J. Nonlinear Var. Anal. 5 (2021), No. 5, pp. 647-664
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.5.02

EXISTENCE AND CONVERGENCE RESULTS FOR A NONLINEAR THERMOELASTIC CONTACT PROBLEM

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Abstract. In this paper, we consider a thermoelastic contact problem in which the heat exchange boundary condition is affected by normal displacement on contact boundary, and the operator in the nonlinear thermoelastic constitutive law is considered to rely on temperature field. First, we deliver the weak formulation of the thermoelastic contact problem which is a coupled system formulated by two variational inequalities with constraints. Then, by employing the Tychonoff fixed point theorem for multivalued operators, an existence theorem for the thermoelastic contact problem is established. Finally, a family of approximate penalized problems corresponding to the thermoelastic contact problem is introduced, and a convergence result is proved. The latter indicates that the solution set of the thermoelastic contact problem can be approached by the solution sets of approximate penalized problems in the sense of the upper semicontinuity property of Kuratowski.
Keywords. Coupled variational inequalities; Kuratowski upper limit; Penalty method; Thermoelastic contact problem; Tychonoff fixed point theorem.

1. INTRODUCTION

In this paper, we investigate a new mathematical model, which describes a stationary nonlinear contact problem in elasticity involving a thermal effect. The model consists of a coupled system of variational inequalities with unilateral constraints in which the unknowns are the displacement field and the temperature field. The first novelty of the paper is the coupling which appears in the nonlinear thermoelastic constitutive law which combines an elasticity operator and a thermal expansion map. Another novelty of the model is the heat exchange boundary condition on the contact surface which is described by the function of the normal displacement, and depends on the difference of the temperature of the body and the one of the foundation. Also, there are two unilateral constraint sets in the model, one is needed for the normal displacement

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Received January 23, 2021; Accepted June 18, 2021.

in the Signorini condition with frictionless effect on the part of the boundary, while the other one for the temperature provides a unilateral obstacle for the temperature inside a domain. The main results concern the weak solvability and the convergence of the penalty method for the system under investigation. The existence proof is based on the application of the Tychonoff fixed point theorem for multivalued functions. For the penalty problem, we establish a convergence result in the sense of the Kuratowski upper semicontinuity property.

Thermoelastic problems with contact arise naturally in industry, particularly, in many production processes of such items as castings, moldings, pistons, thermostats, etc. The quasistatic thermoviscoelastic contact problem were treated in [1], where the friction coefficient depends on the slip. For the comprehensive analysis of thermal effects involved in the contact, we refer to [2, Chapters 3 and 10], where the thermoelastic contact with Signorini's condition was studied. The dynamic thermoviscoelastic problems of contact mechanics which are described by a system of hemivariational inequalities were studied in [3, 4, 5, 6]. Note that the penalty methods for variational and variational-hemivariational inequalities have been extensively studied for various classes of problems; see, e.g., [7, 8, 9, 10] and the references therein.

The paper is structured as follows. In Section 2, we recall basic notation and results, which will be used in next sections. In Section 3, a nonlinear thermoelastic contact problem is introduced, and its weak formulation is obtained in the form of a coupled system of two variational inequalities. Under general assumptions, Section 4 establishes a theorem on existence of solutions for the contact problem via the Tychonoff fixed point theorem for multivalued operators. Finally, in Section 5, we analyse a family of perturbed problems corresponding to the thermoelastic model without constraints and prove a convergence result in the sense of Kuratowski for the solution sets of approximate problems towards a solution of the original contact problem.

2. Preliminary Materials and Notations

In this section, we shortly recall some basic notation and necessary results which will be used in next sections.

Let X be a normed space with norm denoted by $\|\cdot\|_X$, and X^* stand for its topological dual. The duality brackets for the pair (X^*,X) is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$. If no confusion arises, we often skip the subscripts. We adopt the symbols " $\xrightarrow{w}$ " and "$\to$" to stand for the weak and the strong convergences in various spaces.

Let $\Omega\subset\mathbb{R}^d$, $d=2,3$, be a bounded domain (open and connected set) with Lipschitz boundary $\Gamma:=\partial\Omega$. We use the notation $\mathbb{S}^d$ to denote the class of real symmetric matrices of dimension $d\times d$. On $\mathbb{R}^d$ and $\mathbb{S}^d$, we use inner products and norms defined by

$$\xi\cdot\eta=\xi_i\eta_i,\ \|\xi\|=(\xi\cdot\xi)^{1/2}\ \text{ for }\ \xi=(\xi_i),\ \eta=(\eta_i)\in\mathbb{R}^d,$$

$$\sigma\cdot\tau=\sigma_{ij}\tau_{ij},\ \|\sigma\|=(\sigma\cdot\sigma)^{1/2}\ \text{ for }\ \sigma=(\sigma_{ij}),\ \tau=(\tau_{ij})\in\mathbb{S}^d,$$

where the indices i, j, k, l run between 1 and d and the summation convention over repeated indices is applied in here and after. The normal and tangential components of a vector field ξ on the boundary are given by $\xi_\nu=\xi\cdot\nu$ and $\xi_\tau=\xi-\xi_\nu\nu$, respectively, where ν denotes the outward unit normal at the boundary. The notation σ_ν and σ_τ represents the normal and tangential components of the tensor (matrix) σ on the boundary, that is, $\sigma_\nu=(\sigma\nu)\cdot\nu$ and $\sigma_\tau=\sigma\nu-\sigma_\nu\nu$. Moreover, the divergence operators for a tensor σ and a vector q are defined by $\mathrm{Div}\sigma = (\sigma_{ij,j})$ and $\mathrm{div}q = (q_{i,i})$, respectively. Here and in what follows, the index which follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable x. To simplify the notation, we often do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$.

Next, we recall definitions and properties of semicontinuous multivalued operators.

Definition 2.1. Given two topological spaces Y and Z, let $G\colon Y \to 2^Z$ be a multivalued map and $D \subset Y$ be a nonempty set. We say that

(i) G is upper semicontinuous (u.s.c., for short) at $y \in Y$ if, for any open set $O \subset Z$ such that $G(y) \subset O$, there exists a neighborhood $N(y)$ of y such that $G(N(y)) := \cup_{z \in N(y)} G(z) \subset O$. If it holds for all $y \in D$, then G is said to be upper semicontinuous in D;

(ii) G is closed at $y \in Y$ if, for any sequence $\{(y_n, z_n)\} \subset \mathrm{Gr}(G)$ such that $(y_n, z_n) \to (y, z)$ in $Y \times Z$, it holds $(y, z) \in \mathrm{Gr}(G)$, where $\mathrm{Gr}(G)$ denotes the graph of G defined by

$$\mathrm{Gr}(G) := \{\, (y,z) \in Y \times Z \mid z \in G(y) \,\}.$$

If it holds for all $y \in Y$, then G is said to be closed (or G has a closed graph).

The following proposition provides two useful criteria to determinate that a multivalued map is u.s.c.

Proposition 2.1. *Let $F\colon X \to 2^Y$ with X and Y topological spaces. The following statements are equivalent:*

(i) *F is upper semicontinuous in X;*

(ii) *for each closed set $C \subset Y$, $F^-(C) := \{x \in X \mid F(x) \cap C \neq \emptyset\}$ is closed in X;*

(iii) *for each open set $O \subset Y$, $F^+(O) := \{x \in X \mid F(x) \subset O\}$ is open in X.*

In addition, we recall the following definition, see, for example, [11, Definition 4.7.3].

Definition 2.2. Let (X, τ) be a Hausdorff topological space and $\{A_n\} \subset 2^X$ be a sequence of sets. We define the τ-Kuratowski lower limit of the sets A_n by

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{n\to\infty} x_n,\ x_n \in A_n \text{ for all } \ n \geq 1 \right\},$$

and the τ-Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \ldots < n_k < \ldots \right\}.$$

If $A = \tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n$, then A is called the τ-Kuratowski limit of the sequence of sets A_n.

We end this section by recalling the well-known Tychonoff fixed point theorem for multivalued operators. The proof can be found in [12, Theorem 8.6].

