J. Nonlinear Var. Anal. 5 (2021), No. 5, pp. 673-682
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.5.04

OPTIMALITY CONDITIONS FOR VECTOR VARIATIONAL INEQUALITIES VIA IMAGE SPACE ANALYSIS

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Abstract. In this paper, we outline the main features of the image space analysis for vector variational inequalities with cone constraints. Exploiting a suitable separation scheme in the associated image space, we derive saddle point and Karush-Kuhn-Tucker type optimality conditions for the given vector variational inequality.
Keywords. Image space analysis; Separation theorems; Vector variational inequalities.

1. INTRODUCTION

The strong request of mathematical models for optimizing situations with concurrent objective has led to a wide development of Vector Optimization Problems (for short, VOP). Following the development of Vector Optimization, the theory of variational inequalities has been generalized to the vector case with the aim to exploit the advantage of both variational and extremization methods. In particular, from the pioneering work of Giannessi [1], Vector Variational Inequalities (for short, VVI) allow to generalize to the vector case many of the features of (scalar) variational inequalities; see, e.g., [2, 3, 4] and the references therein.

In this paper, we will outline the Image Space Analysis (for short, ISA) [5] for VVI. The ISA has shown to be a unifying scheme for studying constrained extremum problems and, more generally, can be applied to any kind of problem that can be expressed under the form of the impossibility of a parametric system; see e.g., [1, 5, 6, 7, 8, 9, 10] and the references therein. In this approach, the impossibility of such a system is reduced to the disjunction of two suitable subsets of the image space associated with the VVI. Such a disjunction can be proved by showing that the two sets lie in two disjoint level sets of a suitable separating functional. The ISA allows one to develop several topics such as Lagrangian-type necessary optimality conditions, saddle point sufficient conditions, regularity, stability and penalty methods.

The paper is organized as follows. In Section 2, we will preliminarily recall the main features of VVI and consider their connections with VOP. Necessary optimality conditions for a VOP can be formulated in terms of a VVI when the objective function of the vector problem is Gateaux differentiable and the feasible set is convex. Suitable generalized convexity assumptions ensure that a VVI is a sufficient optimality condition for a vector minimum point. In Section 3, we

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Received May 5, 2021; Accepted June 20, 2021.

recall the main features of the ISA for a VVI, while, in Section 4, the last section, we will derive optimality conditions for VVI exploiting a suitable separation scheme in the image space associated with VVI.

2. Main Definitions and Preliminary Results

Let X and Y be Hausdorff locally convex topological vector spaces. For $l \in L(X,Y)$, the set of all linear continuous functions, the value of l at x is denoted by $\langle l,x\rangle$, while Y^* denotes the topological dual of Y. A set $C \subseteq Y$ is said to be a cone if $tC \subseteq C$, $\forall\, t \geq 0$. A convex cone C is said to be *pointed* if $C \cap (-C) = \{0\}$. The polar cone of C is $C^* := \{x^* \in Y^* : \langle x^*,x\rangle \geq 0, \forall x \in C\}$ and we denote $C \setminus \{0\}$ by C_o. Given $A \subseteq Y$, $\operatorname{int} A$ and $\operatorname{cl} A$, denote the topological interior of A, and the closure of A, respectively. The cone generated by the set A is defined by $\operatorname{cone}(A) := \bigcup_{t\geq 0} tA$. Moreover, we define $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x \geq 0\}$. The normal cone to a convex set A at $x \in A$ is defined by $N_A(x) := \{\xi \in Y^* : \langle \xi, a-x\rangle \leq 0,\ \forall\ a \in A\}$. We say that $x \in A$ is a quasi interior point of A, denoted by $x \in \operatorname{qi} A$, if $\operatorname{cl}\operatorname{cone}(A-x) = Y$ or, equivalently, $N_A(x) = \{0\}$; while $x \in A$ is a quasi relative interior point of A, denoted by $x \in \operatorname{qri} A$, if $\operatorname{cl}\operatorname{cone}(A-x)$ is a linear subspace of Y or, equivalently, $N_A(x)$ is a linear subspace of Y^*. Moreover, we have that $\operatorname{qi} A \subseteq \operatorname{qri} A$, and $\operatorname{int} A \neq \emptyset$ implies $\operatorname{int} A = \operatorname{qi} A$. Similarly, if $\operatorname{qi} A \neq \emptyset$, then $\operatorname{qi} A = \operatorname{qri} A$. In particular, if Y is a finite dimensional space, then $\operatorname{qi} A = \operatorname{int} A$ and $\operatorname{qri} A = \operatorname{ri} A$, where $\operatorname{ri} A$ denotes the relative interior of A. For details, we refer to, e.g., [11, 12].

