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BOUNDEDNESS AND APPROXIMATION OF THE CHANDRASEKHAR INTEGRAL OPERATORS IN L^p SPACES

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Abstract. In this paper, we study the solutions of a Fredholm integral equation in L^p spaces. First, the compatibility condition of the kernel function is adopted to extend the Hilbert type inequality in general case, and establish the existence and uniqueness results of the solutions to the Chandrasekhar type integral equation with parameters. Second, we discuss the compactness of integral operators in infinite intervals with the help of the truncation operator method, and study the convergence of the solutions to truncated Fredholm integral equations in the L^p spaces.

Keywords. Approximation solution; Chandrasekhar kernel; Fredholm integral equation; Fixed point theorem; Hilbert-type inequality.

1. INTRODUCTION

Integral equations are widely used in real-world problems such as diffraction problems, elastic problems and electrostatic problems [1]. Correspondingly, the initial value problems and boundary value problems of partial differential equations are also characterized by equivalent integral equations. For example, the potential energy theory of the Laplace operator [2] and the Green function theory of ordinary differential equations [3]. Over the years, many authors have extensively studied the approximate solutions of the one-dimensional Fredholm integral equations of second kind [4, 5, 6, 7, 8, 9, 10].

Among the classical approximation methods, the Galerkin method based on the projection sequence, and the Nyström method based on the digital orthogonality have typical significance. Sloan [11] first proposed the use of iterative techniques to improve Galerkin's method. Chandler [6] proved that if the kernel function and the right-hand side term are smooth, the iterative Galerkin solution of the orthogonal projection in the piecewise polynomial space has twice the convergence order of the general Galerkin solution. Chatelin and Lebbar [12] proved similar results in the iterative collocation method of Gauss points. As the smoothness of the right-hand

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side term of the operator equation is lower than the kernel function of the integral operator, Schock [13] and Sloan [14] proved that the Kantorovich solution has higher-order convergence than the Galerkin's solution. As an alternative to the iterative method, Lin, Zhang and Yan [15] proposed an interpolation post-processing method to improve the collocation solution. Hu [16] discussed the post-interpolation method of the Fredholm integro-differential equation.

Egidi and Maponi [17] considered the L^2 solution to the second kind of Fredholm integral equation with parameters in a finite interval

$$u(s) - \lambda \int_a^b K(s,t)u(t)dt = f(s), \quad a \leqslant s \leqslant b, \tag{1.1}$$

where $K : [a,b] \times [a,b] \to \mathbf{C}$ is an L^2 kernel function, $f : [a,b] \to \mathbf{C}$ is the L^2 right-hand side function, $\lambda \in \mathbf{C}$, and $u : [a,b] \to \mathbf{C}$ is the L^2 solution of the integral equation. The structure of the solution of the integral equation (1.1) can be discussed by the standard Fredholm theory [18], and its approximate solution can be obtained by a variety of numerical methods, such as, the collocation method, the Galerkin method, the Petrov-Galerkin method, the wavelet Petrov-Galerkin method and so on [2, 4, 19, 20]. For each $s,t \in [a,b]$, $K(s,t)$ is the pointwise limit of $K_N(s,t)$, where K_N is a finite rank kernel function. According to the Fredholm theory, we first replace K with K_N to get the solution u_N of the integral equation, and then set $N \to \infty$ to get the solution of Equation (1.1).

Egidi and Maponi [21] proposed a new recursive method to solve the integral equation (1.1). They generalized the well-known Sherman-Morrison formula and explicitly constructed the sequence of approximation functions, but they only used a few examples to illustrate the convergence of this approximation scheme in the case of real-valued functions. In [17], they extended this method to the case of complex-valued functions and gave a proof of the convergence of the approximation algorithm. Although the above two methods of solving approximate solutions are based on Fredholm's theory, they differ significantly. The solution u_N obtained by the finite rank approximation of the kernel function comes from a linear system, while the solution obtained by the Sherman-Morrison method is an iterative solution. When the calculation precision is specified, the recursive solution will be more direct and effective. In addition, Egidi and Maponi [17] also enumerated a typical application of integral equations. They approximated the boundary value problem of Laplace equations by the recursive method, and proved the convergence of the iterative sequence.

One of the main purposes of this paper is to continue the study of the L^p solution to the second type of Fredholm integral equation in the infinite interval

$$\varphi(x) = \psi(x) + \int_0^\infty k(x,y)\varphi(y)dy, \quad 0 < x < \infty,$$

where φ is the solution of the equation and ψ is the initial data in L^p space. As the kernel function $k(x,y)$ takes the special form $k(x,y) = \frac{\mu(x,y)}{\nu(x,y)}$, the second type of Fredholm integral equation becomes the linearized Chandrasekhar integral equation. In a recent research on Chandrasekhar integral equation [22, 23], the authors discussed the convergence of the equation in the L^2 space. This paper will generalize the results obtained previously and discuss the L^p solution of the Fredholm integral equation in infinite intervals. This paper includes two parts. The first part (Sections 2-4) studies the existence and uniqueness results of the L^p solution of the Chandrasekhar-type integral equation with parameters. The second part (Sections 5-7) studies the convergence of the general Fredholm integral equation in the L^p space. More precisely, this paper is organized as follows. In Section 2, we briefly review the classic results of the research on Hilbert-type inequalities. In Section 3, we establish the existence and uniqueness results of the solution to the Chandrasekhar-type integral equation with symmetric kernels by introducing the compatibility condition of the kernel function. Section 4 lists several examples, which support the conditions proposed in Section 3. Section 5 studies the compactness of integral operators and truncation operators in the L^p spaces. In Sections 6 and 7, we discuss the convergence of the approximating solution of the Chandrasekhar-type integral equation as the initial data and the kernel function both have polynomial and exponential decay in the L^p spaces. In astrophysics, these two situations correspond to the radiation effects of near-Earth and far-Earth planets respectively.

2. Hilbert-type Inequalities

In this section, we recall some basic facts on Hilbert type inequalities. The work originates from D. Hilbert, G. Hardy and followed by many other mathematicians during the last 110 years. We refer to readers to [24, 25] w for an excellent exposition of this direction.

2.1. **Hilbert inequalities.** For any $a = \{a_m\}_{m=1}^{\infty} \in l^2$ with the norm $\|a\|_2 = \left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}$, the *Hilbert operator* $L : l^2 \to l^2$ is defined by $c = \{c_n\}_{n=1}^{\infty} \in l^2$ with

$$c_n = (La)(n) = \sum_{m=1}^{\infty} \frac{a_m}{m+n}, \qquad n \in \mathbb{N}.$$

For $b = \{b_n\}_{n=1}^{\infty} \in l^2$, the inner product is given by

$$(La, b) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right) b_n.$$

The *Hilbert inequality* [26] follows that

$$|(La, b)| < \pi \|a\|_2 \|b\|_2$$

with $\|a\|_2 > 0$ and $\|b\|_2 > 0$, asserting that L is bounded on l^2 and its operator norm is $\|L\|_2 = \pi$. Equivalently, $\|La\|_2 < \pi \|a\|_2$, that is,

$$\left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^2 \right\}^{\frac{1}{2}} < \pi \left(\sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{2}}.$$

Similarly, for $f \in L^2(0,\infty)$ with norm $\|f\|_2 = \left(\int_0^{\infty} |f^2(x)|^2 dx\right) \frac{1}{2}$ in real Hilbert space $L^2(0,\infty)$, one defines *Hilbert integral operator* T on $L^2(0,\infty)$ into itself by

$$(Tf)(y) = \int_0^{\infty} \frac{f(x)}{x+y} dx, \qquad y \in (0,\infty).$$

For $g \in L^2(0,\infty)$, one also defines inner product

$$(Tf, g) = \int_0^{\infty} \left(\int_0^{\infty} \frac{f(x)}{x+y} dx \right) g(y) dy = \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy.$$

Given $\|f\|_2 > 0$ and $\|g\|_2 > 0$, *Hilbert integral inequality* [27] follows that

$$|(Tf,g)| < \pi \|f\|_2 \|g\|_2,$$

or $\|Tf\|_2 < \pi\|f\|_2$, that is,

$$\left\{\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^2 dy\right\}^{\frac{1}{2}} < \pi\left(\int_0^\infty |f(x)|^2 dx\right)^{\frac{1}{2}},$$

and the operator norm $\|T\|_2 = \pi$ is the best bound for the above Hilbert integral inequality.

2.2. **Hardy-Hilbert inequalities.** Let l^p be a real Banach space. For $a = \{a_m\}_{m=0}^\infty \in l^p$ with its norm $\|a\|_p = (\sum_{n=0}^\infty |a_n|^p)^{\frac{1}{p}}$, one defines *Hardy-Hilbert operator* on l^p by $\{(La)(n)\}_{n=0}^\infty \in l^p$ with

$$(La)(n) = \sum_{m=0}^\infty \frac{a_m}{m+n+1}, \qquad n \geqslant 0.$$

For $b = \{b_n\}_{n=0}^\infty \in l^{p'}$, the functional operation of La and b is given by

$$(La,b) = \sum_{n=0}^\infty \left(\sum_{m=0}^\infty \frac{a_m}{m+n+1}\right) b_n = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{m+n+1}.$$

The *Hardy-Hilbert inequality* [28, 29] follows that

$$|(La,b)| < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|a\|_p \|b\|_{p'},$$

where $\|a\|_p > 0$ and $\|b\|_{p'} > 0$.

Similarly, one introduces *Hardy-Hilbert integral operator* on $L^p(0,\infty)$. For any $f \in L^p(0,\infty)$ with norm $\|f\|_p = (\int_0^\infty |f(x)|^p dx)^{\frac{1}{p}}$, one defines $Tf \in L^p(0,\infty)$ by

$$(Tf)(y) = \int_0^\infty \frac{f(x)}{x+y}dx, \qquad y \in (0,\infty).$$

Given $g \in L^{p'}(0,\infty)$, the functional operation

$$(Tf,g) = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy$$

reads the *Hardy-Hilbert integral inequality* as

$$|(Tf,g)| < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|f\|_p \|g\|_{p'},$$

where $\|f\|_p > 0$ and $\|g\|_{p'} > 0$.

2.3. **A non-conjugate exponent case.** In this subsection, we collect some results on non-conjugate exponents (p,q) and a parameter λ.