Theorem 2.1. *Let C be a bounded, closed and convex subset of a reflexive Banach space E, and let $S\colon C \to 2^C$ be a multivalued map such that*

(i) *S has bounded, closed and convex values;*

(ii) *S is weakly-weakly u.s.c.*

Then S has a fixed point in C.

3. A Nonlinear Thermoelastic Contact Model

In this section, we presents the physical setting of the thermoelastic contact problem and provide its classical formulation. We list the hypotheses on the data and derive a variational formulation of the mathematical model that is in the form of a system which couples two variational inequalities in which the unknowns are the displacement field and the temperature field.

We suppose that the elastic body occupies a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial\Omega = \Gamma$ which is divided into three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C such that meas $(\Gamma_D) > 0$. The nonlinear thermoelastic contact problem under consideration is formulated by the following mathematical model.

Problem 3.1. Find a displacement field $u\colon \Omega \to \mathbb{R}^d$, a stress field $\sigma\colon \Omega \to \mathbb{S}^d$, a temperature $\theta\colon \Omega \to \mathbb{R}$, and a heat flux $q\colon \Omega \to \mathbb{R}^d$ such that

$$\sigma(x) = \mathcal{A}(x, \varepsilon(u(x))) - \mathcal{M}(x, \theta(x)) \quad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{Div} \sigma(x) + f_0(x) = \mathbf{0} \quad \text{in } \Omega, \tag{3.2}$$

$$q(x) = -\kappa \nabla \theta(x) \quad \text{in } \Omega, \tag{3.3}$$

$$\operatorname{div} q(x) = q_0(x) \quad \text{in } \Omega, \tag{3.4}$$

$$u(x) = \mathbf{0} \quad \text{on } \Gamma_D, \tag{3.5}$$

$$\sigma(x)\nu = f_N(x) \quad \text{on } \Gamma_N, \tag{3.6}$$

$$\theta(x) = 0 \quad \text{on } \Gamma_D \cup \Gamma_N, \tag{3.7}$$

$$\begin{cases} \sigma_\nu(x) \le 0,\ u_\nu(x) \le g,\ \sigma_\nu(x)(u_\nu(x) - g) = 0 \\ \sigma_\tau(x) = \mathbf{0} \end{cases} \quad \text{on } \Gamma_C, \tag{3.8}$$

$$q(x) \cdot \nu = h(u_\nu(x))\, \psi(\theta(x) - \theta_f(x)) \quad \text{on } \Gamma_C, \tag{3.9}$$

$$\theta(x) \le \theta_0(x) \quad \text{in } \Omega. \tag{3.10}$$

We give a short description for the physical phenomena and meaning of the equations and inequalities in Problem 3.1. The equation (3.1) describes a nonlinear thermoelastic constitutive law in which $\mathcal{A}$ and $\mathcal{M}$ stand for a nonlinear elasticity operator and a nonlinear thermal expansion mapping, respectively. The law is justified from the material strength point of view, since the properties of the elasticity operator (or the functions of materials) are often effected by temperature. The quantity $\varepsilon(u)$ denotes the small (or linearized) strain tensor associated with the displacement defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and σ is the stress tensor. If $\mathcal{A}$ and $\mathcal{M}$ are linear with respect to their second variables, then (3.1) reduces to the well-known linear thermoelastic constitutive law, which has been widely applied to investigate various thermoelastic contact problems; see, for example, [3, 13, 14, 15, 16]. Note that the model is considered to be static, so, from the Newton second law, we use the equation of equilibrium (3.2), where f_0 denotes the density of volume forces. Due to the heterogeneity of a material, it permits to use the anisotropic version of the Fourier law of heat conduction (3.3), where κ represents the thermal conductivity function, and q is the heat flux vector. By the law of energy conservation, we can see that the heat flux vector q satisfies the heat conduction equilibrium equation (3.4), where q_0 stands for the density of the applied volume heat sources.

The boundary conditions (3.5) and (3.6) describe the physical situation that the elastic body is fixed on Γ_D and it is subjected to surface tractions of density f_N on Γ_N, respectively. We also suppose that the ambient temperature θ_{ref} is a constant, so, for the sake of convenience, in our model, we just take $\theta_{ref} = 0$, namely, the Dirichlet boundary condition for the temperature field θ on boundary $\Gamma_D \cup \Gamma_N$ is given by (3.7). The Signorini contact condition (3.8) is the so-called unilateral contact with finite penetration and frictionless effect. For more details concerning this boundary condition, we refer to [8, 17, 18]. Note that the heat exchange is performed on the contact boundary Γ_C between the body and the foundation. Here, we adopt a generalized heat exchange boundary condition (3.9), where the temperature of foundation is given by θ_f. On the contact boundary Γ_C, since there is a finite penetration g, then the heat exchange coefficient h is considered to be a function which depends on the normal displacement u_ν.

We mention that in our model we assume that ψ is a continuous function such that $\psi(s) < 0$ for all $s < 0$ and $\psi(s) > 0$ for all $s > 0$. In fact, this assumption is reasonable because of the following two reasons. First, if the temperature θ of the body is higher than the temperature of the foundation θ_f, then by the second law of thermodynamics we can see that the heat flux $q \cdot \nu$ is positive, i.e., $\psi(s) > 0$ for all $s > 0$. Second, when the temperature θ of the body is lower than the temperature of foundation θ_f, we can see that the heat flux $q \cdot \nu$ is negative, i.e., $\psi(s) < 0$ for all $s < 0$. As a concrete example, the function ψ can be specialized as follows

$$\psi(s) = \begin{cases} L & \text{if } s > L, \\ s & \text{if } |s| \le L, \\ -L & \text{otherwise,} \end{cases}$$

where $L > 0$ is a constant. Finally, to guarantee the durable years of the material, we suppose that the temperature θ in the domain Ω satisfies the obstacle condition (3.10).

Next, we introduce the following function spaces needed in the variational formulation of Problem 3.1. Let

$$X = \left\{ v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \right\}, \quad \mathscr{H} = L^2(\Omega;\mathbb{S}^d),$$

and

$$H = L^2(\Omega;\mathbb{R}^d), \quad Y = \left\{ \theta \in H^1(\Omega) \mid \theta = 0 \text{ on } \Gamma_D \cup \Gamma_N \right\}.$$

Obviously, $\mathscr{H}$ is a Hilbert space with the inner product and corresponding norm given by

$$\langle \sigma, \tau \rangle_{\mathscr{H}} := \int_\Omega \sigma(x) : \tau(x)\,dx, \quad \|\sigma\|_{\mathscr{H}} = \langle \sigma, \sigma \rangle_{\mathscr{H}}^{1/2} \text{ for } \sigma, \tau \in \mathscr{H}.$$

Since $\mathrm{meas}(\Gamma_D) > 0$, we can apply the Korn and Poincaré inequalities to find that X endowed with the inner product and the associated the norm

$$\langle w, u \rangle_X := \langle \varepsilon(w), \varepsilon(u) \rangle_{\mathscr{H}}, \quad \|w\|_X := \|\varepsilon(w)\|_{\mathscr{H}} \text{ for } w, u \in X,$$

becomes a Hilbert space. Similarly, we also can see that Y endowed with the inner product and the associated the norm

$$\langle \theta, w \rangle_Y := \int_\Omega \nabla\theta \cdot \nabla w\,dx, \quad \|\theta\|_Y := \|\nabla\theta\|_H \text{ for } \theta, w \in Y,$$

is a Hilbert space, where $\|\cdot\|_H$ is the norm of H which is defined by

$$\|u\|_H := \left(\int_\Omega \|u(x)\|^2\,dx \right)^{\frac{1}{2}} \text{ for all } u \in H.$$

Furthermore, we consider the sets $K_1 \subset X$ and $K_2 \subset Y$ given by

$$K_1 := \{ u \in X \mid u_\nu \le g \ \text{ on } \ \Gamma_C \}, \tag{3.11}$$

$$K_2 := \{ \theta \in Y \mid \theta \le \theta_0 \ \text{ in } \ \Omega \}. \tag{3.12}$$

We impose the following assumptions for the data of Problem 3.1.

