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# FIXED POINTS AND CONVERGENCE RESULTS FOR A CLASS OF CONTRACTIVE MAPPINGS

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**Abstract.** We have recently established the existence of a unique fixed point for nonlinear contractive self-mappings of a closed subset of a Banach space, which is not necessarily bounded. In this paper, we extend this result to contractive mappings, which map a closed subset of a Banach space into the space.
**Keywords.** Banach space; Complete metric space; Contractive mapping; Fixed point.

## 1. Introduction

For more than fifty-five years now, there has been a lot of research activity regarding the fixed point theory of nonexpansive mappings; see, e.g., [1]-[16] and the references therein. This activity stems from Banach's classical theorem [17] regarding the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this area including, in particular, the studies of feasibility, common fixed point problems and variational inequalities, which find significant applications in, *inter alia*, engineering, medical and the natural sciences [15]-[22].

In our recent work [23], we established the existence of a unique fixed point for nonlinear contractive self-mappings of a bounded and closed subset of a Banach space. In the more recent paper [24], we extended this result to the case of *unbounded* sets. More precisely, in [23, 25], we considered the following class of nonlinear mappings.

Let $(X, \|\cdot\|)$ be a Banach space and let $K$ be a bounded, closed and convex subset of $X$. Let $f : X \to [0,\infty)$ be a continuous function such that $f(0) = 0$, the set $f(K-K)$ be bounded, and the following three properties hold:

(i) for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in K$ satisfy $f(x-y) \le \delta$, then $\|x-y\| \le \varepsilon$;

(ii) for each $\lambda \in (0,1)$, there is $\phi(\lambda) \in (0,1)$ such that

$$f(\lambda(x-y)) \le \phi(\lambda) f(x-y) \text{ for all } x, y \in K;$$

(iii) the function defined by $(x,y) \mapsto f(x-y)$, $x, y \in K$ is uniformly continuous on $K \times K$.

Denote by $\mathscr{A}$ the set of all continuous mappings $A : K \to K$ such that

$$f(Ax - Ay) \le f(x-y) \text{ for all } x, y \in K.$$

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For each $A, B \in \mathscr{A}$, set

$$d(A,B) := \sup\{\|Ax - Bx\| : x \in K\}.$$

It is clear that $(\mathscr{A}, d)$ is a complete metric space.

In [25] we established the existence of a set $\mathscr{F}$, which is a countable intersection of open and everywhere dense subsets of $\mathscr{A}$,such that each $C \in \mathscr{F}$ has a unique fixed point and all its iterates converge uniformly to this fixed point. As a matter of fact, the mappings defined above can be considered generalized nonexpansive mappings with respect to $f$. In [26] we constructed an example of a generalized nonexpansive self-mapping of a bounded, closed and convex set in a Hilbert space, which is not nonexpansive in the classical sense.

Note that the classical result of De Blasi and Myjak [27] is a particular case of this result with $f = \|\cdot\|$. Such an approach, where in some problems of functional analysis the norm is replaced by a general function, was used in [28, 29] in the study of generalized best approximation problems, which we now recall.

Given a closed subset $S$ of a Banach space $X$ and a point $x \in X$, we considered in [28, 29] the minimization problem

$$\min\{f(x-y) : y \in S\}. \tag{(P)}$$

This problem was studied by many mathematicians mostly in the case where $f(x) = \|x\|$. In this special case it is well known that if $S$ is convex and $X$ is reflexive, then problem ((P)) always has at least one solution. This solution is unique when $X$ is strictly convex. In [28, 29] we established the generic solvability and well-posedness of problem ((P)) for a general function $f$.

In [23] we improved the results of [25]. Namely, we introduced there a notion of a contractive mapping, and showed that most mappings in $\mathscr{A}$ (in the sense of Baire category) are contractive. Every contractive mapping possesses a unique fixed point and all its iterates converge uniformly to this point. Note that all these results were obtained on a bounded set $K$. In [24] we extended one of the main results of [23] to *unbounded* sets. More precisely, we showed that even if the set $K$ is unbounded, every contractive self-mapping of $K$ possesses a unique fixed point and that all its iterates converge to this fixed point uniformly on bounded subsets of $K$. Moreover, for this result, we do not need property (ii). This result is also a generalization of the result of [10], which was obtained in the case where $f(x) = \|x\|$.

In this paper, we extend the results of [24] to contractive mappings, which map a closed subset of a Banach space into the space. Our main results, Theorems 2.1, 2.2 and 2.3, are stated in Section 2. An auxiliary result is proved in Section 3. We prove our main results in Sections 4, 5 and 6, respectively.