The following result [13, 14] provides a characterization of the quasi interior of a convex cone and of its polar under suitable assumptions.

Proposition 2.1. *Let $C \subseteq Y$ be a closed and convex cone.*
(i) $qi C = \{x \in C : \langle x^*,x\rangle > 0, \forall x^* \in C^* \setminus \{0\}\}$;
(ii) $qi(C^*) = \{x^* \in Y^* : \langle x^*,x\rangle > 0, \forall x \in C_o\}$.

We now give some properties that will be used in what follows (see, e.g., [11, 13]).

Proposition 2.2. *Let $M,N \subseteq Y$ be convex sets.*
(i) $\operatorname{qri} M$ *is a convex set;*
(ii) $\operatorname{qri}(M \times N) = \operatorname{qri} M \times \operatorname{qri} N$*;*
(iii) if $\operatorname{qri} M \neq \emptyset$*, then* $\operatorname{cl}(\operatorname{qri} M) = \operatorname{cl} M$*;*

Let $C \subset Y$ be a pointed, closed and convex cone with $\operatorname{qi} C \neq \emptyset$. Then (Y,C) is an ordered space with a partial ordering defined by

$$y_1 \leq_C y_2 \iff y_2 - y_1 \in C.$$

Given $A \subseteq Y$, by $y_1 \not\leq_A y_2$, we mean $y_2 - y_1 \notin A$. Let $K \subseteq X$ be a closed and convex set and $T : K \to L(X,Y)$. A VVI consists of finding $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \not\leq_{C_o} 0, \quad \forall x \in K. \tag{2.1}$$

A weak VVI (for short, WVVI) consists of finding $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \not\leq_{\operatorname{qi} C} 0, \quad \forall x \in K. \tag{2.2}$$

We recall that a point $y^* \in A \subseteq Y$ is called a minimal point of A if $A \cap (y^* - C_o) = \emptyset$ and a weakly minimal point if $A \cap (y^* - \operatorname{qi} C) = \emptyset$. Setting $A := \langle T(x^*), K - x^*\rangle$, we have that the following equivalences hold:

(i) $x^* \in K$ is a solution to VVI if and only if $y^* = 0$ is a minimal point of A.
(ii) $x^* \in K$ is a solution to WVVI if and only if $y^* = 0$ is a weakly minimal point of A.
Given $f : X \to Y$ and VOP defined by:

$$\min_{x \in K}{}_C f(x), \tag{2.3}$$

a point $x^* \in K$ is said weak efficient if and only if $f(K) \nleq_{\mathrm{qi}C} f(x^*)$ and efficient if and only if $f(K) \nleq_{C_o} f(x^*)$.

We recall that $f : X \to Y$ is C-convex on K if and only if, for any $x_1, x_2 \in K, \lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda f(x_1) + (1-\lambda) f(x_2).$$

For the sake of completeness, we give the proof of the following proposition, which summarizes the relationships between (2.1), (2.2) and (2.3).

Proposition 2.3. *Assume that f is Gâteaux differentiable on K with the Gâteaux derivative Df. Setting $T = Df$, the following statements hold:*
(i) If x^* is a weak efficient solution of (2.3), then x^* solves (2.2).
(ii) If f is C-convex and x^* solves (2.2), then x^* is a weak efficient solution of (2.3).
(iii) If f is C-convex and x^* solves (2.1), then x^* is an efficient solution of (2.3).

