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A MINIMAX APPROACH TO CHARACTERIZE QUASI ε-PARETO SOLUTIONS IN MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. This paper focuses on the study of optimality conditions (both necessary and sufficient) for a weakly quasi ε-Pareto solution to a multiobjective optimization problem by using a minimax programming approach. To establish necessary conditions for approximate solutions of minimax programming problems under a suitable constraint qualification, we use some advanced tools of variational analysis and generalized differentiation. Sufficient conditions for such solutions to the considered problem are also provided by using generalized convex functions defined in terms of the limiting subdifferential for locally Lipschitz functions. In addition, some duality results for minimax programming problems are also provided.
Keywords. Multiobjective optimization; Minimax programming; Optimality conditions; Duality; Limiting subdifferential.

1. INTRODUCTION

Let X denote the Asplund space (i.e., the Banach spaces whose separable subspaces have separable duals), and let Ω be a nonempty locally closed subset of X. Let $K := \{1,\ldots,l\}$ and $J := \{1,\ldots,m\}$ be two index sets. We consider the following constrained multiobjective optimization problem:

$$\min_{\mathbb{R}^l_+} \{f(x) \mid x \in F\}, \tag{MP}$$

where $f := (f_1,\ldots,f_l)$, and the feasible set F is defined by

$$F := \{x \in \Omega \mid g_j(x) \leq 0,\ j \in J\}, \tag{1.1}$$

the functions f_k, $k \in K$, and g_j, $j \in J$ are locally Lipschitz on X.

In most of the cases, it is almost impossible to find a single point that simultaneously minimizes all the objective functions f_k, $k = 1,\ldots,l$. In such cases, we then look for some "best preferred" solution(s) in multiobjective optimization problems. Roughly speaking, a "best preferred" solution means the one that cannot be improved in one objective function without deteriorating their performance in at least one of the rest. That is the concept of the so-called Pareto

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solutions. Indeed, apart from the Pareto solutions, there are also many other solution concepts in multiobjective optimization problems. Among other things, we will focus on the weakly quasi ε-Pareto solutions, below we recall its definition.

Definition 1.1. Let $\varepsilon := (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l_+ \backslash \{0\}$. $\bar{x} \in F$ is said to be a weakly quasi ε-Pareto solution of multiobjective optimization problem (MP) if and only if

$$f(x) + \varepsilon \parallel x - \bar{x} \parallel - f(\bar{x}) \notin -\mathrm{int}\,\mathbb{R}^l_+, \ \forall x \in F.$$

The weakly quasi ε-Pareto solution, which is one kind of approximate Pareto solutions, in multiobjective optimization problems can be treated as a feasible point whose objective values compromise with a prescribed error ε in the optimal values of the multiobjective functions. Therefore, the study of the approximate Pareto solution is very important from both the theoretical and computational points of view. Recently, approximate Pareto solutions have been investigated by many researchers; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein.

In order to find weakly quasi ε-Pareto solutions to multiobjective optimization problems, usually one uses scalarization, which involves formulating a single objective optimization problem (see [13]). In this paper, we will study the optimality conditions for a weakly quasi ε-Pareto solution of problem (MP) by considering its associated *minimax optimization problem* (see [14]). Roughly speaking, let us consider the following minimax programming problem:

$$\min_{x \in F} \max_{k \in K} \ f_k(x), \tag{P}$$

where the functions f_k, $k \in K$ are locally Lipschitz on X, and the feasible set F is defined by (1.1). The problem, in which both a minimization and maximization process are performed, is known in the area of mathematical programming as minimax problems.

Minimax programming problems have been the subject of immense interest in the past few years. Recently, many researchers studied optimality conditions and duality theorems for minimax programming problems. For details, we refer to, for example, [15, 16].

In this paper, we use some advanced tools of variational analysis and generalized differentiation (e.g., the nonsmooth version of Fermat's rule, the sum rule for the Mordukhovich/limiting subdifferential) (see [17]) to establish necessary conditions for quasi α-solutions (see Definition 3.1) of a minimax programming problem with inequality constraints. Sufficient conditions for the such solutions to the considered problem are also provided with the aid of the generalized convexity (see [18]) defined in terms of the limiting subdifferential for locally Lipschitz functions.

In pace with optimality conditions, we propose a dual problem to the primal one and examine weak, strong, and converse-like duality relations under suitable assumptions.

As an application, we employ the necessary and sufficient optimality conditions obtained for the minimax programming problem to derive the corresponding ones for a multiobjective optimization problem. In addition, some examples are also given for illustrating the obtained results.

The rest of the paper is organized as follows. Section 2 provides some preliminaries. In Section 3, we present some results on minimax programming problem, including necessary conditions for quasi α-solutions and sufficient conditions for such solutions. Section 4 is devoted to the duality relations in minimax programming problems. The main results of optimality conditions (both necessary and sufficient) for a weakly quasi ε-Pareto solution to multiobjective optimization problem will be studied in Section 5. Finally, conclusions are given in Section 6, the last section.