## 2. Main Results

Let $(X, \|\cdot\|)$ be a Banach space and let $K$ be a nonempty and closed subset of $X$. Let $f : X \to [0,\infty)$ be a continuous function such that $f(0) = 0$ and the following two properties hold:

(P1) for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in K$ and $y \in X$ satisfy $f(x-y) \le \delta$, then $\|x - y\| \le \varepsilon$;

(P2) for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $z_1, \xi_1 \in K$ and $z_2, \xi_2 \in X$ satisfy

$$\|z_1 - z_2\| \le \delta,\ \|\xi_1 - \xi_2\| \le \delta,$$

then

$$|f(z_1-\xi_1)-f(z_2-\xi_2)|\le\varepsilon.$$

Assume that $A:K\to X$ is a continuous mapping, $\psi:[0,\infty)\to[0,1]$ is a decreasing function satisfying

$$\psi(t)<1 \text{ for all } t>0$$

and that

$$f(Ax-Ay)\le\psi(f(x-y))f(x-y) \text{ for all } x,y\in K. \tag{2.1}$$

In other words, the mapping $A$ is contractive [14]. We denote the identity operator by $A^0$.

We are now ready to state our main results. Their proofs are presented in Sections 4–6 below.

**Theorem 2.1.** *Assume that, for each $\varepsilon>0$, there exists $x\in K$ such that $f(x-Ax)\le\varepsilon$. Then there exists a unique point $x_*\in K$ such that $Ax_*=x_*$. Moreover, for each $\varepsilon>0$, there exists $\delta>0$ such that if $x\in K$ satisfies $f(x-Ax)\le\delta$, then $\|x-x_*\|\le\varepsilon$.*

**Theorem 2.2.** *Let $M>0$. Assume that, for each integer $n\ge1$, there exists a point $x_n\in K$ such that $A^nx_n$ is defined and*

$$f(x_n-Ax_n)\le M. \tag{2.2}$$

*Then there exists a point $x_*\in K$ such that*

$$Ax_*=x_*.$$

**Theorem 2.3.** *Assume that the assumptions of at least one of Theorems 2.1 and 2.2 hold, $M,\varepsilon>0$ and that $x_*\in K$ satisfies*

$$Ax_*=x_*.$$

*Then there exists a natural number $n(\varepsilon)$ such that, for each $x\in K$ satisfying*

$$f(x-x_*)\le M$$

*and each integer $n\ge n(\varepsilon)$ such that $A^nx$ exists, the inequality $\|A^ix-x_*\|\le\varepsilon$ holds for all integers $i\in[n(\varepsilon),n]$.*

## 3. An Auxiliary Result

**Lemma 3.1.** *Let $\varepsilon>0$. Then there exists $\delta>0$ such that, for each $x,y\in K$ satisfying*

$$f(x-Ax)\le\delta,\ f(y-Ay)\le\delta,$$

*the inequality $f(x-y)\le\varepsilon$ holds.*

*Proof.* By (P2), there exists a number $\delta_0\in(0,\varepsilon)$ such that

$$|f(z_1-\xi_1)-f(z_2-\xi_2)|\le2^{-1}\varepsilon(1-\psi(\varepsilon))/2 \tag{3.1}$$

for all $z_1,\xi_1\in K$ and all $z_2,\xi_2\in X$ satisfying

$$\|z_1-z_2\|\le\delta_0,\ \|\xi_1-\xi_2\|\le\delta_0.$$

Property (P1) implies that there exists a number $\delta\in(0,\delta_0)$ such that the following property holds:

(a) if $x\in K$ and $y\in X$ satisfy $f(x-y)\le\delta$, then $\|x-y\|\le\delta_0$.

Let $x,y\in K$ satisfy

$$f(x-Ax)\le\delta,\ f(y-Ay)\le\delta. \tag{3.2}$$

We claim that

$$f(x-y) \leq \varepsilon.$$

Suppose to the contrary that this is not true. Then we have

$$f(x-y) > \varepsilon. \tag{3.3}$$

From (2.1) and (3.3), we have

$$f(Ax-Ay) \leq \psi(f(x-y))f(x-y) \leq \psi(\varepsilon)f(x-y). \tag{3.4}$$

In view of (3.3) and (3.4), we have

$$f(x-y)-f(Ax-Ay) \geq (1-\psi(\varepsilon))f(x-y) > (1-\psi(\varepsilon))\varepsilon. \tag{3.5}$$