Proof. (i) Since $x^* \in K$ is weak efficient, then, for any $x \in K$, we have that $x^* + t(x - x^*) \in K$, $\forall t \in]0,1[$. It follows that

$$f(x^* + t(x - x^*)) - f(x^*) \nleq_{\mathrm{qi}C} 0, \quad \forall t \in]0,1[.$$

Since $tc \in \mathrm{qi}C$, for every $t > 0$ and $c \in \mathrm{qi}C$, this implies

$$\frac{f(x^* + t(x - x^*)) - f(x^*)}{t} \nleq_{\mathrm{qi}C} 0, \quad \forall t \in]0,1[.$$

Taking the limit for $t \downarrow 0$, we obtain (2.2).

(ii) Assume that x^* solves (2.2). Since f is C-convex on K if and only if, for every

$$f(y) - f(x) \geq_C \langle Df(x), y - x \rangle, \quad \forall x, y \in K,$$

it holds that

$$f(x^*) - f(x) \in -\langle Df(x^*), x - x^* \rangle - C, \quad \forall x \in K,$$

which implies $f(x^*) - f(x) \in (Y \setminus \mathrm{qi}C) - C$. Thus

$$f(x^*) - f(x) \ngeq_{\mathrm{qi}C} 0, \quad \forall x \in K,$$

that is, x^* is weak efficient.

(iii) Assume that x^* is not efficient. Then, there exists $x \in K$, such that $f(x^*) - f(x) \geq_{C_o} 0$. Since f is C-convex on K, then, for any $x \in K$,

$$f(x) - f(x^*) - \langle Df(x^*), x - x^* \rangle \geq_C 0.$$

Therefore,

$$-\langle Df(x^*), x - x^* \rangle \geq_C f(x^*) - f(x) \geq_{C_o} 0,$$

which contradicts (2.1). □

3. Image Space Analysis and Separation for VVI

In this section, we recall the main features of the ISA applied to a VVI. To this end, we suppose that the feasible set is defined by

$$K := \{x \in X : g(x) \in D\}, \tag{3.1}$$

where $g : X \to Z$, Z is a locally convex topological vector space, and D is a closed and convex cone in Z. We observe that $x^* \in K$ is a solution to VVI if and only if the following system

$$\langle T(x^*), x^* - x\rangle \geq_{C_o} 0\ ,\ g(x) \in D\ ,\ x \in X, \tag{3.2}$$

is impossible.

From now on, we put $f(x^*, x) := \langle T(x^*), x - x^*\rangle$. Introduce the sets

$$\mathscr{K}_{x^*} := \{(u, v) \in Y \times Z : u = -f(x^*, x)\ v = g(x),\ x \in X\}$$

and

$$\mathscr{H}_{C_o} := \{(u, v) \in Y \times Z : u \in C_o,\ v \in D\}.$$

$\mathscr{K}_{x^*}$ is called the *image* associated with VVI. Hence, $x^* \in K$ is a solution of VVI if and only if

$$\mathscr{K}_{x^*} \cap \mathscr{H}_{C_o} = \emptyset\ . \tag{3.3}$$

Another optimality condition can be obtained by introducing the *extended image* associated with VVI and defined by $\mathscr{E}_{x^*} := \mathscr{K}_{x^*} - cl\,\mathscr{H}_{C_o}$. It is easily proved that (3.3) is equivalent to

$$\mathscr{E}_{x^*} \cap \mathscr{H}_{C_o} = \emptyset. \tag{3.4}$$

Similarly, $x^* \in K$ is a solution of WVVI if and only if

$$\mathscr{K}_{x^*} \cap \mathscr{H}_{\mathrm{qi}C} = \emptyset\ , \tag{3.5}$$

or, equivalently,

$$\mathscr{E}_{x^*} \cap \mathscr{H}_{\mathrm{qi}C} = \emptyset. \tag{3.6}$$

Unlike $\mathscr{K}_{x^*}$, the set $\mathscr{E}_{x^*}$ enjoys the remarkable properties inherithed from the given problem. For example, if $-g$ is D-convex, then the set $\mathscr{E}_{x^*}$ turns out to be convex while $\mathscr{K}_{x^*}$ may be not. For such a reason, in the following, we will consider conditions (3.4), or (3.6) that can be proved by showing that $\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_o}$, or $\mathscr{H}_{\mathrm{qi}C}$, lie in two disjoint level sets of a suitable, possibly nonlinear, functional (see also Proposition 3.1).