If $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}\geqslant 1$, $0<\lambda=2-\frac{1}{p}+\frac{1}{q}\leqslant 1$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}\leqslant K\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}|b_n|^q\right)^{\frac{1}{q}},$$

where $K=K(p,q)$ is sharp if and only if $\frac{1}{p}+\frac{1}{q}=1$, and $\lambda=2-\frac{1}{p}+\frac{1}{q}=1$.

By the same condition, the integral analog also holds,

$$\int_0^{\infty}\int_0^{\infty}\left|\frac{f(x)g(y)}{(x+y)^{\lambda}}\right|dxdy<K\left(\int_0^{\infty}|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}|g(x)|^{p'}\,dy\right)^{\frac{1}{p'}},$$

and moreover, if $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}>1$, $0<\lambda=2-\frac{1}{p}+\frac{1}{q}<1$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{f(x)g(y)}{(x+y)^{\lambda}}\right|dxdy<K\left(\int_{-\infty}^{\infty}|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|g(x)|^{p'}\,dy\right)^{\frac{1}{p'}}.$$

Levin [30] compared different formulations of the bounds in above inequalities , but he could hardly prove the optimality of the bounds. Bonsall [31] discusses the functional inequalities to the general kernels with non-conjugate exponents and parameters.

Remark 2.1. The norms of Hilbert-type integral operators could be exactly estimated, casting a light on the sharp bound of the parameter λ in the linear Fredholm integral equation to ensure the existence of the solutions. Moreover, one may extend the basic models to a relatively wide class of integral kernels.

3. L^p Solutions $(1<p<\infty)$

In this section, we consider the L^p solutions $(1<p<\infty)$ to the linear Fredholm integral equation of the second kind

$$\varphi(y)=\psi(y)+\lambda\int_0^{\infty}k(x,y)\varphi(x)dx,\qquad 0<y<\infty,\tag{3.1}$$

where $\psi\in L^p(0,\infty)$. A boundedness result of the integral operator is provided by Theorem 1 of [32], which guarantees the existence and uniqueness of the solution to the Chandrasekhar-type integral equation. On the premise that the parameters of the equation are not required to accurately reach the optimal range, one can put forward more general conditions.

Let the integral kernel $k(x,y)=k(y,x)$ be symmetric and nonnegative almost every in $(0,\infty)\times(0,\infty)$. For $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, taking $f\in L^p(0,\infty)$ and $g\in L^{p'}(0,\infty)$, one defines

$$Kf(y):=\int_0^{\infty}k(x,y)f(x)dx,\qquad y\in(0,\infty),$$

or equivalently

$$Kg(x):=\int_0^{\infty}k(x,y)g(y)dy,\qquad x\in(0,\infty).$$

For any $\varepsilon\geqslant 0$ and $x>0$, we define

$$k_{\varepsilon}(r,x):=\int_0^{\infty}k(x,y)\omega(x,y)^{\frac{1+\varepsilon}{r}}dy,\qquad r=p\ \text{or}\ p'.$$

It is obvious that $\omega(x,y)=\frac{x}{y}$ satisfies $\omega(x,y)\omega(y,x)=1$, which is called the *compatibility condition* of the convergence factor. In fact, it inspires us to establish a boundedness result of the integral operator K by means of this general condition.

Lemma 3.1. *Suppose that $k_0(p)=k_0(p,x)=\int_0^\infty k(x,y)\omega(x,y)^{\frac{1}{p}}dy$ is independent of $x>0$ and $k_0(p)=k_0(p')$, $\omega(x,y)$ satisfies the compatibility condition $\omega(x,y)\omega(y,x)=1$. Then K is a continuous linear operator mapping $L^p(0,\infty)$ (and $L^{p'}(0,\infty)$) into itself with $\|K\|_{L^p\to L^p}=\|K\|_{L^{p'}\to L^{p'}}\leqslant k_0(p)$.*

Proof. Without loss of generality, let f be nonnegative, otherwise one may replace f by $|f|$. Making use of Hölder's inequality, one has

$$\begin{aligned}
&\left(\int_0^\infty k(x,y)f(x)dx\right)^p\\
&=\left\{\int_0^\infty\left[k(x,y)^{\frac{1}{p}}\omega(x,y)^{\frac{1}{pp'}}f(x)\right]\left[k(y,x)^{\frac{1}{p'}}\omega(y,x)^{\frac{1}{pp'}}\right]dx\right\}^p\\
&\leqslant\left\{\int_0^\infty k(x,y)\omega(x,y)^{\frac{1}{p'}}f(x)^pdx\right\}\left\{\int_0^\infty k(y,x)\omega(y,x)^{\frac{1}{p}}dx\right\}^{p-1}\\
&=(k_0(p))^{p-1}\int_0^\infty k(x,y)\omega(x,y)^{\frac{1}{p'}}f(x)^pdx.
\end{aligned}\tag{3.2}$$

Hence,

$$\begin{aligned}
\|Kf\|_{L^p}&=\left\{\int_0^\infty\left(\int_0^\infty k(x,y)f(x)dx\right)^pdy\right\}^{\frac{1}{p}}\\
&\leqslant(k_0(p))^{\frac{1}{p'}}\left\{\int_0^\infty\int_0^\infty k(x,y)\omega(x,y)^{\frac{1}{p'}}f(x)^pdxdy\right\}^{\frac{1}{p}}\\
&=(k_0(p))^{\frac{1}{p'}}\left\{\int_0^\infty\left[\int_0^\infty k(x,y)\omega(x,y)^{\frac{1}{p'}}dy\right]f(x)^pdx\right\}^{\frac{1}{p}}\\
&=(k_0(p))^{\frac{1}{p'}}\left(k_0(p')\right)^{\frac{1}{p}}\|f\|_{L^p}.
\end{aligned}\tag{3.3}$$

Therefore, $Kf\in L^p(0,\infty)$ and $\|K\|_{L^p\to L^p}\leqslant(k_0(p))^{\frac{1}{p'}}(k_0(p'))^{\frac{1}{p}}=k_0(p)$. Similarly, the integral operator K maps $L^{p'}(0,\infty)$ into itself, and moreover, it follows that $Kg\in L^{p'}(0,\infty)$ and $\|K\|_{L^{p'}\to L^{p'}}\leqslant(k_0(p))^{\frac{1}{p'}}(k_0(p'))^{\frac{1}{p}}=k_0(p)$ holds for any $g\in L^{p'}(0,\infty)$. □

Remark 3.1. Since the integral kernel of the (linearized) Chandrasekhar equation is expressed as the ratio of the characteristic function to the symbolic function, we are naturally interested in the corresponding special form of the convergence factor. Let $\omega(x,y)=\frac{\omega_1(x,y)}{\omega_2(x,y)}$, where $\omega_j>0$, $j=1,2$. Now the compatibility condition $\omega(x,y)\omega(y,x)=1$ is equivalent to $\omega_1(x,y)\omega_1(y,x)=\omega_2(x,y)\omega_2(y,x)$. In particular, $\omega_1(x,y)=\omega_2(y,x)$ (or equivalently $\omega_1(y,x)=\omega_2(x,y)$) fulfills the compatibility condition. Denoting $\omega_1=\Omega$, one hence puts $\omega(x,y)=\frac{\Omega(x,y)}{\Omega(y,x)}$. Generally, the L^p norm of the operator K cannot reach $k_0(p)$. In fact, if it does hold true, $\Omega(x,y)$ will need to be controlled by a single variable function. Specifically, there exist some nonnegative $\Phi_j\in L^p(0,\infty)$ $(j=1,2)$ and a constant $\gamma\geqslant1$ such that $\Omega(x,y)^{-\frac{\gamma}{p}}\leqslant\Phi_1(x)$, $\Omega(y,x)^{-\frac{\gamma}{p}}\geqslant\Phi_2(y)$ and $\|\Phi_1\|_{L^p}\leqslant\|\Phi_2\|_{L^p}$, where Φ_j's $(j=1,2)$ are independent of γ. Thus, $\Phi_1=\Phi_2$ almost everywhere in $(0,\infty)$ and consequently the convergence factor takes the form $\omega(x,y) = \frac{\sigma(x)}{\sigma(y)}$ with some measurable function $\sigma > 0$.

For the particular case, we will be able to further exploit the properties for the norm of the integral operator as follows.

Corollary 3.1. *Suppose that the convergence factor $\omega(x,y) = \frac{\sigma(x)}{\sigma(y)}$ with $\sigma > 0$. If $k_\varepsilon(p) = k_\varepsilon(p,x)$ is independent of $x > 0$, $k_\varepsilon(p) = k_0(p) + o(1)$ $(\varepsilon \to 0^+)$, $k_0(p) = k_0(p')$, and $\sigma^{-\frac{\gamma}{p}} \in L^p(0,\infty)$ holds for any $\gamma > 1$, then K is a continuous linear operator mapping $L^p(0,\infty)$ (and $L^{p'}(0,\infty)$) into itself with $\|K\|_{L^p\to L^p} = k_0(p)$.*

Proof. It suffices to prove $\|K\|_{L^p\to L^p} \geqslant k_0(p)$. By condition, it follows that $\tau_\varepsilon(x) := \sigma(x)^{-\frac{1+\varepsilon}{p}} \in L^p(0,\infty)$ holds for any $\varepsilon > 0$. Denoting $f_\varepsilon(x) = \frac{\tau_\varepsilon(x)}{\|\tau_\varepsilon\|_{L^p}}$ and noticing that the kernel function is symmetric $k(x,y) = k(y,x)$, one has $\|f_\varepsilon\|_{L^p} = 1$ and

$$\begin{aligned}
\|K\|_{L^p\to L^p} &\geqslant \|Kf_\varepsilon\|_{L^p} \\
&= \left\{\int_0^\infty \left(\int_0^\infty k(x,y) f_\varepsilon(x)dx\right)^p dy\right\}^{\frac{1}{p}} \\
&= \frac{1}{\|\tau_\varepsilon\|_{L^p}} \left\{\int_0^\infty \left(\int_0^\infty k(x,y)\sigma(x)^{-\frac{1+\varepsilon}{p}} dx\right)^p dy\right\}^{\frac{1}{p}} \\
&= \frac{1}{\|\tau_\varepsilon\|_{L^p}} \left\{\int_0^\infty \sigma(y)^{-1-\varepsilon} \left(\int_0^\infty k(y,x) \left(\frac{\sigma(y)}{\sigma(x)}\right)^{\frac{1+\varepsilon}{p}} dx\right)^p dy\right\}^{\frac{1}{p}} \\
&= \frac{1}{\|\tau_\varepsilon\|_{L^p}} \left\{\int_0^\infty \tau_\varepsilon(y)^p dy\right\}^{\frac{1}{p}} \cdot k_\varepsilon(p) \\
&= k_\varepsilon(p) \longrightarrow k_0(p),
\end{aligned}$$

as $\varepsilon \to 0$. It means that $\|K\|_{L^p\to L^p} \geqslant k_0(p)$. Hence, combining with the conclusion $\|K\|_{L^p\to L^p} \leqslant k_0(p)$ of Lemma 3.1, one obtains that $\|K\|_{L^p\to L^p} = k_0(p)$. ☐

Remark 3.2. Once $\sigma(x) \sim x^\alpha$ as $x \gg 1$ for some $\alpha \geqslant 1$, it follows that $\|K\|_{L^p\to L^p} = k_0(p)$. For $\alpha = 1$, one may consult [22, 23, 33]. For $\alpha > 1$, the authors show that it is still valid in the very recent work and one may find some $f \in L^p(0,\infty)$ such that $\|f\|_{L^p} = 1$ and $\|Kf\|_{L^p} = k_0(p) = \|K\|_{L^p\to L^p}$.