2. Preliminaries

In this section, we use the following notations and preliminary results (see [17, 19]). Unless otherwise specified, all spaces under consideration are assumed to be Asplund. The canonical pairing between space X and its topological dual X^* is denoted by $\langle\cdot,\cdot\rangle$, while the symbol $\|\cdot\|$ stands for the norm in the considered space. As usual, the polar cone of a set $\Omega \subset X$ is defined by

$$\Omega^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \le 0 \text{ for all } x \in \Omega\}.$$

Given a multifunction $F : X \rightrightarrows X^*$, we denote by

$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) := \left\{x^* \in X^* \mid \exists x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N} := \{1,2,\ldots\}\right\}$$

the Painlevé-Kuratowski upper/outer limit of F as $x \to \bar{x}$, where the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

Given $\Omega \subset X$, and $\bar{x} \in \Omega$, define the Fréchet/regular normal cone to Ω at $\bar{x}$ by

$$\widehat{N}_\Omega(\bar{x}) := \left\{x^* \in X^* \;\middle|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0\right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x};\Omega) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega \subset X$ is obtained from regular normal cones by taking the sequential Painlevé-Kuratowski upper limits as

$$N(\bar{x};\Omega) := \operatorname*{Limsup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x;\Omega).$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x};\Omega) := \emptyset$.

For an extended real-valued function $f : X \to \overline{\mathbb{R}} := [-\infty,\infty]$, its epigraph is defined by

$$\operatorname{epi} f := \{(x,\mu) \in X \times \mathbb{R} \mid \mu \ge f(x)\}.$$

The Mordukhovich/limiting subdifferential of f at $\bar{x} \in X$ with $|f(\bar{x})| < \infty$ is defined by

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*,-1) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f)\}.$$

If $|f(\bar{x})| = \infty$, then one puts $\partial f(\bar{x}) := \emptyset$.

Considering the indicator function $\delta(\cdot;\Omega)$ defined by: $\delta(x;\Omega) := 0$ for $x \in \Omega$ and by $\delta(x;\Omega) := \infty$ otherwise, we have a relation between the Mordukhovich normal cone and the limiting subdifferential of the indicator function as follows (see [17, Proposition 1.79]):

$$\partial\delta(\bar{x};\Omega) = N(\bar{x};\Omega) \text{ for all } \bar{x} \in \Omega. \tag{2.1}$$

The nonsmooth version of Fermat's rule is formulated as follows [17, Proposition 1.114]: If $\bar{x} \in X$ is a local minimizer of $f : X \to \overline{\mathbb{R}}$, then

$$0 \in \partial f(\bar{x}). \tag{2.2}$$

For establishing optimality conditions, the following lemma, which is related to the limiting subdifferential calculus, is very useful.

Lemma 2.1. [17, Theorem 3.36 and Theorem 3.46]

(i) *Let $\phi_i : X \to \overline{\mathbb{R}}$, $i = 1,2,\dots,m$, $m \geq 2$ be lower semicontinuous around $\bar{x} \in X$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then*

$$\partial(\phi_1 + \phi_2 + \ldots + \phi_m)(\bar{x}) \subset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}) + \ldots + \partial\phi_m(\bar{x}). \tag{2.3}$$

(ii) *Let $\phi_i : X \to \overline{\mathbb{R}}$, $i = 1,2,\dots,m$, $m \geq 2$ be lower semicontinuous around $\bar{x}$ for $i \in I_{\max}(\bar{x})$ and upper semicontinuous at $\bar{x}$ for $i \notin I_{\max}(\bar{x})$. Suppose that each ϕ_i, $i = 1,\dots,m$, is Lipschitz continuous around $\bar{x}$. Then*

$$\partial(\max\phi_i)(\bar{x}) \subset \bigcup\left\{\partial\left(\sum_{i\in I_{\max}(\bar{x})}\lambda_i\phi_i\right)(\bar{x}) \;\middle|\; (\lambda_1,...,\lambda_m) \in \Lambda(\bar{x})\right\},$$

where the equality holds and the maximum functions are lower regular at $\bar{x}$ if each ϕ_i is lower regular at this point and sets $I_{\max}(\bar{x})$ and $\Lambda(\bar{x})$ are defined as follows:

$$I_{\max}(\bar{x}) := \{i \in \{1,\dots,m\} \mid \phi_i(\bar{x}) = (\max\phi_i)(\bar{x})\},$$

$$\Lambda(\bar{x}) := \left\{(\lambda_1,\dots,\lambda_m) \mid \lambda_i \geq 0,\ \sum_{i=1}^{m}\lambda_i = 1,\ \lambda_i\left(\phi_i(\bar{x}) - (\max\phi_i)(\bar{x})\right) = 0\right\}.$$

3. Optimality Conditions

In this section, we establish a necessary optimality condition for a quasi α-solution of minimax programming problem. Moreover, by introducing the concepts of the generalized convexity, we give sufficient conditions for such solutions.