To this aim, given $\mathscr{Q}$ and Ξ two suitable subsets of parameters, and $(\theta, \lambda) \in \mathscr{Q} \times \Xi$, we define $\alpha : Y \times \mathscr{Q} \to \mathbb{R}$ and $\gamma : Z \times \Xi \longrightarrow \mathbb{R}$, which fulfill the following conditions:

$$\bigcap_{\theta \in \mathscr{Q}} \mathrm{lev}_{\geq 0}\alpha(\cdot;\theta) \supseteq C_o, \quad \bigcap_{\lambda \in \Xi} \mathrm{lev}_{\geq 0}\gamma(\cdot;\lambda) \supseteq D, \tag{3.7}$$

where $\mathrm{lev}_{\geq 0}\gamma(\cdot;\lambda) := \{v \in \mathbb{R}^m : \gamma(v;\lambda) \geq 0\}$ is the level set of $\gamma(\cdot;\lambda)$, and similarly for $\alpha(\cdot;\theta)$.

Definition 3.1. The function $w : Y \times Z \longrightarrow \mathbb{R}$ defined by:

$$w(u, v; \theta, \lambda) := \alpha(u;\theta) + \gamma(v;\lambda), \tag{3.8}$$

and verifying (3.7), is called a separation function in the image space.

Observe that the conditions in (3.7) imply

$$w(u,v;\theta,\lambda) \geq 0, \quad \forall (u,v) \in \mathscr{H}_{C_o}, \quad \forall (\theta,\lambda) \in \mathscr{Q} \times \Xi.$$

If $\mathscr{Q} \times \Xi = C^* \times D^*$, $\alpha(u;\theta) = \langle \theta, u \rangle$, and $\gamma(v;\lambda) = \langle \lambda, v \rangle$, then w is a linear separation function in the image space.

If there exists $(\theta^*,\lambda^*) \in C^* \times D^*$, $(\theta^*,\lambda^*) \neq 0$ such that

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u,v) \in \mathscr{E}_{x^*}, \tag{3.9}$$

then the sets $\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_o}$ admit a linear separation.

The existence of a linear separation is ensured by the D-convexity of the function $-g$ provided that $0 \notin \operatorname{qi} \mathscr{E}_{x^*}$ since in this case $\mathscr{E}_{x^*}$ is a convex set.

Remark 3.1. Since $cl\mathscr{H}_{\mathrm{qi}C} = cl\mathscr{H}_{C_o} = C \times D$ (see Proposition 2.2), then a linear function separates $\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_0}$ if and only if it separates $\mathscr{E}_{x^*}$ and $\mathscr{H}_{\mathrm{qi}C}$.

The following result shows that a linear function separates $\mathscr{K}_{x^*}$ and $\mathscr{H}_{C_o}$ if and only if it separates $\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_o}$.

Proposition 3.1. *(3.9) is equivalent to*

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u,v) \in \mathscr{K}_{x^*}. \tag{3.10}$$

Proof. Suppose that (3.10) holds. Let $(h_1,h_2) \in \mathscr{H}_{C_o}$. Since

$$\langle \theta^*, -h_1 \rangle + \langle \lambda^*, -h_2 \rangle \leq 0,$$

then

$$\langle \theta^*, u - h_1 \rangle + \langle \lambda^*, v - h_2 \rangle \leq 0, \qquad \forall (u,v) \in \mathscr{K}_{x^*},$$

and (3.9) holds. It is obvious that (3.9) implies (3.10) due to $\mathscr{K}_{x^*} \subseteq \mathscr{E}_{x^*}$. □

The existence of a linear separation does not guarantee the optimality of x^* unless we impose suitable conditions on θ^*.

Proposition 3.2. *Assume that the sets $\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_0}$ admit a linear separation.*
(i) If $\theta^* \in qi(C^*)$, then (3.4) is fulfilled.
(ii) If $\theta^* \in C^* \setminus \{0\}$, then (3.6) is fulfilled.