Let us now consider linear Fredholm integral equation (3.4). One would put it in the previous notations as

$$\varphi = \psi + \lambda K\varphi.$$

Define an operator T from $L^p(0,\infty)$ into itself by

$$T\varphi = \psi + \lambda K\varphi.$$

Noticing that

$$\|T\varphi_1 - T\varphi_2\|_{L^p} = |\lambda|\|K(\varphi_1-\varphi_2)\|_{L^p} \leqslant |\lambda|k_0(p)\|\varphi_1-\varphi_2\|_{L^p},$$

one claims that T is a contraction operator if $|\lambda|k_0(p) < 1$. By using the contraction mapping principle, one reaches the following result.

Theorem 3.1. *For the linear Fredholm integral equation*

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty k(x,y)\varphi(x)dx, \quad 0 < y < \infty, \tag{3.4}$$

if the kernel $k(x,y)$ is symmetric and nonnegative almost every in $(0,\infty)\times(0,\infty)$ and fulfills the condition in Lemma 3.1, i.e. $k_\varepsilon(p) = k_\varepsilon(r,x)$ ($r = p$ or p') is independent of $x > 0$ and $k_\varepsilon(p) = k_0(p) + o(1)$ ($\varepsilon \to 0^+$), then there exists the unique solution in $\overline{\varphi} \in L^p(0,\infty)$ of linear Fredholm integral equation (3.4) as long as $|\lambda| < \frac{1}{k_0(p)}$.

It is worth noting that if the norm of the operator in (3.4) does not reach $k_0(p)$, then one cannot guarantee that the range of the parameter λ in the equation is optimal.

4. Sharp Bounds

To the linear Fredholm integral equation of the second kind

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty k(x,y)\varphi(x)dx, \quad 0 < y < \infty,$$

taking $k(x,y) = \frac{\mu(x,y)}{\nu(x,y)}$, one calls it *Chandrasekhar integral equation* or *Fredholm integral equation with Chandrasekhar kernel*. The prototype of Chandrasekhar's equation corresponds to $\mu(x,y) = 1$ and $\nu(x,y) = x+y$.

In this section, we illustrate several examples to support the conditions of Corollary 3.1. For more applications, one needs to handle two kinds of Eulerian integrals, Beta function (First kind of Eulerian integral) and Gamma function (Second kind of Eulerian integral). For $p > 0$, $q > 0$ and $s > 0$, one defines Beta function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx,$$

and Gamma function

$$\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx.$$

The two kinds of Eulerian integrals are related by

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (p>0,\ q>0), \tag{4.1}$$

and hold some useful equalities

$$B(p,q) = \int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}} = \int_0^1 \frac{t^{p-1}+t^{q-1}}{(1+t)^{p+q}} = B(q,p) \qquad (p>0,\ q>0), \tag{4.2}$$

$$\frac{\Gamma(a+1)}{(b+1)^{a+1}} = \int_0^1 (-\ln u)^a u^b du \qquad (a>-1,\ b>-1), \tag{4.3}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(s\pi)} \qquad (0<s<1). \tag{4.4}$$

For the proof and more properties of Eulerian integrals, one refers to [34].

Example 4.1. $k(x,y) = \frac{1}{|x^\alpha - y^\alpha|^{\frac{1}{\alpha}}} \qquad (\alpha > 1).$

Putting $u=\frac{y}{x}$, $u=\frac{x}{y}$ and $v=\frac{1}{u}$, one observes that

$$\begin{aligned}
k_\varepsilon(p,x) &= \int_0^\infty \frac{1}{|x^\alpha-y^\alpha|^{\frac{1}{\alpha}}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}}dy\\
&= \int_0^x \frac{1}{(x^\alpha-y^\alpha)^{\frac{1}{\alpha}}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}}dy+\int_x^\infty \frac{1}{(y^\alpha-x^\alpha)^{\frac{1}{\alpha}}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}}dy\\
&= \int_0^1 \frac{1}{(1-u^\alpha)^{\frac{1}{\alpha}}}u^{-\frac{1+\varepsilon}{p}}du+\int_0^1 \frac{1}{(1-u^\alpha)^{\frac{1}{\alpha}}}u^{\frac{1+\varepsilon}{p}-1}du\\
&\xrightarrow{\varepsilon\to 0^+} \int_0^1 \frac{1}{(1-u^\alpha)^{\frac{1}{\alpha}}}\left(u^{-\frac{1}{p}}+u^{-\frac{1}{p'}}\right)du\\
&= \frac{1}{\alpha}\int_0^1 (1-v)^{-\frac{1}{\alpha}}\left(v^{\frac{1}{p\alpha}-1}+v^{\frac{1}{p'\alpha}-1}\right)du\\
&= \frac{1}{\alpha}\left[B\left(\frac{1}{p\alpha},1-\frac{1}{\alpha}\right)+B\left(\frac{1}{p'\alpha},1-\frac{1}{\alpha}\right)\right]\\
&= k_0(p)=k_0(p').
\end{aligned}$$

In virtue of Corollary 3.1 and Theorem 3.1, one has

$$\left\{\int_0^\infty\left|\int_0^\infty \frac{f(x)}{(x^\alpha-y^\alpha)^{\frac{1}{\alpha}}}dx\right|^p dy\right\}^{\frac{1}{p}} \leqslant \frac{1}{\alpha}\left[B\left(\frac{1}{p\alpha},1-\frac{1}{\alpha}\right)+B\left(\frac{1}{p'\alpha},1-\frac{1}{\alpha}\right)\right]\|f\|_{L^p},$$

and the linear Chandrasekhar integral equation

$$\varphi(y)=\psi(y)+\lambda\int_0^\infty \frac{\varphi(x)}{(x^\alpha-y^\alpha)^{\frac{1}{\alpha}}}dx$$

has a unique solution in $\overline{\varphi}\in L^p(0,\infty)$ as long as

$$|\lambda|<\left\{\frac{1}{\alpha}\left[B\left(\frac{1}{p\alpha},1-\frac{1}{\alpha}\right)+B\left(\frac{1}{p'\alpha},1-\frac{1}{\alpha}\right)\right]\right\}^{-1}.$$

In particular, taking $p=p'=2$, one has

$$\left\{\int_0^\infty\left|\int_0^\infty \frac{f(x)}{(x^\alpha-y^\alpha)^{\frac{1}{\alpha}}}dx\right|^2 dy\right\}^{\frac{1}{2}} \leqslant \frac{2}{\alpha}B\left(\frac{1}{2\alpha},1-\frac{1}{\alpha}\right)\|f\|_{L^2},$$

and the linear Chandrasekhar integral equation

$$\varphi(y)=\psi(y)+\lambda\int_0^\infty \frac{\varphi(x)}{(x^\alpha-y^\alpha)^{\frac{1}{\alpha}}}dx$$

has a unique solution in $\overline{\varphi}\in L^2(0,\infty)$ as long as $|\lambda|<\left\{\frac{2}{\alpha}B\left(\frac{1}{2\alpha},1-\frac{1}{\alpha}\right)\right\}^{-1}$.

Example 4.2. $k(x,y)=\frac{|\ln(x)-\ln(y)|^\beta}{\max\{x,y\}}\qquad(\beta>-1).$

Putting $u=\frac{y}{x}$, $u=\frac{x}{y}$ and $v=-\ln(u)$, one observes that

$$\begin{aligned}
k_\varepsilon(p,x) &= \int_0^\infty \frac{|\ln(x)-\ln(y)|^\beta}{\max\{x,y\}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}} dy \\
&= \int_0^x \frac{|\ln(x)-\ln(y)|^\beta}{x}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}} dy + \int_x^\infty \frac{|\ln(x)-\ln(y)|^\beta}{y}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}} dy \\
&= \int_0^1 (-\ln(u))^\beta u^{-\frac{1+\varepsilon}{p}} du + \int_0^1 (-\ln(u))^\beta u^{\frac{1+\varepsilon}{p}-1} du \\
&\xrightarrow{\varepsilon\to 0^+} \int_0^1 (-\ln(u))^\beta \left(u^{-\frac{1}{p}}+u^{-\frac{1}{p'}}\right) du \\
&= \int_0^\infty v^\beta e^{-\frac{v}{p}} dv + \int_0^\infty v^\beta e^{-\frac{v}{p'}} dv \\
&= \Gamma(\beta+1)\left[p^{\beta+1}+(p')^{\beta+1}\right] \\
&= k_0(p) = k_0(p').
\end{aligned}$$