Definition 3.1. Consider minimax programming problem (P) with the feasible set F given by (1.1). Let $\alpha \geq 0$, and $\phi(x) := \max_{k\in K} f_k(x)$, $x \in X$. Then $\bar{x} \in F$ is said to be a *quasi α-solution* of (P) if

$$\phi(\bar{x}) \leq \phi(x) + \alpha \parallel x - \bar{x} \parallel \text{ for all } x \in F.$$

Definition 3.2. [18, Remark 3.10] We call that constraint qualification (CQ) is satisfied at $\bar{x} \in F$ if there do not exist $\mu_j \geq 0$, $j \in J(\bar{x})$ not all zero, such that

$$0 \in \sum_{j\in J(\bar{x})}\mu_j\partial g_j(\bar{x}) + N(\bar{x};\Omega),$$

where $J(\bar{x}) := \left\{j \in J \mid g_j(\bar{x}) = 0\right\}$.

To obtain the necessary optimality condition of the Karush–Kuhn–Tucker type, the above constraint qualification ([18, Remark 3.10]) is needed, which reduces to the extended the Mangasarian-Fromovitz constraint qualification (see, e.g., [20]) in the smooth setting.

Now, we establish the Karush–Kuhn–Tucker necessary conditions for a quasi α-solution of problem (P).

Theorem 3.1. *Let the* (CQ) *be satisfied at* $\bar{x} \in \Omega$. *If* $\bar{x}$ *is a quasi* α*-solution of problem* (P)*, then there exist* $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ *with* $\sum_{k\in K} \tau_k = 1$, *and* $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ *such that*

$$\begin{aligned}
&0 \in \sum_{k\in K} \tau_k \partial f_k(\bar{x}) + \sum_{j\in J} \lambda_j \partial g_j(\bar{x}) + \alpha \mathbb{B}_{X^*} + N(\bar{x};\Omega),\\
&\tau_k \left(f_k(\bar{x}) - (\max_{k\in K}) f_k(\bar{x}) \right) = 0,\ k \in K,\\
&\lambda_j g_j(\bar{x}) = 0,\ j \in J.
\end{aligned} \tag{3.1}$$

Proof. Let $\bar{x}$ be a quasi α-solution of problem (P). Then $\bar{x}$ is a minimizer of the following problem

$$\min_{x\in F} \left(\phi(x) + \alpha \parallel x - \bar{x} \parallel \right), \tag{3.2}$$

and $\phi(x) := \max_{k\in K} f_k(x)$. Define a real-valued function ψ by

$$\psi(x) := \max_{j\in J} \left\{ \phi(x) + \alpha \parallel x - \bar{x} \parallel - \phi(\bar{x}), g_j(x) \right\},\ x \in X.$$

We claim that

$$\psi(\bar{x}) = 0 \le \psi(x),\ \forall x \in \Omega. \tag{3.3}$$

Indeed, it is easy to see the equality in (3.3) holds due to $\bar{x} \in F$. Let us justify the inequality therein. If $x \in F$, then $\psi(x) \ge 0$. Otherwise, $\psi(x) < 0$ leads to

$$\phi(x) + \alpha \parallel x - \bar{x} \parallel - \phi(\bar{x}) < 0,$$

which is a contradiction to (3.2). If $x \in \Omega \backslash F$, then there is $j_0 \in J$ such that $g_{j_0}(x) > 0$, which entails that $\psi(x) > 0$.

Thus, (3.3) is valid, and this infers that $\bar{x}$ is a minimizer for ψ on Ω. Then, $\bar{x}$ is a minimizer of the following unconstrained optimization problem

$$\min_{x\in X} \psi(x) + \delta(x;\Omega). \tag{3.4}$$

Invoking now the nonsmooth version of Fermat's rule (2.2) to problem (3.4), we have

$$0 \in \partial \left(\psi + \delta(\cdot;\Omega) \right)(\bar{x}). \tag{3.5}$$

Since the function ψ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot;\Omega)$ is l.s.c around $\bar{x}$, it follows from the sum rule (2.3) applies to (3.5) and from the relation in (2.1) that

$$0 \in \partial \psi(\bar{x}) + N(\bar{x};\Omega). \tag{3.6}$$

Applying further the formula for the limiting subdifferential of maximum functions and the limiting subdifferential sum rule for Lipschitz functions (Lemma 2.1)

$$\begin{aligned}
0 \in \Big\{ \mu_0 \partial \left(\phi + \alpha \parallel \cdot - \bar{x} \parallel \right)(\bar{x}) + \sum_{j\in J(\bar{x})} \mu_j \partial g_j(\bar{x}) \mid \mu_0 \ge 0,\ \mu_j \ge 0,\ \mu_j g_j(\bar{x}) = 0,&\\
j \in J(\bar{x}),\ \mu_0 + \sum_{j\in J(\bar{x})} \mu_j = 1 \Big\} + N(\bar{x};\Omega),&
\end{aligned} \tag{3.7}$$

where $J(\bar{x}) := \{ j \in J \,|\, g_j(\bar{x}) = 0 \}$. In view of $\partial\left(\| \cdot - \bar{x} \|\right)(\bar{x}) = \mathbb{B}_{X^*}$, we get

