with here. In particular, in connection especially with the zeta and theta functions as well as Jacobi's triple-product identities, we refer to [13], [14], [15] and [16] (see also the recently-published survey-cum-expository review articles [17] and [18]), each of which investigated interesting problems related to the subject-matter of our presentation in this paper.

Acknowledgments

This paper was supported by the Key Science Research Project in the Universities of the Henan Province, China (Grant Number: 20B110007).

References

[1] G. E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Vol. 2, Addison-Wesley Publishing Company Incorporated, Reading, Massachusetts, 1976; Reprinted by Cambridge University Press, Cambridge, London and New York, 1998.

[2] F. Franklin, Sur le développement du produit infine $(1-x)(1-x^2)(1-x^3)\cdots$, C.R. Acad. Sci. Sér. I Math. 82 (1881), 448-450.

[3] C. G. J. Jacobi, Fundamenta Nova Theoriae Functionum Ellipticarum, Regiomonti, Sumtibus Fratrum Bornträger, Königsberg, Germany, 1829; Reprinted in Gesammelte Mathematische Werke 1 (1829), 497–538, American Mathematical Society, pp. 97–239, Providence, Rhode Island, 1969.

[4] G.-N. Han, An explicit expansion formula for the powers of the Euler product in terms of partition hook lengths, Preprint 2008); arXiv:0804.1849v2; http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.312.1640.

[5] M. Abramowitz, I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series, Vol. 55, Ninth printing, National Bureau of Standards, Washington, D.C., 1972.

[6] X. You, D.-R. Chen, A new sequence convergent to Euler-Mascheroni constant, J. Inequal. Appl. 2018 (2018), Article ID 75.

[7] C.-P. Chen, N. Elezović, L. Vukšić, Asymptotic formulae associated with the Wallis power function and digamma function, J. Classical Anal. 2 (2013), 151-166.
[8] L. Lin, Asymptotic formulas associated with psi function with applications, J. Math. Anal. Appl. 405 (2013), 52-56.
[9] C.-P. Chen, On the coefficients of asymptotic expansion for the harmonic number by Ramanujan, Ramanujan J. 40 (2016), 279-290.
[10] N. Elezović, Estimations of psi function and harmonic numbers, Appl. Math. Comput. 258 (2015), 192-205.
[11] C.-P. Chen, H. M. Srivastava, New representations for the Lugo and Euler-Mascheroni costants. II, Appl. Math. Lett. 25 (2012), 333-338.
[12] H. M. Srivastava, C.-P. Chen, Sums of the reciprocals of the triangular numbers and related analytical representations, asymptotic expansions and inequalities, J. Nonlinear Convex Anal. 22 (2021), 349-361.
[13] M. P. Chaudhary, S. Chaudhary, G. A. Salilew, A family of theta-function identities in the light of Jacobi's triple-product identity, Appl. Anal. Optim. 4 (2) (2020), 283-289.
[14] H. M. Srivastava, M. P. Chaudhary, S. Chaudhary, Some theta-function identities related to Jacobi's triple-product identity, European J. Pure Appl. Math. 11 (2018), 1-9.
[15] H. M. Srivastava, M. P. Chaudhary, S. Chaudhary, A family of theta-function identities related to Jacobi's triple-product identity, Russian J. Math. Phys. 27 (2020), 139-144.
[16] H. M. Srivastava, R. Srivastava, M. P. Chaudhary, S. Uddin, A family of theta-function identities based upon combinatorial partition identities related to Jacobi's triple-product identity, Mathematics 8 (2020), Article ID 918.
[17] H. M. Srivastava, The Zeta and related functions: Recent developments, J. Adv. Engrg. Comput. 3 (2019), 329-354.
[18] H. M. Srivastava, Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches, Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan. 45 (2019), 234-269.