In virtue of Corollary 3.1 and Theorem 3.1, one has

$$\begin{aligned}
&\left\{\int_0^\infty \left|\int_0^\infty \frac{|\ln(x)-\ln(y)|^\beta}{\max\{x,y\}} f(x)dx\right|^p dy\right\}^{\frac{1}{p}} \\
&\leqslant \Gamma(\beta+1)\left[p^{\beta+1}+(p')^{\beta+1}\right]\left\{\int_0^\infty |f(x)|^p\, dx\right\}^{\frac{1}{p}},
\end{aligned}$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{|\ln(x)-\ln(y)|^\beta}{\max\{x,y\}}\varphi(x)dx$$

has a unique solution in $\overline{\varphi}\in L^p(0,\infty)$ as long as

$$|\lambda| < \left\{\Gamma(\beta+1)\left[p^{\beta+1}+(p')^{\beta+1}\right]\right\}^{-1}.$$

Taking $\beta=1$, one has

$$\left\{\int_0^\infty \left|\int_0^\infty \frac{|\ln(x)-\ln(y)|}{\max\{x,y\}} f(x)dx\right|^p dy\right\}^{\frac{1}{p}} \leqslant \left[p^2+(p')^2\right]\left\{\int_0^\infty |f(x)|^p\, dx\right\}^{\frac{1}{p}},$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{|\ln(x)-\ln(y)|}{\max\{x,y\}}\varphi(x)dx$$

has a unique solution in $\overline{\varphi}\in L^p(0,\infty)$ as long as $|\lambda| < \frac{1}{p^2+(p')^2}$. In particular, $p=p'=2$ and $\beta=1$ lead to

$$\left\{\int_0^\infty \left|\int_0^\infty \frac{|\ln(x)-\ln(y)|}{\max\{x,y\}} f(x)dx\right|^2 dy\right\}^{\frac{1}{2}} \leqslant 8\left\{\int_0^\infty |f(x)|^2\, dx\right\}^{\frac{1}{2}},$$

and the linear Chandrasekhar integral equation

$$\varphi(y)=\psi(y)+\lambda\int_0^\infty \frac{|\ln(x)-\ln(y)|}{\max\{x,y\}}\varphi(x)dx$$

has a unique solution in $\overline{\varphi}\in L^2(0,\infty)$ as long as $|\lambda|<\frac{1}{8}$.

Example 4.3. $k(x,y)=\frac{|\ln(x)-\ln(y)|^\beta}{(x+y)^{1-\alpha}(\min\{x,y\})^\alpha}$ $\quad(\beta>-1,\ \alpha<\min\left\{\frac{1}{p},\frac{1}{p'}\right\})$.

Putting $u=\frac{y}{x}$, $u=\frac{x}{y}$ and making use of Eq. (4.3), one observes that

$$\begin{aligned}
k_\varepsilon(p,x)&=\int_0^\infty \frac{|\ln(x)-\ln(y)|^\beta}{(x+y)^{1-\alpha}(\min\{x,y\})^\alpha}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}}dy\\
&=\int_0^x \frac{|\ln(x)-\ln(y)|^\beta}{(x+y)^{1-\alpha}y^\alpha}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}}dy+\int_x^\infty \frac{|\ln(x)-\ln(y)|^\beta}{(x+y)^{1-\alpha}x^\alpha}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{p}}dy\\
&=\int_0^1 \frac{(-\ln(u))^\beta}{(1+u)^{1-\alpha}}u^{-\alpha-\frac{1+\varepsilon}{p}}du+\int_0^1 \frac{(-\ln(u))^\beta}{(1+u)^{1-\alpha}}u^{-\alpha+\frac{1+\varepsilon}{p}-1}du\\
&\xrightarrow{\varepsilon\to 0^+}\int_0^1 \frac{(-\ln(u))^\beta}{(1+u)^{1-\alpha}}\left(u^{-\alpha-\frac{1}{p}}+u^{-\alpha-\frac{1}{p'}}\right)du\\
&=\int_0^1 (-\ln(u))^\beta(1+u)^{\alpha-1}\left(u^{-\alpha-\frac{1}{p}}+u^{-\alpha-\frac{1}{p'}}\right)du\\
&=\int_0^1 (-\ln(u))^\beta\sum_{k=0}^\infty C_{\alpha-1}^k\left(u^{k-\alpha-\frac{1}{p}}+u^{k-\alpha-\frac{1}{p'}}\right)du\\
&=\sum_{k=0}^\infty C_{\alpha-1}^k\int_0^1 (-\ln(u))^\beta\left(u^{k-\alpha-\frac{1}{p}}+u^{k-\alpha-\frac{1}{p'}}\right)du\\
&=\Gamma(\beta+1)\sum_{k=0}^\infty C_{\alpha-1}^k\left[\frac{1}{\left(k-\alpha+\frac{1}{p}\right)^{\beta+1}}+\frac{1}{\left(k-\alpha+\frac{1}{p'}\right)^{\beta+1}}\right]\\
&=k_0(p)=k_0(p').
\end{aligned}$$

In virtue of Corollary 3.1 and Theorem 3.1, one has

$$\left\{\int_0^\infty\left|\int_0^\infty \frac{|\ln(x)-\ln(y)|^\beta}{(x+y)^{1-\alpha}(\min\{x,y\})^\alpha}f(x)dx\right|^p dy\right\}^{\frac{1}{p}}$$
$$\leqslant\Gamma(\beta+1)\sum_{k=0}^\infty C_{\alpha-1}^k\left[\frac{1}{\left(k-\alpha+\frac{1}{p}\right)^{\beta+1}}+\frac{1}{\left(k-\alpha+\frac{1}{p'}\right)^{\beta+1}}\right]\left\{\int_0^\infty |f(x)|^p dx\right\}^{\frac{1}{p}},$$

and the linear Chandrasekhar integral equation

$$\varphi(y)=\psi(y)+\lambda\int_0^\infty \frac{|\ln(x)-\ln(y)|^\beta}{(x+y)^{1-\alpha}(\min\{x,y\})^\alpha}\varphi(x)dx$$

exists a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as

$$|\lambda| < \left\{ \Gamma(\beta+1) \sum_{k=0}^{\infty} C_{\alpha-1}^{k} \left[\frac{1}{\left(k-\alpha+\frac{1}{p}\right)^{\beta+1}} + \frac{1}{\left(k-\alpha+\frac{1}{p'}\right)^{\beta+1}} \right] \right\}^{-1}.$$

In particular, taking $\beta = 0$, one has

$$k_0(p) = \sum_{k=0}^{\infty} C_{\alpha-1}^{k} \frac{2(k-\alpha)+1}{\left(k-\alpha+\frac{1}{p}\right)\left(k-\alpha+\frac{1}{p'}\right)}.$$

Hence,

$$\left\{ \int_0^\infty \left| \int_0^\infty \frac{f(x)}{(x+y)^{1-\alpha}(\min\{x,y\})^{\alpha}} dx \right|^p dy \right\}^{\frac{1}{p}}$$

$$\leqslant \sum_{k=0}^{\infty} C_{\alpha-1}^{k} \frac{2(k-\alpha)+1}{\left(k-\alpha+\frac{1}{p}\right)\left(k-\alpha+\frac{1}{p'}\right)} \left\{ \int_0^\infty |f(x)|^p dx \right\}^{\frac{1}{p}},$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{(x+y)^{1-\alpha}(\min\{x,y\})^{\alpha}} dx$$

has a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as

$$|\lambda| < \left\{ \sum_{k=0}^{\infty} C_{\alpha-1}^{k} \frac{2(k-\alpha)+1}{\left(k-\alpha+\frac{1}{p}\right)\left(k-\alpha+\frac{1}{p'}\right)} \right\}^{-1}.$$

Remark 4.1. The particular case ($\beta = 0$) of Example 4.3 generalizes the classical Chandrasekhar kernel, replacing $(x+y)^{-1}$ by $\left[(x+y)^{1-\alpha}(\min\{x,y\})^{\alpha}\right]^{-1}$. The norm of the integral operator

$$Kf(y) = \int_0^\infty \frac{f(x)}{(x+y)^{1-\alpha}(\min\{x,y\})^{\alpha}} dx$$

defined on $L^p(0,\infty)$ equals to

$$\|K\|_{L^p \to L^p} = \sum_{k=0}^{\infty} C_{\alpha-1}^{k} \frac{2(k-\alpha)+1}{\left(k-\alpha+\frac{1}{p}\right)\left(k-\alpha+\frac{1}{p'}\right)}.$$

5. The Truncation Operator to the Linearized Chandrasekhar Equation

As the second part of this paper, this section and the subsequent two sections study the approximate solutions to the integral equations of Chandrasekhar type. It needs to be pointed out that the kernel function of the integral operator under consideration does not have to be symmetric, and the equation no longer contains parameters to ensure the existence of the solution of the equation. For integral operators in infinite intervals, the main effort will be devoted to overcoming the non-compactness of integral operators. Anselone [35] utilized the truncation operator in the finite interval to approximate the original operator. It turns out to be an effective method; see, e.g., [22, 23, 36] and the references therein.

In this section, we exploit the properties of the truncation operator in the $L^p[0,\infty)$ space. In detail, we consider the approximating solutions to the following integral equation in $L^p[0,\infty)$,

$$\varphi(x) = \psi(x) + \int_0^\infty k(x,y)\varphi(y)dy, \quad 0 < x < \infty. \tag{5.1}$$

For simplicity, it is shortly rewritten by

$$\varphi = \psi + K\varphi, \tag{5.2}$$

where

$$K\varphi(x) := \int_0^\infty k(x,y)\varphi(y)dy$$

and $k(x,y)$ is a measurable real-valued function on $[0,\infty)\times[0,\infty)$. Apparently, if $k(x,y)$ is symmetric, then Eq. (5.1) is in the exact form of Eq. (3.4).