$$\begin{aligned}
&\partial\big(\max_{k\in K} f_k + \alpha \parallel \cdot - \bar{x} \parallel\big)(\bar{x}) \subset \partial(\max_{k\in K} f_k)(\bar{x}) + \alpha\mathbb{B}_{X^*} \\
&\subset\Big\{ \sum_{k\in K(\bar{x})} \tau_k \partial f_k(\bar{x}) + \alpha\mathbb{B}_{X^*} \,\Big|\, \tau_k \geq 0,\, \tau_k\big(f_k(\bar{x}) - (\max f_k)(\bar{x})\big) = 0,\, k \in K(\bar{x}), \\
&\sum_{k\in K(\bar{x})} \tau_k = 1 \Big\},
\end{aligned} \tag{3.8}$$

where $K(\bar{x}) := \{k \in K \mid f_k(\bar{x}) = \phi(\bar{x})\} \neq \emptyset$. It follows (3.6), (3.7) and (3.8) that

$$\begin{aligned}
&0 \in \Big\{ \sum_{k\in K(\bar{x})} \mu_0 \tau_k \partial f_k(\bar{x}) + \mu_0\alpha\mathbb{B}_{X^*} \,\Big|\, \tau_k \geq 0,\, \tau_k\big(f_k(\bar{x}) - (\max f_k)(\bar{x})\big) = 0,\, k \in K(\bar{x}), \\
&\sum_{k\in K(\bar{x})} \tau_k = 1 \Big\} + \Big\{ \sum_{j\in J(\bar{x})} \mu_j \partial g_j(\bar{x}) \,\Big|\, \mu_0 \geq 0,\, \mu_j \geq 0,\, \mu_j g_j(\bar{x}) = 0,\, j \in J(\bar{x}), \\
&\mu_0 + \sum_{j\in J(\bar{x})} \mu_j = 1 \Big\} + N(\bar{x};\Omega).
\end{aligned}$$

As the (CQ) is satisfied at $\bar{x}$, we have $\mu_0 \neq 0$, and

$$\begin{aligned}
0 \in &\Big\{ \sum_{k\in K(\bar{x})} \tau_k \partial f_k(\bar{x}) + \alpha\mathbb{B}_{X^*} \,\Big|\, \tau_k \geq 0,\, \tau_k\big(f_k(\bar{x}) - (\max f_k)(\bar{x})\big) = 0,\, k \in K,\, \sum_{k\in K(\bar{x})} \tau_k = 1 \Big\} \\
&+ \Big\{ \sum_{j\in J(\bar{x})} \frac{\mu_j}{\mu_0} \partial g_j(\bar{x}) \,\Big|\, \mu_0 > 0,\, \mu_j \geq 0,\, \mu_j g_j(\bar{x}) = 0,\, j \in J(\bar{x}),\, \mu_0 + \sum_{j\in j} \mu_j = 1 \Big\} + N(\bar{x};\Omega).
\end{aligned} \tag{3.9}$$

Now, letting $\tau_k := 0$ for $k \in K \backslash K(\bar{x})$, $\mu_j := 0$ for $j \in J \backslash J(\bar{x})$, and $\lambda_j := \frac{\mu_j}{\mu_0}$, we see that (3.9) clearly implies (3.1), which completes the proof. □

Now, we introduce the concepts of generalized convexity to formulate sufficient conditions for quasi α-solutions of problem (P).

Definition 3.3. [18, Definition 3.11] We say that (f,g) is generalized convex on Ω at $\bar{x} \in \Omega$ if, for any $x \in \Omega$, $\xi_k \in \partial f_k(\bar{x})$, $k \in K$, and $\eta_j \in \partial g_j(\bar{x})$, $j \in J$, there exists $\nu \in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned}
f_k(x) - f_k(\bar{x}) &\geq \langle \xi_k, \nu \rangle, \quad k \in K, \\
g_j(x) - g_j(\bar{x}) &\geq \langle \eta_j, \nu \rangle, \quad j \in J,
\end{aligned}$$

and

$$\langle b, \nu \rangle \leq \parallel x - \bar{x} \parallel, \quad b \in \mathbb{B}_{X^*}.$$

The following theorem describes sufficient optimality conditions for a quasi α-solution of problem (P).

Theorem 3.2. *Let $\bar{x} \in F$ satisfy Karush–Kuhn–Tucker conditions (3.1). If (f,g) is generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a quasi α-solution of problem (P).*

Proof. Let $\phi(x) := \max_{k\in K} f_k(x)$. Since $\bar{x} \in F$ satisfies condition (3.1), there exist $\tau_k \geq 0$, $k \in K$ with $\sum_{k\in K} \tau_k = 1$, $\lambda_j \geq 0$, $j \in J$, $\xi_k \in \partial f_k(\bar{x})$, $k \in K$, $\eta_j \in \partial g_j(\bar{x})$, $j \in J$, and $b \in \mathbb{B}_{X^*}$ such that

$$-\left(\sum_{k\in K} \tau_k \xi_k + \sum_{j\in J} \lambda_j \eta_j + \alpha b\right) \in N(\bar{x};\Omega), \tag{3.10}$$

$$\tau_k \left(f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x})\right) = 0,\ k \in K, \tag{3.11}$$

$$\lambda_j g_j(\bar{x}) = 0,\ j \in J.$$

Assume to the contrary that $\bar{x}$ is not a quasi α-solution of problem (P), we have that there is $\hat{x} \in F$ such that

$$\phi(\bar{x}) > \phi(\hat{x}) + \alpha \parallel \hat{x} - \bar{x} \parallel . \tag{3.12}$$