Proof. (i) Ab absurdo, suppose that $\mathscr{E}_{x^*} \cap \mathscr{H}_{C_o} \neq \emptyset$. This implies that $\mathscr{K}_{x^*} \cap \mathscr{H}_{C_o} \neq \emptyset$. Therefore, there exists $z \in X$ such that $-f(x^*,z) \in C_o, g(z) \in D$. Then, taking into account that $\theta^* \in qi(C^*)$, we conclude from Proposition 2.1 (ii) that

$$0 < \langle \theta^*, -f(x^*,z) \rangle \leq \langle \theta^*, -f(x^*,z) \rangle + \langle \lambda^*, g(z) \rangle \leq 0. \tag{3.11}$$

(ii) Ab absurdo, suppose that $\mathscr{E}_{x^*} \cap \mathscr{H}_{\mathrm{qi}C} \neq \emptyset$. This implies that $\mathscr{K}_{x^*} \cap \mathscr{H}_{\mathrm{qi}C} \neq \emptyset$ so that $\exists z \in X$ such that $-f(x^*,z) \in \operatorname{qi} C, g(z) \in D$. Then, taking into account that $\theta^* \in C^* \setminus \{0\}$, by Proposition 2.1 (i) we obtain again (3.11), which is impossible. □

For $z \in \mathbb{R}^\ell$, let $z_{i^-} := (z_1,..,z_{i-1},z_{i+1},..,z_\ell)$. The following result deepens Proposition 3.2.

Proposition 3.3. *Assume that the sets $\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_o}$ admit a linear separation.*
(i) Let $C := \mathbb{R}^\ell_+$, $qi\,D \neq \emptyset$. Assume that, for every i, the following system is possible:

$$-f_{i^-}(x^*,y) > 0, \quad g(y) \in qi\,D, \quad y \in X.$$

then in (3.9) we can suppose that $\theta^* > 0$.
(ii) If there exists $y \in X$ such that

$$g(y) \in qi\,D, \tag{3.12}$$

then in (3.9) we can suppose that $\theta^* \neq 0$.

In the final part of this section, we give an example of nonlinear separation function. To this aim, for fixed $e \in \mathscr{Q}_o := \{e \in C_0,: Y = \cup_{t\in\mathbb{R}}(te - C)\}$, we consider the function $\xi_e : Y \to \mathbb{R}$ defined in [15]:

$$\xi_e(y) = \min\{t \in \mathbb{R} : y \in te - C\}. \tag{3.13}$$

Proposition 3.4. *ξ_e is well-defined since the minimum in (3.13) is attained.*

Proof. For any $y \in Y$, define $L := \{\lambda \in \mathbb{R} : y \in \lambda e - C\}$. It is enough to show that L is a closed and bounded from below set in $\mathbb{R}$.

Suppose that $\{\lambda_n\} \subset L$, and $\lambda_n \to \lambda^*$, as $n \to +\infty$. Then, $\lambda_n e - y \in C$, for every n and , by the closedness of C, it follows that $\lambda^* e - y \in C$, which proves that L is closed.

We prove that L is bounded from below. Ab absurdo, assume that there exist sequences $\{\lambda_n\} \subset L$ and $\{c_n\} \subset C$ such that $\lambda_n \to -\infty$, as $n \to +\infty$ and $y = \lambda_n e - c_n$, for every $n \in \mathbb{N}$. Then,

$$\frac{y}{\lambda_n} = e - \frac{c_n}{\lambda_n} \in e + C, \text{for } n \text{ sufficiently large.}$$

Taking the limit as $n \to +\infty$, we have $0 \in e + clC = e + C$. Then $e \in -C$, which contradicts the assumption that C is pointed, since $e \in C_0$. □

Remark 3.2. If $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$, $e \in int\,\mathbb{R}^\ell_+$, then:

$$\xi_e(y) = \max_{1\leq i\leq \ell} \frac{y_i}{e_i}. \tag{3.14}$$

We now define the separation function

$$w(u,v;e,\lambda) := \xi_e(u) + \gamma(v;\lambda), \tag{3.15}$$

Proposition 3.5. *Let $(e,\lambda) \in \mathscr{Q}_o \times \Xi$ and assume that $\gamma : \mathbb{R}^m \times \Xi \to \mathbb{R}$ fulfills the second condition in (3.7). Then w is a nonlinear separation function in the image space.*