One introduces a bilateral truncation operator

$$K_T = M_{[0,T]} K M_{[0,T]},$$

where M_D is defined by

$$M_D\varphi(x) = \chi_D(x)\varphi(x)$$

for any $\varphi \in L^p[0,\infty)$ and χ_D is the characteristic function of a subset D. The integral equation associated to the truncation operator K_T follows that

$$\varphi = \psi + K_T\varphi. \tag{5.3}$$

If $(I-K)^{-1}$ and $(I-K_T)^{-1}$ exist for any $T>0$, then the solutions to Eq. (5.2) and Eq. (5.3) are respectively denoted by

$$\hat{\varphi} = (I-K)^{-1}\psi, \qquad \hat{\varphi}_T = (I-K_T)^{-1}\psi. \tag{5.4}$$

For $k(x,y) = k_1(x,y)k_2(x,y)$, setting

$$K_j\varphi(x) := \int_0^\infty k_j(x,y)\varphi(y)dy$$

and

$$A_1 := \|K_1\|_{L_x^\infty\left(L_y^{p'}\right)} = \sup_{x\geqslant 0}\left\{\int_0^\infty |k_1(x,y)|^{p'}dy\right\}^{\frac{1}{p'}},$$

$$A_2 := \|K_2\|_{L_y^\infty(L_x^p)} = \sup_{y\geqslant 0}\left\{\int_0^\infty |k_2(x,y)|^{p}dx\right\}^{\frac{1}{p}},$$

$$A_3 := \|K_2\|_{L_y^\infty(L_x^\infty)} = \sup_{x\geqslant 0, y\geqslant 0} |k_2(x,y)|$$

with $1\leqslant p\leqslant\infty$, one needs to extend the properties of the truncation operator in [36] from the L^2 space to the L^p spaces as follows.

Theorem 5.1. *Suppose that $1\leqslant p<\infty$, $A_j<\infty$ $(j=1,2,3)$, the integral operator K in Eq. (5.2) and K_T in Eq. (5.3) respectively have solutions $\hat{\varphi}$ and $\hat{\varphi}_T$ given by Eq. (5.4). Then the following properties hold.*

(1) K is bounded in $L^p[0,\infty)$.

(2) For any $T>0$, K_T is a compact operator in $L^p[0,\infty)$.

(3) If $(I-K_T)^{-1}$ exists and is uniformly bounded with respect to T, then $(I-K)^{-1}$ exists and $\hat{\varphi}_T \to \hat{\varphi}$ in L^p sense as $T \to \infty$.

Proof. (1) For arbitrary $\varphi \in L^p[0,\infty)$, one has

$$\begin{aligned}\|K\varphi\|_{L^p}^p &= \int_0^\infty \left|\int_0^\infty k(x,y)\varphi(y)dy\right|^p dx \leqslant \int_0^\infty \left(\int_0^\infty |k_1(x,y)k_2(x,y)|\,|\varphi(y)|\,dy\right)^p dx \\ &\leqslant \int_0^\infty \left[\left(\int_0^\infty |k_1(x,y)|^{p'}dy\right)^{\frac{1}{p'}}\left(\int_0^\infty |k_2(x,y)\varphi(y)|^p dy\right)^{\frac{1}{p}}\right]^p dx \\ &\leqslant A_1^p \int_0^\infty\int_0^\infty |k_2(x,y)\varphi(y)|^p dydx = A_1^p\int_0^\infty \left[\int_0^\infty |k_2(x,y)|^p dx\right]|\varphi(y)|^p dy \\ &\leqslant A_1^pA_2^p\|\varphi\|_{L^p}^p.\end{aligned}$$

Hence $\|K\varphi\|_{L^p} \leqslant A_1A_2\|\varphi\|_{L^p}$. Therefore, K is a linear bounded operator from $L^p[0,\infty)$ into itself with $\|K\|_{L^p} \leqslant A_1A_2$. Similarly, one has $\|K_T\|_{L^p} \leqslant A_1A_2$.

(2) For arbitrary $\varphi \in L^p[0,\infty)$ and measurable sets $G,F \subset [0,\infty)$, one has

$$\begin{aligned}&\|M_GKM_F\varphi\|_{L^p}^p \\ &= \int_0^\infty \chi_G(x)\left|\int_0^\infty k_1(x,y)k_2(x,y)\chi_F(y)\varphi(y)dy\right|^p dx \\ &\leqslant \int_0^\infty \chi_G(x)\left[\left(\int_0^\infty |k_1(x,y)|^{p'}dy\right)^{\frac{1}{p'}}\left(\int_0^\infty |k_2(x,y)\varphi(y)|^p\chi_F(y)dy\right)^{\frac{1}{p}}\right]^p dx \\ &\leqslant A_1^pA_3^p\|\varphi\|_{L^p}^p|G|.\end{aligned}$$

Thus $\|M_GKM_F\|_{L^p\to L^p} \leqslant A_1A_3|G|^{\frac{1}{p}}$, and $\lim_{|G|+|F|\to 0}\|M_GKM_F\|_{L^p\to L^p} = 0$. Consequently [37],

$$\lim_{|G|+|F|\to 0}\|M_GK_TM_F\|_{L^p\to L^p} = \lim_{|G|+|F|\to 0}\|M_{G\cap[0,T]}KM_{F\cap[0,T]}\|_{L^p\to L^p} = 0$$

implies that K_T is compact in $L^p[0,T]$. For $y \in L^p[0,\infty)$, it follows that $y\chi_{[0,T]} \in L^p[0,T]$ and

$$\|K_Ty\|_{L^p[0,\infty)} = \left\|K_Ty\chi_{[0,T]}\right\|_{L^p[0,T]}. \tag{5.5}$$

Taking any sequence $\{y_n\}_{n=1}^\infty$ in the unit ball $B(0,1) := \left\{y : \|y\|_{L^p[0,\infty)} < 1\right\}$ and supposing that $\left\{K_Ty_{n_k}\chi_{[0,T]}\right\}$ is a Cauchy sequence in $L^p[0,T]$, one ensures that $\{K_Ty_{n_k}\}_{k=1}^\infty$ is a Cauchy sequence in $L^p[0,\infty)$ via Eq. (5.5). Therefore, K_T is compact in $L^p[0,\infty)$.

(3) For $\varphi \in L^p[0,\infty)$, $K_T\varphi \to K\varphi$ in L^p norm, as $T \to +\infty$. Indeed,

$$\begin{aligned}
&\|K_T\varphi - K\varphi\|_{L^p}^p \\
&= \int_0^\infty \left| \int_0^\infty \chi_{[0,T]}(x)k(x,y)\chi_{[0,T]}(y)\varphi(y)dy - \int_0^\infty k(x,y)\varphi(y)dy \right|^p dx \\
&= \int_0^\infty \left| \int_0^\infty k(x,y) \left[\chi_{[0,T]}(x)\varphi(y)\chi_{[0,T]}(y) - \varphi(y)\right] dy \right|^p dx \\
&\leqslant \int_0^\infty \left(\int_0^\infty |k_1(x,y)k_2(x,y)| \left|\chi_{[0,T]}(x)\varphi(y)\chi_{[0,T]}(y) - \varphi(y)\right| dy \right)^p dx \\
&\leqslant \int_0^\infty \left(\int_0^\infty |k_1(x,y)|^{p'} dy \right)^{\frac{p}{p'}} \left(\int_0^\infty |k_2(x,y)|^p \left|\chi_{[0,T]}(x)\varphi(y)\chi_{[0,T]}(y) - \varphi(y)\right|^p dy \right) dx \\
&\leqslant A_1^p \int_0^\infty \int_0^\infty |k_2(x,y)|^p \left|\chi_{[0,T]}(x)\varphi(y)\chi_{[0,T]}(y) - \varphi(y)\right|^p dydx \\
&= A_1^p \int_0^\infty \int_0^\infty |k_2(x,y)|^p \left|\chi_{[T,\infty]}(x)\varphi(y) + \chi_{[0,T]}(T)\varphi(y)\chi_{[T,\infty]}(y)\right|^p dydx \\
&\leqslant 2^p A_1^p \left(\int_T^\infty \int_0^\infty |k_2(x,y)|^p |\varphi(y)|^p dydx + \int_0^T \int_T^\infty |k_2(x,y)|^p |\varphi(y)|^p dydx \right),
\end{aligned}$$

where in the last inequality, one utilizes the basic fact that $(a+b)^p \leqslant 2^p(a^p+b^p)$ holds for all $a>0$, $b>0$ and $p>0$.

On the other hand, $\int_T^\infty \int_0^\infty |k_2(x,y)|^p |\varphi(y)|^p dydx$ and $\int_0^T \int_T^\infty |k_2(x,y)|^p |\varphi(y)|^p dydx$ are both dominated by

$$\int_0^\infty \int_0^\infty |k_2(x,y)|^p |\varphi(y)|^p dydx \leqslant A_2^p \|\varphi\|_{L^p}^p < \infty.$$

From the dominated convergence theorem, $\|K_T\varphi - K\varphi\|_{L^p} \to 0$ as $T \to \infty$. Since $(I-K_T)^{-1}$ exists and is uniformly bounded with respect to T, we have that $(I-K)^{-1}$ exists and $(I-K_T)^{-1} \to (I-K)^{-1}$ as $T \to \infty$ (cf. Lemma 1.5 in [35]). Therefore, $\hat{\varphi}_T \to \hat{\varphi}$ in L^p sense as $T \to \infty$, which completes the proof. □

Remark 5.1. For $p=1$, the bounds A_j's ($j=1,2,3$) take the form

$$\begin{aligned}
A_1 &:= \|K_1\|_{L_x^\infty(L_y^\infty)} = \sup_{x\geqslant 0, y\geqslant 0} |k_1(x,y)| < \infty, \\
A_2 &:= \|K_2\|_{L_y^\infty(L_x^1)} = \sup_{y\geqslant 0} \left\{ \int_0^\infty |k_2(x,y)| dx \right\} < \infty, \\
A_3 &:= \|K_2\|_{L_y^\infty(L_x^\infty)} = \sup_{x\geqslant 0, y\geqslant 0} |k_2(x,y)| < \infty.
\end{aligned}$$

The case for $p=\infty$ is delicate. The first assertion keeps valid, while the second one is invalid unless an additional condition $\|K_1\|_{L_x^\infty(L_y^\infty)} := \sup_{x\geqslant 0, y\geqslant 0} |k_1(x,y)| < \infty$ is provided. Due to the construction of the truncation operator, the third assertion generally does not hold in L^∞ sense.

6. Approximating Solutions of Polynomial Decay

In [33], the authors consider the polynomial decay estimate of the L^2 solution to the integral equation as the characteristic part of the kernel function and the right-hand side term are dominated by the polynomial upper bound. This section will directly extend this result to the L^p situation, and get a complete picture of the physical parameter settings.