From the generalized convex on Ω at $\bar{x}$, we deduce from (3.10) that, for such $\hat{x}$, there is $\nu \in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{k\in K} \tau_k \langle \xi_k, \nu\rangle + \sum_{j\in J} \lambda_j \langle \eta_j, \nu\rangle + \alpha\langle b, \nu\rangle \\ &\leq \sum_{k\in K} \tau_k \left(f_k(\hat{x}) - f_k(\bar{x})\right) + \sum_{j\in J} \lambda_j \left(g_j(\hat{x}) - g_j(\bar{x})\right) + \alpha \parallel \hat{x} - \bar{x} \parallel . \end{aligned}$$

Hence,

$$\sum_{k\in K} \tau_k f_k(\bar{x}) + \sum_{j\in J} \lambda_j g_j(\bar{x}) \leq \sum_{k\in K} \tau_k f_k(\hat{x}) + \sum_{j\in J} \lambda_j g_j(\hat{x}) + \alpha \parallel \hat{x} - \bar{x} \parallel . \tag{3.13}$$

Observe that $\lambda_j g_j(\bar{x}) = 0$ and $\lambda_j g_j(\hat{x}) \leq 0$ for all $j \in J$. So, taking into consideration (3.13), we have

$$\sum_{k\in K} \tau_k f_k(\bar{x}) \leq \sum_{k\in K} \tau_k f_k(\hat{x}) + \alpha \parallel \hat{x} - \bar{x} \parallel .$$

On the other hand, by (3.11), it holds that

$$\begin{aligned} \sum_{k\in K} \tau_k \phi(\bar{x}) &= \sum_{k\in K} \tau_k f_k(\bar{x}) \\ &\leq \sum_{k\in K} \tau_k f_k(\hat{x}) + \alpha \parallel \hat{x} - \bar{x} \parallel \\ &\leq \sum_{k\in K} \tau_k \phi(\hat{x}) + \alpha \parallel \hat{x} - \bar{x} \parallel . \end{aligned}$$

This implies

$$\phi(\bar{x}) \leq \phi(\hat{x}) + \alpha \parallel \hat{x} - \bar{x} \parallel \tag{3.14}$$

due to $\sum_{k\in K} \tau_k = 1$. Obviously, (3.14) contradicts (3.12), which completes the proof. □

Now, we give an example that illustrates that we cannot remove the generalized convex property of (f, g) on Ω at the considered point in Theorem 3.2.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) := (f_1(x), f_2(x))$, where

$$f_1(x) := \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $f_2(x) := -x^2$. We consider the feasible set F given by (1.1) with $\Omega := \mathbb{R}$ and $J := \{1\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) := \begin{cases} \frac{1}{2}(x-1), & x \geq 1, \\ 0, & 0 < x < 1, \\ x, & x \leq 0. \end{cases}$$

Then, $F = (-\infty, 1]$. Note that f_k, $k = 1, 2$, are locally Lipschitz at $\bar{x} = (0,0)$, and

$$\partial f_1(\bar{x}) = [-1,1], \quad \partial f_2(\bar{x}) = \{0\}, \text{ and } \partial g(\bar{x}) = \{0,1\}.$$

Choosing $x = \frac{3}{5\pi} \in F$, $\xi_1 := 0 \in \partial f_1(\bar{x})$, and $\xi_2 := 0$, we have

$$f_1(x) - f_1(\bar{x}) = \Big(\frac{3}{5\pi}\Big)^2 \sin\frac{5\pi}{3} < 0 = \langle \xi_1, \nu \rangle,$$
$$f_2(x) - f_2(\bar{x}) = -\Big(\frac{3}{5\pi}\Big)^2 < 0 = \langle \xi_2, \nu \rangle,$$

for $\nu \in N(\bar{x};\mathbb{R})^\circ$. Take arbitrarily α satisfying $0 < \alpha < \frac{2}{3\pi}$. It is not difficult to infers that $\bar{x}$ satisfies Karush–Kuhn–Tucker conditions (3.1); e.g., $\tau_1 = 1$, $\tau_2 = 0$, $\lambda = 1$. However, $\bar{x}$ is not a quasi α-solution of minimax problem. To see the following, we choose $\hat{x} = \frac{2}{3\pi} \in F$. Then,

$$\max_{k=1,2} f_k(\hat{x}) + \alpha|\hat{x} - \bar{x}| < \max_{k=1,2} f_k(\bar{x}).$$

One can check that the (f,g) does not satisfy the generalized convexity.

4. Duality Relations

In this section, we formulate a dual problem for the minimax programming problem (P) in some ways of Wolfe [21], and explore weak, strong and converse-like duality relations between them.