Proof. We show that the first condition in (3.7) holds by proving that

$$\xi_e(u) > 0, \quad \forall u \in C_o, \ \forall e \in \mathscr{Q}_o. \tag{3.16}$$

Ab absurdo, assume that there exist $u \in C_o$ and $e \in \mathscr{Q}_o$ such that $\xi_e(u) \leq 0$, that is, there exist $\bar{t} \leq 0$ and $\bar{c} \in C$ such that

$$u = \bar{t}e - \bar{c}. \tag{3.17}$$

Since $\bar{t} \leq 0$, and C is a convex cone, then (3.17) implies that $u \in -C$, which is impossible, because C is pointed and $u \in C_o$. □

Remark 3.3. We observe that from (3.16) it follows that

$$w(u,v;e,\lambda) > 0, \quad \forall (u,v) \in \mathscr{H}_{C_o}.$$

4. Optimality Conditions

In this section, we will derive optimality conditions for VVI exploiting the separation scheme in the image space which allows us to consider a generalized Lagrangian function associated with VVI and defined by

$$\mathscr{L}_{x^*}(\theta,\lambda,x) = -\alpha(-f(x^*,x);\theta) - \gamma(g(x);\lambda), \tag{4.1}$$

where α and γ fulfill (3.7) and recalling that we have set $f(x^*,x) := \langle T(x^*), x-x^*\rangle$.

Theorem 4.1. *Let $x^* \in K$. If there exist $(\theta^*,\lambda^*) \in \mathscr{Q} \times \Xi$ such that $\mathrm{lev}_{>0}\alpha(\cdot;\theta^*) \supseteq C_o$, $\alpha(0;\theta^*) = 0$, $\gamma(g(x^*);\lambda^*) = 0$, and*

$$\mathscr{L}_{x^*}(\theta^*,\lambda^*,x^*) \leq \mathscr{L}_{x^*}(\theta^*,\lambda^*,x), \quad \forall x \in X, \tag{4.2}$$

then x^* is a solution of VVI.

Proof. By the assumptions, it follows from $f(x^*,x^*) = 0$ that $\mathscr{L}_{x^*}(\theta^*,\lambda^*,x^*) = 0$, and (4.2) is equivalent to

$$-\alpha(-f(x^*,x);\theta^*) - \gamma(g(x);\lambda^*) \geq 0, \quad \forall x \in X. \tag{4.3}$$

Assume now that x^* is not a solution of VVI. Then, there exists $\hat{x} \in K$ such that $-f(x^*,\hat{x}) \in C_0$. In view of $\mathrm{lev}_{>0}\alpha(\cdot;\theta^*) \supseteq C_o$, taking into account the second condition in (3.7) we obtain

$$-\alpha(-f(x^*,\hat{x});\theta^*) - \gamma(g(\hat{x});\lambda^*) < 0,$$

which contradicts (4.3). □

In particular, for separation function (3.15), we have that the assumptions of Theorem 4.1 are fulfilled. We have the following optimality conditions.

Theorem 4.2. *Let $x^* \in K$. If there exist $e^* \in \mathscr{Q}_o$ and $\lambda^* \in \Xi$ such that*

$$w(u,v;e^*,\lambda^*) \leq 0, \quad \forall (u,v) \in \mathscr{K}_{x^*}, \tag{4.4}$$

or, equivalently,

$$\xi_{e^*}(-f(x^*,x)) + \gamma(g(x);\lambda^*) \leq 0, \quad \forall x \in X, \tag{4.5}$$

then x^* is a solution of VVI.

Proof. We prove that the assumptions of Theorem 4.1, where we set $\alpha(u;e) := \xi_e(u)$ and $\mathscr{Q} = \mathscr{Q}_0$, are fulfilled. It follows from (3.16) that $\mathrm{lev}_{>0}\xi_{e^*} \supseteq C_o$. We now prove that $\xi_e(0) = 0$, for every $e \in C_0$. It is easy to see that $\xi_e(0) \leq 0$ for every $e \in C$. Assume that $\xi_e(0) < 0$ for some $e \in C_0$. Then, there exists $\bar{t} < 0$ and $\bar{c} \in C$ such that (3.17) holds with $u = 0$, i.e.,

$$\bar{c} = \bar{t}e \in -C_0,$$

which is impossible, because C is pointed.