$$\begin{cases} \mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \text{(a) for all } \varepsilon \in \mathbb{S}^d, x \mapsto \mathcal{A}(x,\varepsilon) \text{ is measurable on } \Omega, \text{ and for a.e. } x \in \Omega \\ \qquad \varepsilon \mapsto \mathcal{A}(x,\varepsilon) \text{ is continuous.} \\ \text{(b) there exist a function } \alpha_{\mathcal{A}} \in L^2(\Omega)_+ \text{ and a constant } c_{\mathcal{A}} > 0 \text{ such that} \\ \qquad\qquad \|\mathcal{A}(x,\varepsilon)\| \le \alpha_{\mathcal{A}}(x) + c_{\mathcal{A}}\|\varepsilon\| \\ \qquad \text{for all } \varepsilon \in \mathbb{S}^d \text{ and a.e. } x \in \Omega. \\ \text{(c) for a.e. } x \in \Omega \text{ the function } \varepsilon \mapsto \mathcal{A}(x,\varepsilon) \text{ is monotone, i.e.,} \\ \qquad\qquad (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \ge 0 \\ \qquad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ and a.e. } x \in \Omega. \\ \text{(d) there exist constants } m_{\mathcal{A}} > 0, \beta_{\mathcal{A}} \ge 0 \text{ and } \gamma_{\mathcal{A}} \in \mathbb{R} \text{ such that} \\ \qquad\qquad \mathcal{A}(x,\varepsilon) : \varepsilon \ge m_{\mathcal{A}}\|\varepsilon\|^2 - \beta_{\mathcal{A}}\|\varepsilon\| + \gamma_{\mathcal{A}} \\ \qquad \text{for all } \varepsilon \in \mathbb{S}^d \text{ and a.e. } x \in \Omega. \end{cases} \tag{3.13}$$

$$\begin{cases} \mathcal{M}\colon \Omega \times \mathbb{R} \to \mathbb{S}^d \text{ is such that} \\ \text{(a) for all } s \in \mathbb{R}, x \mapsto \mathcal{M}(x,s) \text{ is measurable on } \Omega. \\ \text{(b) for a.e. } x \in \Omega \text{ the function } s \mapsto \mathcal{M}(x,s) \text{ is continuous.} \\ \text{(c) there exists a constant } c_{\mathcal{M}} > 0 \text{ such that} \\ \qquad\qquad \|\mathcal{M}(x,s)\| \le c_{\mathcal{M}}(1+|s|) \\ \qquad \text{for all } s \in \mathbb{R} \text{ and a.e. } x \in \Omega. \end{cases} \tag{3.14}$$

$$\begin{cases} \text{The thermal conductivity tensor } \kappa(x) = (\kappa_{ij}(x)) \text{ is such that} \\ \qquad \kappa_{ij} \in L^\infty(\Omega) \text{ and } \kappa_{ij}\xi_i\xi_j \ge m_\kappa\|\xi\|^2 \text{ for all } \xi = (\xi_i) \in \mathbb{R}^d \\ \qquad \text{a.e. } x \in \Omega, \text{ with } m_\kappa > 0. \end{cases} \tag{3.15}$$

$$\begin{cases} h\colon \mathbb{R} \to [0,\infty) \text{ is continuous such that there exists } c_h > 0 \text{ satisfying} \\ \qquad 0 \le h(s) \le c_h \text{ for all } s \in \mathbb{R}. \end{cases} \tag{3.16}$$

$$\begin{cases} \psi\colon \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing such that} \\ \qquad \psi(s) < 0 \text{ for all } s < 0, \text{ and } \psi(s) > 0 \text{ for all } s > 0, \\ \text{and there exists } c_\psi > 0 \text{ satisfying} \\ \qquad |\psi(s)| \le c_\psi(1+|s|) \text{ for all } s \in \mathbb{R}. \end{cases} \tag{3.17}$$

$$\begin{cases} f_0 \in L^2(\Omega;\mathbb{R}^d), \;\; f_N \in L^2(\Gamma_N;\mathbb{R}^d), \\ g \in L^\infty(\Gamma_C) \text{ with } g(x) \geq 0 \text{ on } \Gamma_C \text{ and } g \neq 0, \\ q_0 \in L^2(\Omega), \;\; \theta_f \in L^2(\Gamma_C), \\ \theta_0 \in L^2(\Omega) \text{ with } \theta_0(x) \geq 0 \text{ for a.e. } x \in \Omega \text{ and } \theta_0 \not\equiv 0. \end{cases} \tag{3.18}$$

We complete this section with the variational formulation of Problem 3.1. Assume that u, σ, θ and q are smooth functions that satisfy (3.1)–(3.10). Let $v \in K_1$. We multiply (3.2) by $v-u$ and use Green's formula to find

$$\int_\Omega \sigma(x) : (\varepsilon(v) - \varepsilon(u))\, dx = \int_\Omega f_0 \cdot (v-u)\, dx + \int_\Gamma \sigma(x)\nu \cdot (v-u)\, d\Gamma.$$

Combining this equality with the nonlinear thermoelastic constitutive law (3.1) implies

$$\begin{aligned} &\int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : (\varepsilon(v) - \varepsilon(u))\, dx \\ &\qquad = \int_\Omega f_0 \cdot (v-u)\, dx + \int_\Gamma \sigma(x)\nu \cdot (v-u)\, d\Gamma. \end{aligned}$$

Keeping in mind that

$$\begin{aligned} \int_\Gamma \sigma(x)\nu \cdot (v-u)\, d\Gamma &= \int_{\Gamma_D} \sigma(x)\nu \cdot (v-u)\, d\Gamma \\ &\quad + \int_{\Gamma_N} \sigma(x)\nu \cdot (v-u)\, d\Gamma + \int_{\Gamma_C} \sigma(x)\nu \cdot (v-u)\, d\Gamma, \end{aligned}$$

and

$$\int_{\Gamma_C} \sigma(x)\nu \cdot (v-u)\, d\Gamma = \int_{\Gamma_C} \sigma_\tau(x) \cdot (v_\tau - u_\tau) + \sigma_\nu(x)(v_\nu - u_\nu)\, d\Gamma,$$

it follows from the boundary conditions (3.5)–(3.6) that (see [19, formula (6.33)])

$$\begin{aligned} &\int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : (\varepsilon(v) - \varepsilon(u))\, dx \\ &\quad = \int_\Omega f_0(x) \cdot (v-u)\, dx + \int_{\Gamma_N} f_N \cdot (v-u)\, d\Gamma + \int_{\Gamma_C} \sigma_\tau(x) \cdot (v_\tau - u_\tau)\, d\Gamma \\ &\qquad + \int_{\Gamma_C} \sigma_\nu(x)(v_\nu - u_\nu)\, d\Gamma. \end{aligned} \tag{3.19}$$

The condition (3.8) implies

$$\begin{aligned} \int_{\Gamma_C} \sigma_\nu(x)(v_\nu - u_\nu)\, d\Gamma &= \int_{\Gamma_C} \sigma_\nu(x)(v_\nu - g) + \sigma_\nu(x)(g - u_\nu)\, d\Gamma \\ &= \int_{\Gamma_C} \sigma_\nu(x)(v_\nu - g)\, d\Gamma \geq 0, \end{aligned}$$

where the last inequality is obtained by using the definition of K_1. The latter together with (3.19) and the frictionless condition (3.8) entails

$$\begin{aligned} &\int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : (\varepsilon(v) - \varepsilon(u))\, dx \\ &\qquad \geq \int_\Omega f_0 \cdot (v-u)\, dx + \int_{\Gamma_N} f_N \cdot (v-u)\, d\Gamma. \end{aligned} \tag{3.20}$$

On the other hand, let $\eta \in K_2$. We multiply (3.4) by $\eta - \theta$ and use the integration by parts formula as well as the Fourier law of heat conduction (3.3) to obtain

$$\int_\Omega \kappa\nabla\theta\cdot\nabla(\eta-\theta)\,dx + \int_\Gamma q\cdot\nu(\eta-\theta)\,d\Gamma = \int_\Omega q_0(x)(\eta-\theta)\,dx.$$

Taking account of the above equality and the boundary conditions (3.7) and (3.9), one has

$$\int_\Omega \kappa\nabla\theta\cdot\nabla(\eta-\theta)\,dx + \int_{\Gamma_C} h(u_\nu(x))\,\psi(\theta(x)-\theta_f(x))(\eta-\theta)\,d\Gamma$$
$$= \int_\Omega q_0(x)(\eta-\theta)\,dx. \tag{3.21}$$

Combining (3.20) and (3.21), we are now in a position to give the variational formulation of Problem 3.1 as follows.