Let $z \in X$, $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ with $\sum_{k\in K} \tau_k = 1$, and $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. In connection with the minimax programming problem (P), we consider a dual problem in the following form:

$$\max_{(z,\tau,\lambda)\in F_D} \Big\{ L(z,\tau,\lambda) := \phi(z) + \sum_{j\in J} \lambda_j g_j(z) \Big\}. \tag{D}$$

Here, we denote $\phi(z) := \max_{k\in K} f_k(z)$ and

$$F_D := \Big\{ (z,\tau,\lambda) \in \Omega \times \Big(\mathbb{R}^l_+ \backslash \{0\}\Big) \times \mathbb{R}^m_+ \mid 0 \in \sum_{k\in K} \tau_k \partial f_k(z) + \sum_{j\in J} \lambda_j \partial g_j(z) + \alpha \mathbb{B}_{X^*} + N(z;\Omega),$$
$$\tau_k \left(f_k(z) - \phi(z)\right) = 0,\ k \in K,\ \sum_{k\in K} \tau_k = 1 \Big\}.$$

Definition 4.1. A point $(\bar{z}, \bar{\tau}, \bar{\lambda}) \in F_D$ is called a quasi α-solution of problem (D) if and only if

$$L(\bar{z},\bar{\tau},\bar{\lambda}) + \alpha \parallel (\bar{z},\bar{\tau},\bar{\lambda}) - (z,\tau,\lambda) \parallel \geq L(z,\tau,\lambda),\ \forall (z,\tau,\lambda) \in F_D.$$

We provide weak duality theorem between (P) and (D).

Theorem 4.1. *Let $x \in F$, and let $(z,\tau,\lambda) \in F_D$. If (f,g) is generalized convex on Ω at z, then*

$$\phi(x) + \alpha \parallel x - z \parallel \geq L(z,\tau,\lambda).$$

Proof. Since $(z,\tau,\lambda)\in F_D$, there exist $\tau:=(\tau_1,\dots,\tau_l)\in\mathbb{R}^l_+$ with $\sum_{k\in K}\tau_k=1$, $\lambda:=(\lambda_1,\dots,\lambda_m)\in\mathbb{R}^m_+$, $\xi_k\in\partial f_k(z)$, $k\in K$, $\eta_j\in\partial g_j(z)$, $j\in J$, and $b\in\mathbb{B}_{X^*}$ such that

$$\begin{aligned}&-\left(\sum_{k\in K}\tau_k\xi_k+\sum_{j\in J}\lambda_j\eta_j+\alpha b\right)\in N(z;\Omega),\\ &\tau_k\left(f_k(z)-\phi(z)\right)=0,\ k\in K.\end{aligned}\tag{4.1}$$

By the generalized convex on Ω at z, there exists $\nu\in N(z;\Omega)^\circ$ such that

$$\begin{aligned}0&\le\sum_{k\in K}\tau_k\langle\xi_k,\nu\rangle+\sum_{j\in J}\lambda_j\langle\eta_j,\nu\rangle+\alpha\langle b,\nu\rangle\\ &\le\sum_{k\in K}\tau_k\left(f_k(x)-f_k(z)\right)+\sum_{j\in J}\lambda_j\left(g_j(x)-g_j(z)\right)+\alpha\parallel x-z\parallel.\end{aligned}$$

Hence,

$$\sum_{k\in K}\tau_kf_k(z)+\sum_{j\in J}\lambda_jg_j(z)\le\sum_{k\in K}\tau_kf_k(x)+\sum_{j\in J}\lambda_jg_j(x)+\alpha\parallel x-z\parallel.\tag{4.2}$$

In addition, $x\in F$. It follows that $\sum_{j\in J}\lambda_jg_j(x)\le 0$. So, we get from (4.2) and (4.1) that

$$\begin{aligned}\sum_{k\in K}\tau_k\phi(z)+\sum_{j\in J}\lambda_jg_j(z)&=\sum_{k\in K}\tau_kf_k(z)+\sum_{j\in J}\lambda_jg_j(z)\\ &\le\sum_{k\in K}\tau_kf_k(x)+\sum_{j\in J}\lambda_jg_j(x)+\alpha\parallel x-z\parallel\\ &\le\sum_{k\in K}\tau_k\phi(x)+\alpha\parallel x-z\parallel.\end{aligned}$$

This implies that

$$\phi(z)+\sum_{j\in J}\lambda_jg_j(z)\le\phi(x)+\alpha\parallel x-z\parallel.$$

The proof is complete. □

The forthcoming theorem describes the strong duality relation between (P) and (D).

Theorem 4.2. *Let $\bar{x}\in F$ be a quasi α-solution of problem (P) such that the (CQ) is satisfied at this point. Then there exists $(\bar{\tau},\bar{\lambda})\in\left(\mathbb{R}^l_+\setminus\{0\}\right)\times\mathbb{R}^m_+$ such that $(\bar{x},\bar{\tau},\bar{\lambda})\in F_D$ and*

$$\phi(\bar{x})=L(\bar{x},\bar{\tau},\bar{\lambda}).$$

Furthermore, if (f,g) is generalized convex on Ω at any $z\in\Omega$, then $(\bar{x},\bar{\tau},\bar{\lambda})$ is a quasi α-solution of problem (D).