Lemma 6.1. *For $1 \leqslant p < \infty$, suppose that K and L are linear bounded operators in $L^p[0,\infty)$, $(I-K)^{-1}$ exists and $\Lambda := \|(I-K)^{-1}(L-K)L\| < 1$, then $(I-L)^{-1}$ exists in $L^p[0,\infty)$ with norm*

$$\|(I-L)^{-1}\| \leqslant \frac{1+\|(I-K)^{-1}\|\|L\|}{1-\Lambda}$$

and, for any $f \in L^p[0,\infty)$,

$$\|(I-L)^{-1}f-(I-K)^{-1}f\| \leqslant \frac{\|(I-K)^{-1}\|\|Lf-Kf\|+\Lambda\|(I-K)^{-1}f\|}{1-\Lambda}.$$

Setting

$$\Omega(T) := \left\{\int_T^\infty \int_0^T |k_2(x,y)|^p\, dydx\right\}^{\frac{1}{p}},$$

for any $f \in L^p[0,\infty)$, one denotes

$$\Omega_f(T) := \left\{\int_T^\infty \int_0^T |k_2(x,y)f(y)|^p\, dydx\right\}^{\frac{1}{p}},$$
$$\omega_f(T) := \left\{\int_T^\infty |f(y)|^p\, dy\right\}^{\frac{1}{p}}.$$

The following technical lemma attributes the L^p error estimate to the decay of the characteristic part of the kernel function and the right-hand side term at infinity, which improves the result in L^2 space [36].

Lemma 6.2. *Suppose that $1 \leqslant p < \infty$, $A_j < \infty$ $(j = 1,2,3)$, the integral operator K in Eq. (5.2) and K_T in Eq. (5.3) respectively have solutions $\hat{\varphi}$ and $\hat{\varphi}_T$ given by Eq. (5.4). If $\Omega(T) < \frac{1}{A_1^2 A_3 \left\|(I-K)^{-1}\right\|}$, then one has the following L^p estimate*

$$\|\hat{\varphi}-\hat{\varphi}_T\|_{L^p} \leqslant \frac{A_1\left\|(I-K)^{-1}\right\|}{1-A_1^2A_3\Omega(T)\left\|(I-K)^{-1}\right\|}\left[\Omega_\psi(T)+A_2\omega_\psi(T)+A_1A_3\Omega(T)\|\hat{\varphi}\|_{L^p}\right]. \quad (6.1)$$

Proof. Taking any $\varphi \in L^p[0,\infty)$,

$$\begin{aligned}(K_T-K)K_T\varphi &= \left(M_{[0,T]}KM_{[0,T]}-K\right)\chi_{[0,T]}\int_0^T k(x,y)\varphi(y)dy \\ &= \left(M_{[0,T]}-I\right)\int_0^T k(z,x)\left[\int_0^T k(x,y)\varphi(y)dy\right]dx,\end{aligned}$$

one has

$$\begin{aligned}
&\left\|(I-K)^{-1}(K_T-K)K_T\varphi\right\|_{L^p}\\
&\leqslant \|(I-K)^{-1}\|_{L^p\to L^p}\|(K_T-K)K_T\varphi\|_{L^p}\\
&\leqslant \|(I-K)^{-1}\|_{L^p\to L^p}\left\{\int_T^\infty\left[\int_0^T|k(z,x)|\left(\int_0^T|k(x,y)||\varphi(y)|dy\right)dx\right]^p dz\right\}^{\frac{1}{p}}\\
&\leqslant \|(I-K)^{-1}\|_{L^p\to L^p}\left[\int_T^\infty\left(\int_0^T|k_2(z,x)|^p dx\right)\left(A_1^2A_3\|\varphi\|_{L^p}\right)^p dz\right]^{\frac{1}{p}}\\
&\leqslant A_1^2A_3\|\varphi\|_{L^p}\|(I-K)^{-1}\|_{L^p\to L^p}\Omega(T).
\end{aligned}$$

Hence

$$\Lambda := \left\|(I-K)^{-1}(K_T-K)K_T\right\|_{L^p\to L^p} \leqslant A_1^2A_3\|(I-K)^{-1}\|_{L^p\to L^p}\Omega(T).$$

On the other hand,

$$(K-K_T)\psi(x) = M_{[T,\infty)}(x)\int_0^T k(x,y)\psi(y)dy + \int_T^\infty k(x,y)\psi(y)dy,$$

thus

$$\begin{aligned}
\|(K-K_T)\psi\|_{L^p} &\leqslant \left\{\int_T^\infty\left|\int_0^T k(x,y)\psi(y)dy\right|^p dx\right\}^{\frac{1}{p}}\\
&\quad+\left\{\int_0^\infty\left|\int_T^\infty k(x,y)\psi(y)dy\right|^p dx\right\}^{\frac{1}{p}}\\
&=: I_1+I_2.
\end{aligned}$$

In view of

$$I_1 \leqslant A_1\left\{\int_T^\infty\int_0^T|k_2(x,y)|^p|\psi(y)|^p dydx\right\}^{\frac{1}{p}} = A_1\Omega_\psi(T)$$

and

$$\begin{aligned}
I_2 &\leqslant A_1\left(\int_0^\infty\int_T^\infty|k_2(x,y)|^p|\psi(y)|^p dydx\right)^{\frac{1}{p}}\\
&= A_1\left[\int_T^\infty|\psi(y)|^p\left(\int_0^\infty|k_2(x,y)|^p dx\right)dy\right]^{\frac{1}{p}}\\
&\leqslant A_1A_2\left(\int_T^\infty|\psi(y)|^p dy\right)^{\frac{1}{p}} = A_1A_2\omega_\psi(T),
\end{aligned}$$

one immediately has

$$\|(K-K_T)\psi\|_{L^p} \leqslant A_1\left[A_2\omega_\psi(T)+\Omega_\psi(T)\right].$$

Since $\Lambda < 1$ as $\Omega(T) < \frac{1}{A_1^2 A_3 \|(I-K)^{-1}\|}$ for sufficiently large T, one may take $L = K_T$, $f = \psi$ and make use of Lemma 6.1 to reach

$$\|\hat{\varphi} - \hat{\varphi}_T\|_{L^p} \leqslant \frac{A_1 \|(I-K)^{-1}\|_{L^p \to L^p}}{1 - A_1^2 A_3 \|(I-K)^{-1}\|_{L^p \to L^p} \Omega(T)} \cdot \left[A_2 \omega_\psi(T) + \Omega_\psi(T) + A_1 A_3 \|\hat{\varphi}\|_{L^p} \Omega(T)\right],$$

which completes the proof. □

We are now in a position to formulate the main result as the characteristic part of kernel function k_2 and initial data ψ are both polynomially dominated. The notation $X \lesssim Y$ means that there exists some constant $C > 0$ such that $X \leqslant CY$.

Theorem 6.1. *Suppose that $1 \leqslant p < \infty$, $A_j < \infty$ ($j = 1,2,3$), the integral operator K in Eq. (5.2) and K_T in Eq. (5.3) respectively have solutions $\hat{\varphi}$ and $\hat{\varphi}_T$ given by Eq. (5.4). If $|k_2(x,y)|^p \lesssim \frac{1}{(1+x)^\alpha (1+y)^\beta}$ and $|\psi(y)|^p \lesssim \frac{1}{(1+y)^\gamma}$ for all $(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+$ with $\alpha > 1$, $\beta \geqslant 0$, $\alpha + \beta > 2$ and $\gamma > 1$, then, as the truncating endpoint T is sufficiently large, the following polynomial error estimates hold.*

- *For $\beta > 1$,*

$$\|\hat{\varphi} - \hat{\varphi}_T\|_{L^p} \lesssim \frac{A_1 \|(I-K)^{-1}\|_{L^p \to L^p} \left[(\alpha-1)(\beta-1)(1+T)^{\alpha-1}\right]^{\frac{1}{p}}}{[(\alpha-1)(\beta-1)(1+T)^{\alpha-1}]^{\frac{1}{p}} - A_1^2 A_3 \|(I-K)^{-1}\|_{L^p \to L^p}} \times \left\{ \frac{A_2}{[(\gamma-1)(1+T)^{\gamma-1}]^{\frac{1}{p}}} + \frac{(\beta-1)^{\frac{1}{p}} + A_1 A_3 (\beta+\gamma-1)^{\frac{1}{p}} \|\hat{\varphi}\|_{L^p}}{[(\alpha-1)(\beta-1)(\beta+\gamma-1)(1+T)^{\alpha-1}]^{\frac{1}{p}}} \right\}. \tag{6.2}$$

- *For $\beta = 1$,*

$$\|\hat{\varphi} - \hat{\varphi}_T\|_{L^p} \lesssim \frac{A_1 \|(I-K)^{-1}\|_{L^p \to L^p} \left[(\alpha-1)(1+T)^{\alpha-1}\right]^{\frac{1}{p}}}{[(\alpha-1)(1+T)^{\alpha-1}]^{\frac{1}{p}} - A_1^2 A_3 \|(I-K)^{-1}\|_{L^p \to L^p} [\ln(1+T)]^{\frac{1}{p}}} \times \left\{ \frac{A_2}{[(\gamma-1)(1+T)^{\gamma-1}]^{\frac{1}{p}}} + \frac{1 + A_1 A_3 \gamma^{\frac{1}{p}} \|\hat{\varphi}\|_{L^p} [\ln(1+T)]^{\frac{1}{p}}}{[(\alpha-1)\gamma(1+T)^{\alpha-1}]^{\frac{1}{p}}} \right\}. \tag{6.3}$$

- *For $\beta < 1$,*

$$\|\hat{\varphi} - \hat{\varphi}_T\|_{L^p} \lesssim \frac{A_1 \|(I-K)^{-1}\|_{L^p \to L^p} \left[(\alpha-1)(1-\beta)(1+T)^{\alpha+\beta-2}\right]^{\frac{1}{p}}}{\left[(\alpha-1)(1-\beta)(1+T)^{\alpha+\beta-2}\right]^{\frac{1}{p}} - A_1^2 A_3 \|(I-K)^{-1}\|_{L^p \to L^p}} \times \left\{ \frac{A_2}{[(\gamma-1)(1+T)^{\gamma-1}]^{\frac{1}{p}}} + \frac{(1-\beta)^{\frac{1}{p}} (1+T)^{\frac{\beta-1}{p}} + A_1 A_3 (\beta+\gamma-1)^{\frac{1}{p}} \|\hat{\varphi}\|_{L^p}}{\left[(\alpha-1)(1-\beta)(\beta+\gamma-1)(1+T)^{\alpha+\beta-2}\right]^{\frac{1}{p}}} \right\}. \tag{6.4}$$

Proof. It suffices to estimate $\Omega(T)$, $\Omega_\psi(T)$, and $\omega_\psi(T)$ for different β's.