Finally, we show that condition (4.5) implies (4.2), where we set

$$\mathscr{L}_{x^*}(e,\lambda,x) := -\xi_e(-f(x^*,x)) - \gamma(g(x);\lambda).$$

Indeed, (4.5) is equivalent to $-\mathscr{L}_{x^*}(e^*,\lambda^*,x) \leq 0, \forall x \in X$. Setting $x = x^*$ in (4.5), we obtain $\gamma(g(x^*);\lambda^*) \leq 0$. Since $x^* \in K$, we have $\gamma(g(x^*);\lambda^*) \geq 0$ which yields $\gamma(g(x^*);\lambda^*) = 0$. Then

$$\mathscr{L}_{x^*}(e^*,\lambda^*,x^*) = 0 \leq \mathscr{L}_{x^*}(e^*,\lambda^*,x), \quad \forall x \in X.$$

□

Remark 4.1. The optimality condition (4.4) can also be deduced from Remark 3.3. In fact, it follows from this remark that (4.4) implies that (3.3) is fulfilled, which yields the optimality of x^*.

In case that α and γ are linear functions and $\mathscr{Q} \times \Xi = C^* \times D^*$, we have that (4.1) collapses to the classic Lagrangian function

$$L_{x^*}(\theta, \lambda, x) = \langle \theta, f(x^*, x) \rangle - \langle \lambda, g(x) \rangle.$$

Definition 4.1. $(\theta^*, \lambda^*, x^*) \in C^* \times D^* \times X$, with $(\theta^*, \lambda^*) \neq 0$, is a *saddle point* for L_{x^*} on $D^* \times X$ if

$$L_{x^*}(\theta^*, \lambda, x^*) \leq L_{x^*}(\theta^*, \lambda^*, x^*) \leq L_{x^*}(\theta^*, \lambda^*, x), \quad \forall \lambda \in D^*, \ \forall x \in X. \tag{4.6}$$

Theorem 4.3. *$\mathscr{E}_{x^*}$ and $\mathscr{H}_{C_0}$ admit a linear separation and $x^* \in K$ if and only if there exist $\theta^* \in C^*$ and $\lambda^* \in D^*$ such that $(\theta^*, \lambda^*, x^*)$ is a saddle point for L_{x^*} on $D^* \times X$.*

From Proposition 3.2 (i), we obtain the following optimality condition.

Theorem 4.4. *If there exist $\theta^* \in qi(C^*)$ and $\lambda^* \in D^*$ such that $(\theta^*, \lambda^*, x^*)$ is a saddle point for L_{x^*} on $D^* \times X$, then x^* is a solution of VVI.*

If we assume that the image space is finite dimensional i.e., $Y = \mathbb{R}^\ell$, $Z = \mathbb{R}^m$, and that $-g$ is a differentiable $D-$convex function on X, then the saddle point optimality condition is equivalent to Karush-Kuhn-Tucker conditions.

Theorem 4.5. *$(\theta^*, \lambda^*, x^*)$ is a saddle point for L_{x^*} on $D^* \times X$ if and only if it is a solution of the following system:*

$$\begin{cases} \sum_{i=1}^{\ell} \theta_i T_i(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0, \\ \langle \lambda, g(x) \rangle = 0, \\ \theta \in C^*, \ \lambda \geq 0, \ (\theta, \lambda) \neq 0, \ g(x) \in D, \ x \in X. \end{cases} \tag{4.7}$$

Proof. (Only if) Assume that there exist $\theta^* \in C^*$ and $\lambda^* \in D^*$ such that $(\theta^*, \lambda^*, x^*)$ is a saddle point for L_{x^*} on $D^* \times X$, i.e.,

$$\begin{aligned} & \langle \theta^*, f(x^*, x^*) \rangle - \langle \lambda, g(x^*) \rangle \leq \langle \theta^*, f(x^*, x^*) \rangle - \langle \lambda^*, g(x^*) \rangle \\ \leq \ & \langle \theta^*, f(x^*, x) \rangle - \langle \lambda^*, g(x) \rangle, \forall (\lambda, x) \in D^* \times X. \end{aligned} \tag{4.8}$$