Problem 3.2. Find a displacement field $u \in K_1$ and a temperature $\theta \in K_2$ such that

$$\int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : (\varepsilon(v)-\varepsilon(u))\,dx$$
$$\geq \int_\Omega f_0\cdot(v-u)\,dx + \int_{\Gamma_N} f_N\cdot(v-u)\,d\Gamma \ \text{ for all } \ v\in K_1,$$

and

$$\int_\Omega \kappa\nabla\theta\cdot\nabla(\eta-\theta)\,dx + \int_{\Gamma_C} h(u_\nu(x))\,\psi(\theta(x)-\theta_f(x))(\eta-\theta)\,d\Gamma$$
$$= \int_\Omega q_0(x)(\eta-\theta)\,dx \ \text{ for all } \ \eta\in K_2.$$

The latter consists of two coupled variational inequalities with constraints. The existence of solutions to Problem 3.2 will be shown in the next section.

4. Existence of Weak Solutions

In this section ,we examine the weak solvability of Problem 3.2. In the proof, we combine a recent result from [20, Theorem 3.3] (see [21] as well) and the Tychonoff fixed point principle for multivalued maps given in Theorem 2.1.

The existence theorem for Problem 3.2 reads as follows.

Theorem 4.1. *Assume that (3.13)–(3.18) are satisfied. Then, the solution set of Problem 3.2 is nonempty.*

Proof. We introduce the operators $G\colon Y\times X\to X^*$ and $F\colon X\times Y\to Y^*$ defined by

$$\langle G(\theta,u),v\rangle_X := \int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : \varepsilon(v)\,dx \tag{4.1}$$

and

$$\langle F(u,\theta),\eta\rangle_Y := \int_\Omega \kappa\nabla\theta\cdot\nabla\eta\,dx + \int_{\Gamma_C} h(u_\nu(x))\,\psi(\theta(x)-\theta_f(x))\eta\,d\Gamma \tag{4.2}$$

for all u, $v\in X$ and θ, $\eta\in Y$, respectively. Employing the Riesz representation theorem, we are able to find a function $f\in X^*$ such that

$$\langle f,v\rangle_{X^*\times X} = \int_\Omega f_0\cdot v\,dx + \int_{\Gamma_N} f_N\cdot v\,d\Gamma \ \text{ for all } \ v\in X.$$

By the definitions above, it is easy to see that Problem 3.2 could be rewritten as the following coupled variational inequality problem: find $u \in K_1$ and $\theta \in K_2$ such that

$$\langle G(\theta,u)-f, v-u\rangle_X \geq 0 \text{ for all } v \in K_1, \tag{4.3}$$

and

$$\langle F(u,\theta)-q_0, \eta-\theta\rangle_Y = 0 \text{ for all } \eta \in K_2. \tag{4.4}$$

The proof of solvability of problem (4.3)–(4.4) is divided into four steps.

Step 1. For each $u \in K_1$, problem (4.4) has a unique solution $\theta = \theta(u) \in K_2$.

For any $u \in K_1$ fixed, hypothesis (3.15) reveals that $Y \ni \theta \mapsto F(u,\theta) \in Y^*$ is continuous and strongly monotone. Indeed, let $\{\theta_n\} \subset Y$ and $\theta \in Y$ be such that $\theta_n \to \theta$ in Y as $n \to \infty$. Then, one has

$$\begin{aligned}\|F(u,\theta_n)-F(u,\theta)\|_{Y^*} &= \sup_{\|\eta\|_Y=1} |\langle F(u,\theta_n)-F(u,\theta),\eta\rangle_Y| \\ &\leq c_h\left(\int_{\Gamma_C}|\psi(\theta(x)-\theta_f(x))-\psi(\theta_n(x)-\theta_f(x))|^2\,d\Gamma\right)^{1/2} \\ &\quad +\|\kappa\|\left(\int_\Omega\|\nabla(\theta-\theta_n)\|^2\,dx\right)^{1/2},\end{aligned}$$

where $c_h > 0$ is given in (3.16). Passing to the limit as $n \to \infty$ and using the Lebesgue dominated convergence theorem as well as the continuity of ψ, one has

$$\|F(u,\theta_n)-F(u,\theta)\|_{Y^*} \to 0 \text{ as } n \to \infty,$$

thus, $Y \ni \theta \mapsto F(u,\theta) \in Y^*$ is continuous. For any θ_1, $\theta_2 \in K_2$, the hypotheses (3.15) and (3.16) imply

$$\begin{aligned}\langle F(u,\theta_1)-F(u,\theta_2),\theta_1-\theta_2\rangle_Y &= \int_\Omega \kappa\nabla(\theta_1-\theta_2)\cdot\nabla(\theta_1-\theta_2)\,dx \\ &+\int_{\Gamma_C} h(u_\nu)\,[\psi(\theta_1(x)-\theta_f(x))-\psi(\theta_2(x)-\theta_f(x))]\,d\Gamma \geq m_\kappa\|\theta_1-\theta_2\|_Y^2.\end{aligned}$$

Hence, we deduce that $Y \ni \theta \mapsto F(u,\theta) \in Y^*$ is strongly monotone with constant $m_\kappa > 0$. Further, we show that $Y \ni \theta \mapsto F(u,\theta) \in Y^*$ is a coercive operator. Since $\psi(0) = 0$, by (3.15), it follows that

$$\begin{aligned}&\langle F(u,\theta),\theta\rangle_Y \\ &\quad\geq m_\kappa\|\theta\|_Y^2+\int_{\Gamma_C}h(u_\nu)\,\psi(\theta-\theta_f)(\theta-\theta_f)\,d\Gamma+\int_{\Gamma_C}h(u_\nu)\,\psi(\theta-\theta_f)\theta_f\,d\Gamma \\ &\quad\geq m_\kappa\|\theta\|_Y^2-c_hc_\psi\int_{\Gamma_C}(1+|\theta|+|\theta_f|)|\theta_f|\,d\Gamma \\ &\quad\geq m_\kappa\|\theta\|_Y^2-c_hc_\psi\|\theta\|_{L^2(\Gamma_C)}\|\theta_f\|_{L^2(\Gamma_C)}-c_hc_\psi\|\theta_f\|_{L^2(\Gamma_C)}^2-c_hc_\psi\|\theta_f\|_{L^1(\Gamma_C)}\end{aligned}$$

for all $\theta \in K_2$, where the last inequality is obtained by using the Hölder inequality. From the last estimate the coercivity of the map $Y \ni \theta \mapsto F(u,\theta) \in Y^*$ follows.

Since all conditions of [20, Theorem 3.3] are satisfied, we use this theorem to conclude that for each $u \in K_1$, problem (4.4) admits at least one solution. Let θ_1, $\theta_2 \in K_2$ be two solutions of problem (4.4). Then

$$0 \geq \langle F(u,\theta_1) - F(u,\theta_2), \theta_1 - \theta_2 \rangle_Y \geq m_\kappa \|\theta_1 - \theta_2\|_Y^2 + \int_{\Gamma_C} h(u_\nu)\,[\psi(\theta_1 - \theta_f) - \psi(\theta_2 - \theta_f)](\theta_1 - \theta_2)\,d\Gamma \geq m_\kappa \|\theta_1 - \theta_2\|_Y^2.$$

Hence, $\theta_1 = \theta_2$. Therefore, for each $u \in K_1$, problem (4.4) has a unique solution $\theta = \theta(u) \in K_2$.

In what follows, we denote by $H\colon K_1 \to K_2$ the solution map of problem (4.4), that is, $H(u) = \theta(u)$ for all $u \in K_1$.