• For $\beta > 1$, one estimates the upper bounds

$$\Omega^p(T) = \int_T^\infty \int_0^T |k_2(x,y)|^p \, dydx \lesssim \int_T^\infty \int_0^T \frac{dydx}{(1+x)^\alpha (1+y)^\beta}$$
$$= \left(\int_T^\infty \frac{dx}{(1+x)^\alpha}\right)\left(\int_0^T \frac{dy}{(1+y)^\beta}\right) \lesssim \frac{1}{(\alpha-1)(\beta-1)(1+T)^{\alpha-1}},$$

$$\Omega_\psi^p(T) = \int_T^\infty \int_0^T |k_2(x,y)\psi(y)|^p \, dydx \lesssim \int_T^\infty \int_0^T \frac{dydx}{(1+x)^\alpha (1+y)^{\beta+\gamma}}$$
$$= \left(\int_T^\infty \frac{dx}{(1+x)^\alpha}\right)\left(\int_0^T \frac{dy}{(1+y)^{\beta+\gamma}}\right) \lesssim \frac{1}{(\alpha-1)(\beta+\gamma-1)(1+T)^{\alpha-1}},$$

and

$$\omega_\psi^p(T) = \int_T^\infty |\psi(y)|^p \, dy \lesssim \int_T^\infty \frac{dy}{(1+y)^\gamma} = \frac{1}{(\gamma-1)(1+T)^{\gamma-1}}.$$

As $T \to \infty$, $\Omega(T)$, $\Omega_\psi(T)$, and $\omega_\psi(T)$ are sufficiently small to make sure that Lemma 6.2 is applicable. Finally, a direct substitution yields the polynomial error estimate (6.2) for the truncation solutions to Eq. (5.2).

• For $\beta = 1$, one similarly has the bounds

$$\Omega^p(T) = \int_T^\infty \int_0^T |k_2(x,y)|^p \, dydx \lesssim \int_T^\infty \int_0^T \frac{dydx}{(1+x)^\alpha (1+y)}$$
$$= \left(\int_T^\infty \frac{dx}{(1+x)^\alpha}\right)\left(\int_0^T \frac{dy}{1+y}\right) \lesssim \frac{\ln(1+T)}{(\alpha-1)(1+T)^{\alpha-1}},$$

$$\Omega_\psi^p(T) = \int_T^\infty \int_0^T |k_2(x,y)\psi(y)|^p \, dydx \lesssim \int_T^\infty \int_0^T \frac{dydx}{(1+x)^\alpha (1+y)^{1+\gamma}}$$
$$= \left(\int_T^\infty \frac{dx}{(1+x)^\alpha}\right)\left(\int_0^T \frac{dy}{(1+y)^{1+\gamma}}\right) \lesssim \frac{1}{(\alpha-1)\gamma(1+T)^{\alpha-1}},$$

and

$$\omega_\psi^p(T) = \int_T^\infty |\psi(y)|^p \, dy \lesssim \int_T^\infty \frac{dy}{(1+y)^\gamma} = \frac{1}{(\gamma-1)(1+T)^{\gamma-1}}.$$

Although the convergence in y direction is reduced, as long as T is sufficiently large, one can still ensure that $\Omega(T)$ is small enough, and the upper bounds of $\Omega_\psi(T)$ and $\omega_\psi(T)$ are similar to the above case. Therefore, Lemma 6.2 is applicable to our situation and the error estimate (6.3) of the truncated solution to Eq. (5.2) is obtained from it.

• For $\beta < 1$, the bounds become

$$\Omega^p(T) = \int_T^\infty \int_0^T |k_2(x,y)|^p \, dydx \lesssim \int_T^\infty \int_0^T \frac{dydx}{(1+x)^\alpha (1+y)^\beta}$$
$$= \left(\int_T^\infty \frac{dx}{(1+x)^\alpha}\right)\left(\int_0^T \frac{dy}{(1+y)^\beta}\right) \lesssim \frac{1}{(\alpha-1)(1-\beta)(1+T)^{\alpha+\beta-2}},$$

$$\Omega_\psi^p(T) = \int_T^\infty \int_0^T |k_2(x,y)\psi(y)|^p \, dydx \lesssim \int_T^\infty \int_0^T \frac{dydx}{(1+x)^\alpha (1+y)^{\beta+\gamma}}$$
$$= \left(\int_T^\infty \frac{dx}{(1+x)^\alpha}\right) \left(\int_0^T \frac{dy}{(1+y)^{\beta+\gamma}}\right) \lesssim \frac{1}{(\alpha-1)(\beta+\gamma-1)(1+T)^{\alpha-1}},$$

and

$$\omega_\psi^p(T) = \int_T^\infty |\psi(y)|^p \, dy \lesssim \int_T^\infty \frac{dy}{(1+y)^\gamma} = \frac{1}{(\gamma-1)(1+T)^{\gamma-1}}.$$

The condition $\alpha+\beta > 2$ ensures that $\Omega(T)$ has polynomial decay as T is sufficiently large, while the estimates of $\Omega_\psi(T)$ and $\omega_\psi(T)$ maintain the same form of convergence. We use Lemma 6.2 again to obtain the error estimate (6.4) of the truncated solution to Eq. (5.2). □

Remark 6.1. First, the range of parameters in this theorem is necessary. In fact, $A_3 < \infty$ implies $\alpha \geqslant 0$ and $\beta \geqslant 0$, and $A_2 < \infty$ further requires $\alpha > 1$. $\gamma > 1$ guarantees $\omega_\psi(T) < \infty$. Second, as the kernel function has the form $k(x,y) = k_1(x,y)k_2(x,y)$, $k_1(x,y)$ depend on the y variable only, and $k_2(x,y)$ depends on the x variable only. These components are regarded as the symbolic function and characteristic function in the case of separation of kernel function variables. This specific form of the kernel function corresponds to the case of $\beta = 0$, which has an independent significance in the theory of celestial radiation.

7. Approximating Solutions of Exponential Decay

We study the large-time error estimation of the equation approximation solution as the characteristic part of the kernel function and the right-hand side term have exponential decay in this section. Taking account of no parameters in the integral equation, one may use the iterative kernel method to approximate the solution of the equation. In fact, this successive approximation method has a wide range of practicability, and applicable to both finite interval and infinite interval cases [23, 36].

Since the truncation operator is the main analysis tool, one needs to structure the truncated solution of the equation. Denoting $\bar{\varphi}_T$ as the solution to the truncation equation

$$\varphi - K_T\varphi = \psi\chi_{[0,T]},$$

one easily checks that the approximating solution to Eq. (5.3) is represented by

$$\hat{\varphi}_T(x) = \begin{cases} \bar{\varphi}_T(x), & x \in [0,T], \\ \psi(x), & x \in (T,\infty). \end{cases} \tag{7.1}$$

7.1. **Exponential error estimates.** Similar to Lemma 6.2 in the polynomial case, one needs an error estimate compatible with the upper bound of exponential decay.

Lemma 7.1. *Suppose that $1 \leqslant p < \infty$, $A_j < \infty$ $(j = 1,2,3)$, the integral operator K in Eq. (5.2) and K_T in Eq. (5.3) respectively have solutions $\hat{\varphi}$ and $\hat{\varphi}_T$ given by Eq. (5.4). If $(I-K_T)^{-1}$ exists and is uniformly bounded with respect to T, then one has the following L^p estimate*

$$\|\hat{\varphi} - \hat{\varphi}_T\|_{L^p} \leqslant 2A_1 \left\|(I-K)^{-1}\right\|_{L^p \to L^p} \left[\Omega_{\hat{\varphi}_T}(T) + A_2\omega_\psi(T)\right]. \tag{7.2}$$

Proof. Since $\hat{\varphi}$ and $\hat{\varphi}_T$ fulfill the equations

$$\hat{\varphi} - K\hat{\varphi} = \hat{\varphi}_T - K_T\hat{\varphi}_T = \psi,$$

i.e., $\hat{\varphi}_T - \hat{\varphi} = K_T\hat{\varphi}_T - K\hat{\varphi}$, it follows that $(\hat{\varphi}_T - \hat{\varphi}) - K(\hat{\varphi}_T - \hat{\varphi}) = (K_T - K)\hat{\varphi}_T$. Hence $\hat{\varphi}_T - \hat{\varphi} = (I - K_T)^{-1}(K_T - K)\hat{\varphi}_T$ and then

$$\|\hat{\varphi}_T - \hat{\varphi}\|_{L^p} \leqslant \|(I-K_T)^{-1}\|_{L^p \to L^p} \|(K_T - K)\hat{\varphi}_T\|_{L^p}. \tag{7.3}$$

By the construction of $\hat{\varphi}_T$ in Eq. (7.1), one has

$$\begin{aligned}
&\|(K_T - K)\hat{\varphi}_T\|_{L^p}^p \\
&= \int_0^\infty \left| \int_0^\infty k(x,y) \left[\chi_{[0,T]}(x)\hat{\varphi}_T(y)\chi_{[0,T]}(y) - \hat{\varphi}_T(y)\right] dy \right|^p dx \\
&\leqslant A_1^p \int_0^\infty \int_0^\infty |k_2(x,y)|^p |\chi_{[0,T]}(x)\varphi_T(y)\chi_{[0,T]}(y) - \varphi_T(y)|^p dydx \\
&= A_1^p \int_0^\infty \int_0^\infty |k_2(x,y)|^p \left|\hat{\varphi}_T(y)\chi_{[T,\infty)}(y) + \chi_{[T,\infty)}(x)\hat{\varphi}_T(y)\chi_{[0,T]}(y)\right|^p dydx \\
&\leqslant 2^p A_1^p \left[\int_0^\infty \int_T^\infty |k_2(x,y)|^p |\hat{\varphi}_T(y)|^p dydx + \int_T^\infty \int_0^T |k_2(x,y)|^p |\hat{\varphi}_T(y)|^p dydx\right] \\
&= 2^p A_1^p \left[\int_T^\infty \left(\int_0^\infty |k_2(x,y)|^p dx\right) |\psi(y)|^p dy + \int_T^\infty \int_0^T |k_2(x,y)|^p |\hat{\varphi}_T(y)|^p dydx\right] \\
&\leqslant 2^p A_1^p \left[A_2^p \omega_\psi^p(T) + \Omega_{\hat{\varphi}_T}^p(T)\right].
\end{aligned}$$

Consequently

$$\|(K_T - K)\hat{\varphi}_T\|_{L^p} \leqslant 2A_1 \left[A_2\omega_\psi(T) + \Omega_{\hat{\varphi}_T}(T)\right].$$

Finally, it follows that

$$\|\hat{\varphi}_T - \hat{\varphi}\|_{L^p} \leqslant 2A_1 \left\|(I-K)^{-1}\right\|_{L^p \to L^p} \left[A_2\omega_\psi(T) + \Omega_{\hat{\varphi}_T}(T)\right],$$

which together with Eq. (7.3) conclude the proof. □

Now we extend the L^2 case in [23] to the L^p case. Compared with [36], [23] removes the displacement dominance restriction on the characteristic part of the kernel function and increases the choice of the parameters.