Property (a) and (3.2) imply that

$$\|x-Ax\| \leq \delta_0, \ \|y-Ay\| \leq \delta_0. \tag{3.6}$$

It now follows from (3.1) and (3.6) that

$$|f(Ax-Ay)-f(x-y)| \leq \varepsilon(1-\psi(\varepsilon))/2.$$

This, however, contradicts (3.5). The contradiction we have reached completes the proof of Lemma 3.1. □

## 4. Proof of Theorem 2.1

For each integer $n \geq 1$, there exists a point $x_n \in K$ such that

$$f(x_n - Ax_n) \leq n^{-1}. \tag{4.1}$$

Lemma 3.1 implies that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{4.2}$$

Property (P1) and (4.1) imply that

$$\lim_{n\to\infty} \|x_n - Ax_n\| = 0. \tag{4.3}$$

Since the mapping $A$ is continuous, it follows from (4.2) and (4.3) that

$$Ax_* = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_n = x_*.$$

Now we see that Theorem 2.1 follows from Lemma 3.1 and property (P1).

## 5. Proof of Theorem 2.2

In view of (2.1), for every integer $n \geq 2$ and every integer $i \in \{0,\dots,n-2\}$, we have

$$f(A^{i+1}x_n - A^{i+2}x_n) \leq \psi(f(A^i x_n - A^{i+1}x_n))f(A^i x_n - A^{i+1}x_n). \tag{5.1}$$

Let $\varepsilon > 0$ be given. Theorem 2.1 implies that it is sufficient to show that there exists a point $x \in K$ such that

$$f(x-Ax) \leq \varepsilon.$$

Suppose to the contrary that this is not true. Then for each integer $n \geq 2$ and each $i \in \{0,\dots,n-1\}$,

$$f(A^i x_n - A^{i+1}x_n) \geq \varepsilon. \tag{5.2}$$

By (2.1), (2.2) and (5.2), for each integer $n \geq 2$ and each $i \in \{0,\dots,n-2\}$, we have

$$\begin{aligned} f(A^{i+1}x_n - A^{i+2}x_n) &\leq \psi(f(A^i x_n - A^{i+1}x_n)) f(A^i x_n - A^{i+1}x_n) \\ &\leq \psi(\varepsilon) f(A^i x_n - A^{i+1}x_n), \end{aligned}$$

and

$$\begin{aligned} f(A^{n-1}x_n - A^n x_n) &\leq \psi(\varepsilon)^{n-1} f(Ax_n - x_n) \\ &\leq M\psi(\varepsilon)^{n-1} \to 0 \text{ as } n \to \infty. \end{aligned}$$

This, however, contradicts (5.2). The contradiction we have reached completes the proof of Theorem 2.2.

## 6. Proof of Theorem 2.3

Property (P1) implies that there exists a number $\varepsilon_1 \in (0,\varepsilon)$ such that the following property holds:

(a) if $x \in K$ and $y \in X$ satisfy $f(x-y) \leq \varepsilon_1$, then $\|x-y\| \leq \varepsilon$.

Choose a natural number $n(\varepsilon) > 2$ such that

$$\psi(\varepsilon_1)^{n(\varepsilon)-1} M < \varepsilon. \tag{6.1}$$

Assume than $n \geq n(\varepsilon)$, $x \in K$ satisfies

$$f(x - x_*) \leq M, \tag{6.2}$$

and that $A^n x$ exists. By property (a), it is sufficient to show that

$$f(A^i x - x_*) \leq \varepsilon_1 \text{ for all integers } i \in [n(\varepsilon), n].$$

In view of (2.1), in order to meet this goal, it suffices to show that there exists an integer $i \in [0, n(\varepsilon)]$ such that

$$f(A^i x - x_*) \leq \varepsilon_1.$$

Suppose to the contrary that this does not hold. Then we have

$$f(A^i x - x_*) > \varepsilon_1, \ i = 0, \dots, n(\varepsilon). \tag{6.3}$$

It now follows from (2.1), (6.2) and (6.3) that, for all $i = 0, \dots, n(\varepsilon) - 1$,

$$f(A^{i+1}x - x_*) \leq \psi(f(A^i x - x_*)) f(A^i x - x_*) \leq \psi(\varepsilon_1) f(A^i x - x_*),$$

and

$$\varepsilon < f(A^{n(\varepsilon)}(x) - x_*) \leq \psi(\varepsilon_1)^{n(\varepsilon)-1} f(x - x_*) \leq \psi(\varepsilon_1)^{n(\varepsilon)-1} M.$$

This, however, contradicts (6.1). The contradiction we have reached completes the proof of Theorem 2.3.

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