Step 2. For each $\theta \in K_2$, the solution set of problem (4.3) is nonempty, bounded, closed and convex.

We again invoke [20, Theorem 3.3] to verify this assertion. Let $\theta \in K_2$ be fixed. It follows from hypothesis (3.13)(a) that the function $u \mapsto G(\theta,u)$ is continuous. The monotonicity of $u \mapsto G(\theta,u)$ is a direct consequence of condition (3.13)(c). Moreover, for any $u \in K_1$, we use hypotheses (3.13)(d) and (3.14)(c) to obtain

$$\begin{aligned}
\langle G(\theta,u),u\rangle_X &= \int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : \varepsilon(u)\,dx \\
&\geq \int_\Omega \Big(m_\mathcal{A} \|\varepsilon(u)\|_{\mathbb{S}^d}^2 - \beta_\mathcal{A} \|\varepsilon(u)\|_{\mathbb{S}^d} + \gamma_\mathcal{A} \Big)\,dx - \int_\Omega c_\mathcal{M}(1+|\theta|)\|\varepsilon(u)\|_{\mathbb{S}^d}\,dx \\
&\geq (m_\mathcal{A} - \varepsilon)\|u\|_X^2 - \frac{c_\mathcal{M}^2 \|\theta\|_{L^2(\Omega)}^2}{2\varepsilon} + c(\varepsilon)
\end{aligned}$$

with some $c(\varepsilon) > 0$ and $\varepsilon = \frac{1}{2} m_\mathcal{A}$, where the last inequality is obtained by using the Young inequality. Therefore, we can see that the map $u \mapsto G(\theta,u)$ is coercive.

We apply [20, Theorem 3.3] to conclude that for each $\theta \in K_2$ the solution set of problem (4.3) is nonempty, bounded, closed and convex. Let $\Lambda\colon K_2 \to 2^{K_1}$ be the solution map of problem (4.3).

By virtue of definitions of H and Λ, we note that if $u \in K_1$ is a fixed point of $\Lambda \circ H\colon K_1 \to 2^{K_1}$, then it holds $\theta(u) = H(u)$, $u \in \Lambda(\theta(u))$ and $(u,\theta(u)) \in K_1 \times K_2$ is also a solution of problem (4.3)–(4.4). Based on this observation, we shall apply the Tychonoff fixed point principle of Theorem 2.1, to prove that multivalued map $\Lambda \circ H\colon K_1 \to 2^{K_1}$ has at least one fixed point in K_1.

Step 3. The multivalued map $\widetilde{\Lambda} := \Lambda \circ H\colon K_1 \to 2^{K_1}$ is weakly-weakly u.s.c.

It is clear from Proposition 2.1 that it is sufficient to show that, for any weakly closed set $D \subset K_1$, the set

$$\widetilde{\Lambda}(D)^- := \Big\{ u \in K_1 \mid \widetilde{\Lambda}(u) \cap D \neq \emptyset \Big\}$$

is weakly closed in X. Let $\{u_n\} \subset \widetilde{\Lambda}(D)^-$ and $u_n \xrightarrow{w} u$ in X as $n \to \infty$ for some $u \in X$. Then, for each $n \in \mathbb{N}$, there exists $v_n \in \Lambda(u_n) \cap D$. Our goal is to verify that $u \in \widetilde{\Lambda}(D)^-$. Let $\theta_n = H(u_n)$. We claim that H is a completely continuous operator, i.e., it is weakly-strongly continuous. First, we note that the sequence $\{\theta_n\}$ is bounded in Y. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_\Omega \kappa \nabla\theta_n \cdot \nabla(\eta - \theta_n)\,dx + \int_{\Gamma_C} h(u_{n\nu}(x))\,\psi(\theta_n(x) - \theta_f(x))(\eta - \theta_n)\,d\Gamma \\
&\quad = \int_\Omega q_0(x)(\eta - \theta_n)\,dx \qquad (4.5)
\end{aligned}$$

for all $\eta \in K_2$. Inserting $\eta = 0$ into the inequality above, due to $\theta_0 \geq 0$, we have

$$\begin{aligned}
\|q_0\|_{L^2(\Omega)}\|\theta_n\|_{L^2(\Omega)} &\geq \int_\Omega q_0(x)\theta_n\,dx \\
&= \int_\Omega \kappa\nabla\theta_n\cdot\nabla\theta_n\,dx + \int_{\Gamma_C} h(u_{n,\nu}(x))\psi(\theta_n(x)-\theta_f(x))\theta_n\,d\Gamma \\
&\geq m_\kappa\|\theta_n\|_Y^2 + \int_{\Gamma_C} h(u_{n,\nu}(x))\psi(\theta_n(x)-\theta_f(x))(\theta_n-\theta_f)\,d\Gamma \\
&\quad + \int_{\Gamma_C} h(u_{n,\nu}(x))\psi(\theta_n(x)-\theta_f(x))\theta_f\,d\Gamma \\
&\geq m_\kappa\|\theta_n\|_Y^2 - c_h c_\psi \int_{\Gamma_C}(1+|\theta_n|+|\theta_f|)|\theta_f|\,d\Gamma \\
&\geq m_\kappa\|\theta_n\|_Y^2 - c_h c_\psi\|\theta_n\|_{L^2(\Gamma_C)}\|\theta_f\|_{L^2(\Gamma_C)} - c_h c_\psi\|\theta_f\|^2_{L^2(\Gamma_C)} - c_h c_\psi\|\theta_f\|_{L^1(\Gamma_C)}.
\end{aligned} \tag{4.6}$$

This implies that the sequence $\{\theta_n\}$ is bounded in Y. From the reflexivity of Y and the compactness of the embedding of Y to $L^2(\Omega)$, by passing to a subsequence if necessary, we may assume that

$$\theta_n \xrightarrow{w} \theta \ \text{ in } \ Y \ \text{ and } \ \theta_n \to \theta \ \text{ in } \ L^2(\Omega), \ \text{ as } \ n\to\infty$$

for some $\theta \in Y$.

Next, we shall show that $\theta = H(u)$. The convexity and closedness of K_2 ensures that $\theta \in K_2$. Letting $\eta = \theta$ in (4.5) and passing to the upper limit as $n\to\infty$, it gives

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega \kappa\nabla\theta_n\cdot\nabla(\theta_n-\theta)\,dx \\
&\quad = \lim_{n\to\infty}\int_{\Gamma_C} h(u_{n\nu}(x))\psi(\theta_n(x)-\theta_f(x))(\theta_n-\theta)\,d\Gamma + \lim_{n\to\infty}\int_\Omega q_0(x)(\theta_n-\theta)\,dx = 0,
\end{aligned}$$

where we have used the Lebesgue dominated convergence theorem, the boundedness of the function h, and the continuity of ψ. The latter combined with the inequality

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega \kappa\nabla\theta_n\cdot\nabla(\theta_n-\theta)\,dx \\
&\quad = \limsup_{n\to\infty}\int_\Omega \kappa\nabla(\theta_n-\theta)\cdot\nabla(\theta_n-\theta)\,dx + \lim_{n\to\infty}\int_\Omega \kappa\nabla\theta\cdot\nabla(\theta_n-\theta)\,dx \\
&\quad \geq m_\kappa \limsup_{n\to\infty}\|\theta_n-\theta\|_Y^2
\end{aligned}$$

implies

$$\theta_n \to \theta \ \text{ in } \ Y, \ \text{ as } \ n\to\infty. \tag{4.7}$$

Letting $n\to\infty$ in (4.5), we obtain that θ is a solution of problem (4.4) corresponding to u. The uniqueness of solution of problem (4.4) points out that $\theta = H(u)$. Since every weakly convergent subsequence of $\{\theta_n\}$ towards the same limit $\theta = H(u)$, we conclude that $H(u_n) \to H(u)$ in Y as $n\to\infty$, see, e.g., [22, Theorem 1.20]. This shows that H is a weakly-strongly continuous map.