Proof. Applying Theorem 3.1, we find $\tau:=(\tau_1,\dots,\tau_l)\in\mathbb{R}^l_+$ with $\sum_{k\in K}\tau_k=1$, and $\lambda:=(\lambda_1,\dots,\lambda_m)\in\mathbb{R}^m_+$ such that

$$\begin{aligned}&0\in\sum_{k\in K}\tau_k\partial f_k(\bar{x})+\sum_{j\in J}\lambda_j\partial g_j(\bar{x})+\alpha\mathbb{B}_{X^*}+N(\bar{x};\Omega),\\ &\tau_k\left(f_k(\bar{x})-\max_{k\in K}f_k(\bar{x})\right)=0,\ k\in K,\\ &\lambda_jg_j(\bar{x})=0,\ j\in J.\end{aligned}\tag{4.3}$$

Putting $\bar{\tau}_k := \frac{\tau_k}{\sum_{k\in K}\tau_k}$, $k \in K$ with $\sum_{k\in K}\bar{\tau}_k = 1$, and $\bar{\lambda}_j := \frac{\lambda_j}{\sum_{k\in K}\tau_k}$, $j \in J$, we have $\bar{\tau} := (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+\backslash\{0\}$, and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_m) \in \mathbb{R}^m_+$. Observe that the assertion in (4.3) is also valid when τ_k's, and λ_j's are replaced by $\bar{\tau}_k$'s, and $\bar{\lambda}_j$'s, respectively. So we conclude that $(\bar{x}, \bar{\tau}, \bar{\lambda}) \in F_D$. Obviously,

$$L(\bar{x}, \bar{\tau}, \bar{\lambda}) = \phi(\bar{x}) + \sum_{j\in J}\lambda_j g_j(\bar{x}) = \phi(\bar{x}).$$

Thus, when (f, g) is generalized convex on Ω at any $z \in \Omega$, we apply the weak duality result in Theorem 4.1 to conclude that

$$L(\bar{x}, \bar{\tau}, \bar{\lambda}) + \alpha \parallel (\bar{x}, \bar{\tau}, \bar{\lambda}) - (z, \tau, \lambda) \parallel = \phi(\bar{x}) + \alpha \parallel (\bar{x}, \bar{\tau}, \bar{\lambda}) - (z, \tau, \lambda) \parallel \geq L(z, \tau, \lambda)$$

for any $(z, \tau, \lambda) \in F_D$. It means that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a quasi α-solution of problem (D). □

The forthcoming theorem declares converse-like duality relation between (P) and (D).

Theorem 4.3. Let $(\bar{x}, \bar{\tau}, \bar{\lambda}) \in F_D$ be such that $\phi(\bar{x}) = L(\bar{x}, \bar{\tau}, \bar{\lambda})$. If $\bar{x} \in F$ and (f, g) is generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a quasi α-solution of problem (P).

Proof. Since $(\bar{x}, \bar{\tau}, \bar{\lambda}) \in F_D$, there exist $\bar{\tau} \in \mathbb{R}^l_+$ with $\sum_{k\in K}\bar{\tau}_k = 1$, $\bar{\lambda} \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(\bar{x})$, $k \in K$, $\bar{\eta}_j \in \partial g_j(\bar{x})$, $j \in J$, and $\bar{b} \in \mathbb{B}_{X^*}$ such that

$$-\left(\sum_{k\in K}\bar{\tau}_k\bar{\xi}_k + \sum_{j\in J}\bar{\lambda}_j\bar{\eta}_j + \alpha\bar{b}\right) \in N(\bar{x};\Omega), \tag{4.4}$$

$$\bar{\tau}_k\left(f_k(\bar{x}) - \phi(\bar{x})\right) = 0,\ k \in K. \tag{4.5}$$

Since $\phi(\bar{x}) = L(\bar{x}, \bar{\tau}, \bar{\lambda})$, then $\sum_{j\in J}\lambda_j g_j(\bar{x}) = 0$. Due to $\bar{x} \in F$, we deduce $\lambda_j g_j(\bar{x}) = 0,\ j \in J$. This together with (4.4), (4.5) that $\bar{x}$ satisfies condition (3.1). To finish the proof, it remains to apply Theorem 3.2. □

5. Multiobjective Optimization Problems

In this section, we establish necessary and sufficient conditions for weakly quasi ε-Pareto solutions of (MP) by employing the optimality conditions obtained by (P) in Section 3.

The following result is a Karush–Kuhn–Tucker (KKT) necessary condition for weakly quasi ε-Pareto solutions of problem (MP).

Theorem 5.1. *Let $\varepsilon := (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l_+$ and the* (CQ) *be satisfied at $\bar{x} \in \Omega$. If $\bar{x}$ is a weakly quasi ε-Pareto solution of problem* (MP)*, then there exist $\tau \in \mathbb{R}^l_+$ with $\sum_{k\in K}\tau_k = 1$, and $\lambda \in \mathbb{R}^m_+$ such that*

$$\begin{aligned} &0 \in \sum_{k\in K}\tau_k\partial f_k(\bar{x}) + \sum_{j\in J}\lambda_j\partial g_j(\bar{x}) + \tilde{\alpha}\mathbb{B}_{X^*} + N(\bar{x};\Omega), \\ &\lambda_j g_j(\bar{x}) = 0,\ j \in J, \end{aligned} \tag{5.1}$$

where $\tilde{\alpha} := \max_{k\in K}\{\alpha_k\}$.