From the first inequality in (4.8) we have

$$-\langle \lambda, g(x^*) \rangle \leq -\langle \lambda^*, g(x^*) \rangle, \quad \forall \lambda \in D^*. \tag{4.9}$$

Notice that $D = (D^*)^*$, since D is a closed and convex cone in Z. Letting $\lambda := 0$ in (4.9) leads to $\langle \lambda^*, g(x^*) \rangle \leq 0$. We first prove that $x^* \in K$, i.e., $g(x^*) \in D$.

Ab absurdo, suppose that $g(x^*) \notin D = (D^*)^*$. Then there exists $\bar{\lambda} \in D^*$ such that $\langle \bar{\lambda}, g(x^*) \rangle < 0$. Since $t\bar{\lambda} \in D^*$ for any $t > 0$, then $-t\langle \bar{\lambda}, g(x^*) \rangle \to +\infty$, as $t \to +\infty$, which contradicts (4.9). Since $x^* \in K$, then $\langle \lambda^*, g(x^*) \rangle \geq 0$ and so $\langle \lambda^*, g(x^*) \rangle = 0$. By the second inequality in (4.8) it follows that

$$0 \leq \langle \theta^*, f(x^*, x) \rangle - \langle \lambda^*, g(x) \rangle, \quad \forall x \in X. \tag{4.10}$$

Since $f(x^*,x^*)=0$ and $\langle \lambda^*, g(x^*)\rangle = 0$, (4.10) yields that x^* is a global minimum point of $L_{x^*}(\theta^*,\lambda^*,x)$ on X. Being $f(x^*,\cdot)$ linear and g differentiable, the function $L_{x^*}(\theta^*,\lambda^*,\cdot)$ is differentiable and it follows that

$$0=\nabla_x L_{x^*}(\theta^*,\lambda^*,x^*)=\sum_{i=1}^{\ell}\theta_i^* T_i(x^*)-\sum_{i=1}^{m}\lambda_i^*\nabla g_i(x^*).$$

(If) Let (θ^*,λ^*,x^*) be a solution of system (4.7). Since $f(x^*,\cdot)$ is linear and $-g$ is D-convex on X, then $x\mapsto L_{x^*}(\theta^*,\lambda^*,x)=\langle\theta^*,f(x^*,x)\rangle-\langle\lambda^*,g(x)\rangle$ is convex on X. Now the first equality in (4.7) is equivalent to $\nabla_x L_{x^*}(\theta^*,\lambda^*,x^*)=0$ and thus

$$L_{x^*}(\theta^*,\lambda^*,x^*)\le L_{x^*}(\theta^*,\lambda^*,x),\quad \forall x\in X.$$

Since $f(x^*,x^*)=0$, then complementarity condition $\langle\lambda^*,g(x^*)\rangle=0$ and $x^*\in K$ yield

$$\begin{aligned} L_{x^*}(\theta^*,\lambda,x^*) &= \langle\theta^*,f(x^*,x^*)\rangle-\langle\lambda,g(x^*)\rangle\\ &\le 0\\ &= \langle\theta^*,f(x^*,x^*)\rangle-\langle\lambda^*,g(x^*)\rangle\\ &= L_{x^*}(\theta^*,\lambda^*,x^*),\quad \forall\lambda\in D^*. \end{aligned}$$

This shows that (θ^*,λ^*,x^*) is a saddle point for L_{x^*} on $D^*\times X$. □

Corollary 4.1. *If there exist $\theta^*\in int(C^*)$ and $\lambda^*\in\mathbb{R}^m$ such that (θ^*,λ^*,x^*) solves system (4.7), then x^* is a solution of VVI.*

Remark 4.2. Analogous optimality conditions for WVVI can be obtained by replacing in Theorem 4.4 the assumption $\theta^*\in qi(C^*)$ with $\theta^*\in C^*\setminus\{0\}$, and similarly in Corollary 4.1, recalling that $qi(C^*)=int(C^*)$ when Y is finite dimensional.

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