Furthermore, we are going to prove that the sequence $\{v_n\}$ is bounded in X. Because of $\mathbf{0} \in K_1$, a short computation gives

$$\begin{aligned}
&\frac{m_{\mathcal{A}}}{2}\|v_n\|_X^2 - \frac{c_{\mathcal{M}}^2\|\theta_n\|_{L^2(\Omega)}^2}{m_{\mathcal{A}}} + c_0 \\
&\quad \le \int_\Omega (\mathcal{A}(x,\boldsymbol{\varepsilon}(v_n(x))) - \mathcal{M}(x,\theta_n(x))) : \boldsymbol{\varepsilon}(v_n)\,dx \\
&\quad\quad \le \int_\Omega f_0\cdot v_n\,dx + \int_{\Gamma_N} f_N\cdot v_n\,d\Gamma \\
&\quad\quad\quad \le \|f_0\|_H\|v_n\|_H + \|f_N\|_{L^2(\Gamma_N;\mathbb{R}^d)}\|v_n\|_{L^2(\Gamma_N;\mathbb{R}^d)}
\end{aligned} \tag{4.8}$$

with some $c_0 > 0$. The boundedness of $\{\theta_n\}$ and the estimate (4.8) imply that $\{v_n\}$ is bounded in X. Without any loss of generality, we may assume that

$$v_n \xrightarrow{w} z \text{ in } X, \text{ as } n \to \infty \tag{4.9}$$

for some $z \in X$. Owing to the weak closedness of the set D, we have $z \in D$. Subsequently, for each $n \in \mathbb{N}$, exploiting the monotonicity of $\varepsilon \mapsto \mathcal{A}(x,\varepsilon)$, we obtain

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\boldsymbol{\varepsilon}(v(x))) - \mathcal{M}(x,\theta_n(x))) : (\boldsymbol{\varepsilon}(v) - \boldsymbol{\varepsilon}(v_n))\,dx \\
&\quad \ge \int_\Omega f_0\cdot(v - v_n)\,dx + \int_{\Gamma_N} f_N\cdot(v - v_n)\,d\Gamma
\end{aligned}$$

for all $v \in K_1$. Using the continuity of the function $s \mapsto \mathcal{M}(x,s)$ and (4.7), by passing to the limit as $n \to \infty$, one finds

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\boldsymbol{\varepsilon}(v(x))) - \mathcal{M}(x,\theta(x))) : (\boldsymbol{\varepsilon}(v) - \boldsymbol{\varepsilon}(z))\,dx \\
&\quad \ge \int_\Omega f_0\cdot(v - z)\,dx + \int_{\Gamma_N} f_N\cdot(v - z)\,d\Gamma
\end{aligned}$$

for all $v \in K_1$. Due to the continuity of $\varepsilon \mapsto \mathcal{A}(x,\varepsilon)$, we are now in a position to invoke the Minty approach to obtain

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\boldsymbol{\varepsilon}(z(x))) - \mathcal{M}(x,\theta(x))) : (\boldsymbol{\varepsilon}(v) - \boldsymbol{\varepsilon}(z))\,dx \\
&\quad \ge \int_\Omega f_0\cdot(v - z)\,dx + \int_{\Gamma_N} f_N\cdot(v - z)\,d\Gamma
\end{aligned}$$

for all $v \in K_1$, i.e., $z \in \widetilde{\Lambda}(u)$. Therefore, it holds $u \in \widetilde{\Lambda}(D)^-$. Applying Proposition 2.1, we conclude that the map $\widetilde{\Lambda}$ is weakly-weakly u.s.c.

Step 4. There exists a constant $M > 0$ such that $\widetilde{\Lambda}(B_M) \subset B_M$, where the closed ball in X is denoted by $B_M := \{u \in X \mid \|u\|_X \le M\}$.

We argue by contradiction. Suppose that for each $n \in \mathbb{N}$, we are able to find $u_n \in X$ with $\|u_n\|_X = n$ such that $v_n \in \widetilde{\Lambda}(u_n)$ and $\|v_n\|_X > n$. We set $\theta_n = H(u_n)$. The estimate (4.6) guarantees that $\{\theta_n\}$ is bounded in Y. Therefore, from (4.8), we have

$$\frac{m_{\mathcal{A}}}{2}\|v_n\|_X - \frac{c_{\mathcal{M}}^2\|\theta_n\|_{L^2(\Omega)}^2}{m_{\mathcal{A}}\|v_n\|_X} + \frac{c_0}{\|v_n\|_X} \le c_1$$

for some $c_1 > 0$. Taking the limit, as $n \to \infty$, in the inequality above, we obtain a contradiction. So, we conclude that there exists a constant $M > 0$ such that $\widetilde{\Lambda}$ maps B_M into itself.

Having verified all hypotheses of Theorem 2.1, we use it to find an element $u \in K_1$ satisfying $u \in \widetilde{\Lambda}(u)$. This means that $(u, \theta) := (u, H(u))$ is a solution of the problem (4.3)–(4.4). Consequently, the solution set of Problem 3.2 is nonempty. This completes the proof. $\square$

5. Convergence of a Penalty Method

In the section, we analyse the convergence of a penalty method for the thermoelastic contact problem, Problem 3.1. We introduce two sequences $\{\delta_n\}$ and $\{\lambda_n\}$ of penalty parameters such that

$$\delta_n, \lambda_n > 0, \quad \delta_n \to 0 \quad \text{and} \quad \lambda_n \to 0 \ \text{ as } \ n \to \infty. \tag{5.1}$$

Then, for δ_n, $\lambda_n > 0$, we formulate an approximate penalized problem associated with Problem 3.1 in which the Signorini contact condition is replaced by a normal compliance condition $\frac{1}{\delta_n} p_2(u_\nu - g)$ on Γ_C, and the obstacle condition for temperature is replaced by a nonlinear perturbation term $\frac{1}{\lambda_n} p_1(\theta - \theta_0)$ in Ω. Physically, the penalty parameter $\delta_n > 0$ can be interpreted as a deformability coefficient of the foundation, and $\frac{1}{\delta_n}$ is the surface stiffness coefficient, see [22, Chapter 5]. We shall prove the Kuratowski upper semicontinuity convergence property for the solution set to the penalized problem, see the convergence (5.13) below.

The double penalized problem under consideration reads as follows.

Problem 5.1. Find a displacement field $u\colon \Omega \to \mathbb{R}^d$, a stress field $\sigma\colon \Omega \to \mathbb{S}^d$, a temperature $\theta\colon \Omega \to \mathbb{R}$, and a heat flux $q\colon \Omega \to \mathbb{R}^d$ such that

$$\sigma(x) = \mathcal{A}(x, \varepsilon(u(x))) - \mathcal{M}(x, \theta(x)) \quad \text{in } \Omega, \tag{5.2}$$

$$\operatorname{Div} \sigma(x) + f_0(x) = \mathbf{0} \quad \text{in } \Omega, \tag{5.3}$$

$$q(x) = -\kappa \nabla \theta(x) \quad \text{in } \Omega, \tag{5.4}$$

$$\operatorname{div} q(x) + \frac{1}{\lambda_n} p_1(\theta - \theta_0) = q_0(x) \quad \text{in } \Omega, \tag{5.5}$$

$$u(x) = \mathbf{0} \quad \text{on } \Gamma_D, \tag{5.6}$$

$$\sigma(x)\nu = f_N(x) \quad \text{on } \Gamma_N, \tag{5.7}$$

$$\theta(x) = 0 \quad \text{on } \Gamma_D \cup \Gamma_N, \tag{5.8}$$

$$\begin{cases} -\sigma_\nu(x) = \dfrac{1}{\delta_n} p_2(u_\nu(x) - g) \\ \sigma_\tau(x) = \mathbf{0} \end{cases} \quad \text{on } \Gamma_C, \tag{5.9}$$

$$q(x) \cdot \nu = h(u_\nu(x)) \psi(\theta(x) - \theta_f(x)) \quad \text{on } \Gamma_C. \tag{5.10}$$

Here the functions $p_1 \colon \Omega \times \mathbb{R} \to [0,\infty)$ and $p_2 \colon \Gamma_C \times \mathbb{R} \to [0,\infty)$ are assumed to satisfy the following conditions.