Proof. Let $\bar{x}$ be a weakly quasi ε-Pareto solution of problem (MP) and let

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}),\ k \in K,\ x \in X.$$

We will show that $\bar{x}$ is a quasi $\tilde{\alpha}$-solution of the minimax programming problem

$$\min_{x\in F}\max_{k\in K}\widehat{f}_k(x). \tag{5.2}$$

To do this, let us put $\widehat{\phi}(x) := \max_{k\in K}\widehat{f}_k(x)$ and prove that

$$\widehat{\phi}(\bar{x}) \le \widehat{\phi}(x) + \tilde{\alpha}\parallel x-\bar{x}\parallel,\ \forall x\in F. \tag{5.3}$$

Indeed, if (5.3) is not valid, then there exists $x_0 \in F$ such that $\widehat{\phi}(x_0)+\tilde{\alpha}\parallel x_0-\bar{x}\parallel < \widehat{\phi}(\bar{x})$. Since $\widehat{\phi}(\bar{x}) = 0$, it holds that $\max_{k\in K}\{f_k(x_0)-f_k(\bar{x})\}+\tilde{\alpha}\parallel x_0-\bar{x}\parallel < 0$. Thus,

$$f(x_0)-f(\bar{x})+\varepsilon\parallel x_0-\bar{x}\parallel\in -\operatorname{int}\mathbb{R}^l_+,$$

which contradicts the fact that $\bar{x}$ is a weakly quasi ε-Pareto solution of problem (MP). So, we can employ the KKT condition in Theorem 3.1, but applied to problem (5.2). Then we find $\tau\in\mathbb{R}^l_+$ with $\sum_{k\in K}\tau_k = 1$, and $\lambda\in\mathbb{R}^m_+$ such that

$$\begin{aligned}
&0\in\sum_{k\in K}\tau_k\partial f_k(\bar{x})+\sum_{j\in J}\lambda_j\partial g_j(\bar{x})+\tilde{\alpha}\mathbb{B}_{X^*}+N(\bar{x};\Omega),\\
&\tau_k\left(\widehat{f}_k(\bar{x})-\max_{k\in K}\widehat{f}_k(\bar{x})\right)=0,\ k\in K,\\
&\lambda_j g_j(\bar{x})=0,\ j\in J.
\end{aligned} \tag{5.4}$$

It is now clear that (5.4) implies (5.1). Thus, the proof is complete. □

The following theorem describes sufficient optimality conditions for a weakly quasi $\tilde{\varepsilon}$-Pareto solution of problem (MP).

Theorem 5.2. *Let $\tilde{\varepsilon} := (\tilde{\alpha},\ldots,\tilde{\alpha})\in\mathbb{R}^l_+$ and $\bar{x}\in F$ satisfy condition* (5.1)*. If (f,g) is generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a weakly quasi $\tilde{\varepsilon}$-Pareto solution of problem* (MP)*.*

Proof. Set $\widehat{f}_k(x) := f_k(x)-f_k(\bar{x}),\ k\in K,\ x\in X$. As $\bar{x}\in F$ satisfies condition (5.1), it is not difficult to see that $\bar{x}$ also satisfies condition (5.4). Let $\widehat{f} := (\widehat{f}_1,\ldots,\widehat{f}_l)$. Since (f,g) is generalized convex on Ω at $\bar{x}$, it follows that $(\widehat{f},g)$ is generalized convex on Ω at this point as well. We apply the sufficient criteria in Theorem 3.2 to conclude that $\bar{x}$ is a quasi $\tilde{\alpha}$-solution of the minimax programming problem

$$\min_{x\in F}\max_{k\in K}\widehat{f}_k(x).$$

It means that $\widehat{\phi}(\bar{x})\le\widehat{\phi}(x)+\tilde{\alpha}\parallel x-\bar{x}\parallel,\ \forall x\in F$, where $\widehat{\phi}(x) := \max_{k\in K}\widehat{f}_k(x)$. In other words, we obtain

$$0\le\max_{k\in K}\{f_k(x)-f_k(\bar{x})\}+\tilde{\alpha}\parallel x-\bar{x}\parallel,\ \forall x\in F,$$

which entails that $f(x)-f(\bar{x})+\tilde{\varepsilon}\parallel x-\bar{x}\parallel\notin-\operatorname{int}\mathbb{R}^l_+,\ \forall x\in F$. Consequently, $\bar{x}$ is a weakly quasi $\tilde{\varepsilon}$-Pareto solution of problem (MP). □

6. The Conclusion

In this paper, under the minimax programming approach, we explored the optimality conditions (both necessary and sufficient) for a weakly quasi ε-Pareto solution to a multiobjective optimization problem. It is worth noting that the approach, which is motivated by the point-based technique based on scalarization idea, usually is used in dealing with multiobjective optimization problems.

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