Theorem 7.1. *Suppose that $1 \leqslant p < \infty$, $A_j < \infty$ $(j = 1,2,3)$ with $A_1A_2 < 1$, $(I - K_T)^{-1}$ exists in $L^p[0,\infty)$ and is uniformly bounded in T, the integral operator K in Eq. (5.2) and K_T in Eq. (5.3) respectively have solutions $\hat{\varphi}$ and $\hat{\varphi}_T$ given by Eq. (5.4). If*

$$|k_2(x,y)|^p \leqslant e^{-\alpha x - \beta y}, \qquad |\psi(y)|^p \leqslant e^{-\gamma y}, \quad x \geqslant 0, y \geqslant 0 \tag{7.4}$$

for some $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, $\lambda := \frac{A_1}{(\beta+\gamma)^{\frac{1}{p}}} < 1$, then, for sufficiently large $T > 0$, it follows that

$$\|\hat{\varphi} - \hat{\varphi}_T\|_{L^p} \leqslant \frac{2A_1}{1 - A_1A_2} \left[\frac{A_2}{\gamma^{\frac{1}{p}}} e^{-\frac{\gamma}{p}T} + \frac{e^{-\frac{\alpha}{p}T}}{(1-\lambda)\alpha^{\frac{1}{p}}(\alpha+\beta)^{\frac{1}{p}}}\right]. \tag{7.5}$$

Proof. For every $\varphi \in L^p[0,\infty)$, one has

$$\begin{aligned}
&\|K_T\varphi\|_{L^p}^p \\
&\leqslant \int_0^T \left(\int_0^T |k_1(x,y)k_2(x,y)||\varphi(y)|dy\right)^p dx \\
&\leqslant \int_0^T \left[\left(\int_0^T |k_1(x,y)|^{p'}dy\right)^{\frac{1}{p'}} \left(\int_0^T |k_2(x,y)|^p|\varphi(y)|^p dy\right)^{\frac{1}{p}}\right]^p dx \\
&\leqslant A_1^p \int_0^T \left(\int_0^T |k_2(x,y)|^p|\varphi(y)|^p dy\right) dx \\
&\leqslant A_1^p A_2^p \|\varphi\|_{L^p}^p.
\end{aligned}$$

Thus $\|K_T\|_{L^p\to L^p} \leqslant A_1A_2 < 1$.

On the other hand,

$$|\hat{\varphi}_T|^p = \left|(I-K_T)^{-1}\psi\right|^p = \left|\sum_{j=0}^{\infty} K_T^j\psi\right|^p \leqslant \left(\sum_{j=0}^{\infty}\left|K_T^j\psi\right|\right)^p.$$

In view of

$$\begin{aligned}
&\int_0^T |k(t_2,t_1)||\psi(t_1)|dt_1 \\
&\leqslant \left(\int_0^T |k_1(t_2,t_1)|^{p'}dt_1\right)^{\frac{1}{p'}} \left(\int_0^T |k_2(t_2,t_1)|^p|\psi(t_1)|^p dt_1\right)^{\frac{1}{p}} \\
&\leqslant A_1 \left(\int_0^T |k_2(t_2,t_1)|^p|\psi(t_1)|^p dt_1\right)^{\frac{1}{p}},
\end{aligned}$$

one claims that, for any positive integer $m \geqslant 2$,

$$\begin{aligned}
&\underbrace{\int_0^T \cdots \int_0^T}_{m\ tuples} |k(t,t_m)||k(t_m,t_{m-1})|\cdots|k(t_2,t_1)||\psi(t_1)|dt_1\cdots dt_m \\
&\leqslant A_1^m \left\{\underbrace{\int_0^T \cdots \int_0^T}_{m\ tuples} |k_2(t,t_m)|^p|k_2(t_m,t_{m-1})|^p\cdots|k_2(t_2,t_1)|^p|\psi(t_1)|^p dt_1\cdots dt_m\right\}^{\frac{1}{p}}.
\end{aligned}$$

It follows that

$$\underbrace{\int_0^T \cdots \int_0^T}_{(m+1)\ tuples} |k(t,t_{m+1})| \cdots |k(t_2,t_1)||\psi(t_1)| dt_1 \cdots dt_{m+1}$$

$$= \int_0^T |k_1 k_2|(t,t_{m+1}) \left[\underbrace{\int_0^T \cdots \int_0^T}_{m\ tuples} |k(t_{m+1},t_m)| \cdots |k(t_2,t_1)||\psi(t_1)| dt_1 \cdots dt_m \right] dt_{m+1}$$

$$\leqslant A_1^m \left(\int_0^T |k_1(t,t_{m+1})|^{p'} dt_{m+1} \right)^{\frac{1}{p'}}$$

$$\cdot \left(\underbrace{\int_0^T \cdots \int_0^T}_{(m+1)\ tuples} |k_2(t,t_{m+1})|^p \cdots |k_2(t_2,t_1)|^p |\psi(t_1)|^p dt_1 \cdots dt_{m+1} \right)^{\frac{1}{p}}$$

$$\leqslant A_1^{m+1} \left(\underbrace{\int_0^T \cdots \int_0^T}_{(m+1)\ tuples} |k_2(t,t_{m+1})|^p \cdots |k_2(t_2,t_1)|^p |\psi(t_1)|^p dt_1 \cdots dt_{m+1} \right)^{\frac{1}{p}}.$$

Therefore,

$$|K_T^i \psi(x)| = \left| \chi_{[0,T)}(x) \underbrace{\int_0^T \cdots \int_0^T}_{i\ tuples} |k(x,x_i)| \cdots |k(x_2,x_1)||\psi(x_1)| dx_1 \cdots dx_i \right|$$

$$\leqslant A_1^i \chi_{[0,T)}(x) \left\{ \underbrace{\int_0^T \cdots \int_0^T}_{i\ tuples} |k_2(x,x_i)|^p \cdots |k_2(x_2,x_1)|^p |\psi(x_1)|^p dx_1 \cdots dx_i \right\}^{\frac{1}{p}}.$$

The assumptions

$$|k_2(x,y)|^p \leqslant e^{-\alpha x - \beta y}, \qquad |\psi(y)|^p \leqslant e^{-\gamma y},$$

lead to that

$$\int_0^T |k_2(x,y)|^p |\psi(y)|^p dy \leqslant \int_0^T e^{-\alpha x} \cdot e^{-(\beta+\gamma)y} dy \leqslant \frac{e^{-\alpha x}}{\beta+\gamma}.$$

It consequently follows that

$$|K_T^i \psi(x)| \leqslant A_1^i \chi_{[0,T)}(x) \left[\frac{e^{-\alpha x}}{(\beta+\gamma)^i} \right]^{\frac{1}{p}} = \lambda^i \chi_{[0,T)}(x) e^{-\frac{\alpha}{p}x}$$

and

$$|\hat{\varphi}_T(x)|^p \leqslant \left(\sum_{j=0}^{\infty} \left| K_T^j \psi(x) \right| \right)^p \leqslant \left(\sum_{j=0}^{\infty} \lambda^j \right)^p \chi_{[0,T)}(x) e^{-\alpha x} = \frac{e^{-\alpha x}}{(1-\lambda)^p} \chi_{[0,T)}(x).$$

We now turn to estimate ω_ψ and $\Omega_{\hat{\varphi}_T}$ based on the bounds of $\hat{\varphi}_T$. Indeed, it holds

$$\omega_\psi^p(T) \leqslant \int_T^\infty e^{-\gamma y} dy = \frac{e^{-\gamma T}}{\gamma}, \tag{7.6}$$

and

$$\begin{aligned}\Omega_{\hat{\varphi}_T}^p(T) &= \int_T^\infty \int_0^T |k_2(x,y)\hat{\varphi}_T(y)|^p \, dydx \\ &\leqslant \frac{1}{(1-\lambda)^p} \int_T^\infty \int_0^T e^{-\alpha x - \beta y} \cdot e^{-\alpha y} dydx \\ &= \frac{1}{(1-\lambda)^p} \left(\int_T^\infty e^{-\alpha x} dx\right)\left(\int_0^T e^{-(\alpha+\beta)y} dy\right) \\ &\leqslant \frac{e^{-\alpha T}}{\alpha(\alpha+\beta)(1-\lambda)^p}.\end{aligned} \tag{7.7}$$

Since $\|K\|_{L^p\to L^p} \leqslant A_1A_2 < 1$, it follows that

$$\|(I-K)^{-1}\|_{L^p\to L^p} \leqslant 1 + \|K\| + \|K\|^2 + \cdots \leqslant \frac{1}{1-A_1A_2}. \tag{7.8}$$

Finally, we reach the estimate in (7.5) by substituting (7.6), (7.7), and (7.8) into (7.2) to complete the proof. □

Remark 7.1. The convergence results in Sections 5-6 show that since the truncation operator is defined pointwisely, under the most natural conditions (the conditions proposed in the theorem), the approximate solution of the integral equation generally does not converge in L^∞ sense, unless other conditions are provided to ensure uniform convergence of the operator.

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