$$\begin{cases} p_1 \colon \Omega \times \mathbb{R} \to [0,\infty) \text{ is such that} \\ \text{(a)} \;\; x \mapsto p_1(x,s) \text{ is measurable in } \Omega \text{ for all } s \in \mathbb{R}. \\ \text{(b)} \;\; p_1(x,s) = 0 \text{ if and only if } s \le 0 \text{ for a.e. } x \in \Omega. \\ \text{(c)} \;\; \text{there exists } L_{p_1} > 0 \text{ such that} \\ \qquad\qquad |p_1(x,s_1) - p_1(x,s_2)| \le L_{p_1}|s_1 - s_2| \\ \quad\;\; \text{for all } s_1, s_2 \in \mathbb{R} \text{ and a.e. } x \in \Omega. \\ \text{(d)} \;\; s \mapsto p_1(x,s) \text{ is monotone for a.e. } x \in \Omega. \end{cases} \tag{5.11}$$

$$\begin{cases} p_2 \colon \Gamma_C \times \mathbb{R} \to [0,\infty) \text{ is such that} \\ \text{(a)} \;\; x \mapsto p_2(x,s) \text{ is measurable on } \Gamma_C \text{ for all } s \in \mathbb{R}. \\ \text{(b)} \;\; p_2(x,s) = 0 \text{ if and only if } s \le 0 \text{ for a.e. } x \in \Gamma_C. \\ \text{(c)} \;\; \text{there exists } L_{p_2} > 0 \text{ such that} \\ \qquad\qquad |p_2(x,s_1) - p_2(x,s_2)| \le L_{p_2}|s_1 - s_2| \\ \quad\;\; \text{for all } s_1, s_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_C. \\ \text{(d)} \;\; s \mapsto p_2(x,s) \text{ is monotone for a.e. } x \in \Gamma_C. \end{cases} \tag{5.12}$$

A particular case for function p_1 or p_2 is the one $s \mapsto s^+ := \max\{s,0\}$ for all $s \in \mathbb{R}$.

Using a standard procedure, we obtain the following variational formulation of Problem 5.1.

Problem 5.2. Find a displacement field $u \in X$ and a temperature $\theta \in Y$ such that

$$\begin{aligned} &\int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : \varepsilon(v)\,dx + \frac{1}{\delta_n}\int_{\Gamma_C} p_2(u_\nu - g)v_\nu\,d\Gamma \\ &\quad = \int_\Omega f_0 \cdot v\,dx + \int_{\Gamma_N} f_N \cdot v\,d\Gamma \;\text{ for all }\; v \in X, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \kappa\nabla\theta \cdot \nabla\eta\,dx + \frac{1}{\lambda_n}\int_\Omega p_1(\theta - \theta_0)\eta\,dx + \int_{\Gamma_C} h(u_\nu(x))\psi(\theta(x) - \theta_f(x))\eta\,d\Gamma \\ &\quad = \int_\Omega q_0(x)\eta\,dx \;\text{ for all }\; \eta \in Y. \end{aligned}$$

The main result on the existence and convergence of solutions to Problem 5.2 are provided in the following theorem.

Theorem 5.1. *Suppose that* (3.13)–(3.18) *and* (5.11)–(5.12) *hold. If, in addition, the sequences* $\{\lambda_n\}$ *and* $\{\delta_n\}$ *satisfy* (5.1)*, then*

(i) *for each* $n \in \mathbb{N}$*, the solution set of Problem* 5.2*, denoted by* Π_n*, is nonempty;*

(ii) *it holds*

$$w\text{–}\limsup_{n\to\infty} \Pi_n \subset \Pi, \tag{5.13}$$

where Π *is the solution set of Problem* 3.2.

Proof. (i) For each $n \in \mathbb{N}$, we introduce the operators $G_n\colon Y \times X \to X^*$ and $F_n\colon X \times Y \to Y^*$ defined by

$$\langle G_n(\theta,u),v\rangle_X := \int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : \varepsilon(v)\,dx + \frac{1}{\delta_n}\int_{\Gamma_C} p_2(u_\nu - g)v_\nu\,d\Gamma \tag{5.14}$$

and

$$\langle F_n(u,\theta),\eta\rangle_Y := \int_\Omega \kappa\nabla\theta\cdot\nabla\eta\,dx + \int_{\Gamma_C} h(u_\nu(x))\psi(\theta(x)-\theta_f(x))\eta\,d\Gamma + \frac{1}{\lambda_n}\int_\Omega p_1(\theta-\theta_0)\eta\,dx \tag{5.15}$$

for all u, $v \in X$ and θ, $\eta \in Y$. The continuity and monotonicity of the functions $s \mapsto p_1(s)$ and $s \mapsto p_2(s)$ guarantee that the operators $u \mapsto G_n(\theta,u)$ and $\theta \mapsto F_n(u,\theta)$ are both continuous and monotone. Let $u \in X$ and $\theta \in Y$ be arbitrary. Then, we have

$$\int_{\Gamma_C} p_2(u_\nu - g)u_\nu\,d\Gamma = \int_{\Gamma_C}(p_2(u_\nu-g) - p_2(0-g))u_\nu\,d\Gamma \ge 0, \tag{5.16}$$

$$\int_\Omega p_1(\theta-\theta_0)\theta\,dx = \int_\Omega (p_1(\theta-\theta_0) - p_1(0-\theta_0))\theta\,dx \ge 0, \tag{5.17}$$

where we have used the monotonicity of p_1 and p_2 as well as the fact that $p_1(s) = p_2(s) = 0$ for all $s \le 0$. Hence, we can apply the same arguments as in the proof of Theorem 4.1 to conclude that, for each $n \in \mathbb{N}$, the solution set of Problem 5.2 is nonempty, that is, $\Pi_n \neq \emptyset$.

(ii) First, we show that the set $\cup_{n\in\mathbb{N}}\Pi_n$ is uniformly bounded in $X \times Y$. For each $n \in \mathbb{N}$, let $(u_n,\theta_n) \in \Pi_n$ be arbitrary. Then, by using (4.6), (5.17) and the Hölder inequality, one finds

$$\begin{aligned} m_\kappa\|\theta_n\|_Y^2 &\le \int_\Omega \kappa\nabla\theta_n\cdot\nabla\theta_n\,dx + \frac{1}{\lambda}\int_\Omega p_1(\theta_n-\theta_0)\theta_n\,dx \\ &= \int_\Omega q_0(x)\theta_n\,dx - \int_{\Gamma_C} h(u_{n,\nu}(x))\psi(\theta_n(x)-\theta_f(x))\theta_n\,d\Gamma \le \|q_0\|_{L^2(\Omega)}\|\theta_n\|_{L^2(\Omega)} \\ &\quad + c_h c_\psi\|\theta_n\|_{L^2(\Gamma_C)}\|\theta_f\|_{L^2(\Gamma_C)} + c_h c_\psi\|\theta_f\|^2_{L^2(\Gamma_C)} + c_h c_\psi\|\theta_f\|_{L^1(\Gamma_C)}. \end{aligned}$$

Hence, it follows that the set $\cup_{n\in\mathbb{N}}\{\theta_n \in Y \mid (u_n,\theta_n) \in \Pi_n\}$ is uniformly bounded in Y. On the other hand, using (4.8) and (5.16), we deduce

$$\begin{aligned} &\frac{m_{\mathcal{A}}}{2}\|u_n\|_X^2 - \frac{c_{\mathcal{M}}^2\|\theta_n\|^2_{L^2(\Omega)}}{m_{\mathcal{A}}} + c_0 \\ &\quad\le \int_\Omega(\mathcal{A}(x,\varepsilon(u_n(x))) - \mathcal{M}(x,\theta_n(x))) : \varepsilon(u_n)\,dx + \frac{1}{\delta_n}\int_{\Gamma_C} p_2(u_{n,\nu}-g)u_{n,\nu}\,d\Gamma \\ &\quad= \int_\Omega f_0\cdot u_n\,dx + \int_{\Gamma_N} f_N\cdot u_n\,d\Gamma \\ &\quad\le \|u_n\|_H\|f_0\|_H + \|f_N\|_{L^2(\Gamma_2;\mathbb{R}^d)}\|u_n\|_{L^2(\Gamma_2;\mathbb{R}^d)} \end{aligned}$$

with some $c_0 > 0$. The estimate above together with the boundedness of $\{\theta_n\}$ implies that the set $\cup_{n\in\mathbb{N}}\{u_n \in X \mid (u_n,\theta_n) \in \Pi_n\}$ is bounded in X. Therefore, we conclude that $\cup_{n\in\mathbb{N}}\Pi_n$ is uniformly bounded in $X \times Y$.

Let $\{(u_n,\theta_n)\} \subset X$ be a sequence such that $(u_n,\theta_n) \in \Pi_n$ for all $n \in \mathbb{N}$. The boundedness of $\cup_{n\in\mathbb{N}}\Pi_n$ allows to assume that, passing to a subsequence if necessary,

$$u_n \xrightarrow{w} u \text{ in } X, \text{ and } \theta_n \xrightarrow{w} \theta \text{ in } Y, \text{ as } n \to \infty \tag{5.18}$$

with some $(u,\theta) \in X \times Y$. This means that the set $w\text{–}\limsup_{n\to\infty}\Pi_n$ is nonempty.

Furthermore, we are going to prove that $w\text{–}\limsup_{n\to\infty}\Pi_n \subset \Pi$. Let $(u,\theta) \in w\text{–}\limsup_{n\to\infty}\Pi_n$. Without any loss of generality, we are able to find a sequence $\{(u_n,\theta_n)\} \subset X \times Y$ such that $(u_n,\theta_n) \in \Pi_n$ for each $n \in \mathbb{N}$ and (5.18) holds. The compactness of the embeddings of X to H and Y to $L^2(\Omega)$, and the compactness of the trace operator of X to $L^2(\Gamma_C;\mathbb{R}^d)$ imply

$$u_n \to u \text{ in } H \text{ and } L^2(\Gamma_C;\mathbb{R}^d), \text{ and } \theta_n \to \theta \text{ in } L^2(\Omega), \text{ as } n \to \infty. \tag{5.19}$$

We claim that $\theta \in K_2$. A simple calculation gives

$$\begin{aligned}\int_\Omega p_1(\theta_n-\theta_0)\eta\,dx = -\lambda_n\int_{\Gamma_C} h(u_{n\nu}(x))\,\psi(\theta_n(x)-\theta_f(x))\eta\,d\Gamma \\ -\lambda_n\int_\Omega \kappa\nabla\theta_n\cdot\nabla\eta\,dx + \lambda_n\int_\Omega q_0(x)\eta\,dx\end{aligned}$$

for all $\eta \in Y$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in the last equality, from (3.16) and (5.1), we obtain

$$\int_\Omega p_1(\theta-\theta_0)\eta\,dx = 0 \text{ for all } \eta \in Y.$$

Hence $p_1(\theta-\theta_0) = 0$ for a.e. $x \in \Omega$, i.e., $\theta(x) \le \theta_0(x)$ for a.e. $x \in \Omega$ (see (5.11)(b)). It proves the claim and $\theta \in K_2$.

Next, for every $n \in \mathbb{N}$, from hypothesis (3.15), the monotonicity of p_1 and the fact that $p_1(\theta - \theta_0) = 0$ for a.e. in Ω (thanks to $\theta \in K_2$), it follows that

$$\begin{aligned} m_\kappa\|\theta_n-\theta\|_Y^2 + \int_\Omega \kappa\nabla\theta\cdot\nabla(\theta_n-\theta)\,dx &\le \int_\Omega \kappa\nabla\theta_n\cdot\nabla(\theta_n-\theta)\,dx \\ \le \int_\Omega \kappa\nabla\theta_n\cdot\nabla(\theta_n-\theta)\,dx &+ \frac{1}{\lambda_n}\int_\Omega p_1(\theta_n-\theta_0)(\theta_n-\theta)\,dx \\ = -\int_{\Gamma_C} h(u_{n\nu}(x))\psi(\theta_n(x)-\theta_f(x))(\theta_n-\theta)\,d\Gamma &+ \int_\Omega q_0(x)(\theta_n-\theta)\,dx. \end{aligned} \tag{5.20}$$

Passing to the upper limit, as $n \to \infty$, in (5.20), and using (5.19) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\|\theta_n-\theta\|_Y^2 \le 0.$$

This means that $\theta_n \to \theta$ in Y, as $n \to \infty$.

Subsequently, we pass to the limit as $n \to \infty$ in the following equality

$$\begin{aligned}\int_{\Gamma_C} p_2(u_{n\nu}-g)v_\nu\,d\Gamma = -\delta_n\int_\Omega(\mathcal{A}(x,\varepsilon(u_n(x))) - \mathcal{M}(x,\theta_n(x))) : \varepsilon(v)\,dx \\ +\delta_n\int_\Omega f_0\cdot v\,dx + \delta_n\int_{\Gamma_N} f_N\cdot v\,d\Gamma \text{ for all } v \in X,\end{aligned}$$

and use the boundedness of $\{(u_n,\theta_n)\}$, hypotheses (3.13) (b), and (3.14)(c) to find

$$\int_{\Gamma_C} p_2(u_\nu-g)v_\nu\,d\Gamma = 0 \text{ for all } v \in X.$$

This indicates that $u_\nu \le g$ for a.e. on Γ_C, thus, $u \in K_1$. The monotonicity of p_2 and (5.12)(b) show that

$$\frac{1}{\delta_n}\int_{\Gamma_C} p_2(u_{n\nu}-g)(u_{n\nu}-v_\nu)\,d\Gamma \ge \frac{1}{\delta_n}\int_{\Gamma_C} p_2(v_\nu-g)(u_{n\nu}-v_\nu)\,d\Gamma = 0$$

for all $v \in K_1$. Then, we have

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,\varepsilon(v(x))) : \varepsilon(u_n-v)\,dx \le \int_\Omega \mathcal{A}(x,\varepsilon(u_n(x))) : \varepsilon(u_n-v)\,dx \\
&= \int_\Omega \mathcal{A}(x,\varepsilon(u_n(x))) : \varepsilon(u_n-v)\,dx + \frac{1}{\delta_n}\int_{\Gamma_C} p_2(u_{n\nu}-g)(u_{n\nu}-v_\nu)\,d\Gamma \\
&= \int_\Omega \mathcal{M}(x,\theta_n(x))) : \varepsilon(u_n-v)\,dx + \int_\Omega f_0\cdot(u_n-v)\,dx + \int_{\Gamma_N} f_N\cdot(u_n-v)\,d\Gamma
\end{aligned}$$

for all $v \in K_1$. We use (3.14)(b) and pass to the limit, as $n\to\infty$, in the inequality above. We obtain

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\varepsilon(v(x))) - \mathcal{M}(x,\theta(x)))) : \varepsilon(v-u)\,dx \\
&\qquad \ge \int_\Omega f_0\cdot(v-u)\,dx + \int_{\Gamma_N} f_N\cdot(v-u)\,d\Gamma
\end{aligned}$$

for all $v \in K_1$. Employing the Minty technique, we deduce

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\varepsilon(u(x))) - \mathcal{M}(x,\theta(x))) : \varepsilon(v-u)\,dx \\
&\qquad \ge \int_\Omega f_0\cdot(v-u)\,dx + \int_{\Gamma_N} f_N\cdot(v-u)\,d\Gamma
\end{aligned}$$

for all $v \in K_1$. Similarly, we also can obtain

$$\int_\Omega \kappa\nabla\theta\cdot\nabla\eta\,dx + \int_{\Gamma_C} h(u_\nu(x))\,\psi(\theta(x)-\theta_f(x))\eta\,d\Gamma = \int_\Omega q_0(x)\eta\,dx$$

for all $\eta \in K_2$. This means that $(u,\theta)\in\Pi$, i.e., the inclusion (5.13) holds. □

Acknowledgments

This project was funded by the NNSF of China Grant Nos. 12001478, 12026255 and 12026256, the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Preludium Project No. 2017/25 /N/ST1/00611, and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It was also supported by Natural Science Foundation of Guangxi Grants Nos. 2020GXNSFBA297137 and 2018GXNSFAA281353, and the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019.

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