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A MODEL OF DEFORMATIONS OF A DISCONTINUOUS STIELTJES STRING WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. Variational methods are used to study a model of the deformation of a discontinuous Stieltjes string (a chain of strings held together by springs) located along the segment $[0,l]$. The model is described by the integro-differential equation $-\left(pu'_\mu\right)(x)+\left(pu'_\mu\right)(0)+\int_0^x ud[Q]=F(x)-F(0)$ with derivatives with respect to the measure μ generated by a given strictly increasing function $\mu(x)$ on the segment $[0,l]$, where the function $u(x)$ determines the deformation of the string, $p(x)$ characterizes the elasticity of the string, the functions $Q(x)$ and $F(x)$ describe the elastic response of the external environment and the external load, respectively. The integral $\int_0^x ud[Q]$ is understood in the generalized sense according to Stieltjes. We are looking for solutions $u(x)$ in the class of μ–absolutely continuous functions on $[0,l]$, whose derivatives have bounded variation on $[0,l]$. We assume that one of the boundary conditions is nonlinear and has the form $-p(l-0)u'_\mu(l-0)-\gamma u(l)\in N_{[-k,k]}u(l)$, where $N_{[-k,k]}u(l)$ denotes the outward normal cone at the point $u(l)$ to the segment $[-k,k]$. This condition arises due to the presence of the limiter $[-k,k]$ on the motion of the elastically fixed right end of the string (by a spring with elasticity γ) so that $|u(l)|\leq k$. In this paper, necessary and sufficient conditions for the minimization of the energy functional of the Stieltjes string system are established, the critical loads at which the contact of the end of the string with the boundary points of the limiter occurs are determined, and the dependence of the solution on the length of the limiter is studied.

Keywords. Discontinuous Stieltjes string; Energy functional; Stieltjes integral; Variation.

1. INTRODUCTION

The differential equation

$$-\left(pu'\right)'+qu=f \tag{1.1}$$

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appears frequently in various natural science models (see, for example, [1]-[34] and the references therein). In particular, the spectral problem for the Shrödinger equation

$$-u'' + qu = \lambda u \tag{1.2}$$

becomes actual in mathematical physics, where q is a singular potential, containing singularities of the delta function type and other more complicated singularities which are generated by discontinuities of the solutions.

The presence of the δ–function $\delta(x-\xi)$ in the coefficient q of the equation

$$-u'' + qu = 0$$

is described by physicists by the equality

$$u'(\xi+0) - u'(\xi-0) = \gamma u(\xi),$$

executed at the point ξ and called the δ*–interaction*. Mathematically, the symbol δ' denotes the derivative of the δ–function, while for physicists $\delta'(x-\xi)$ means that at the point ξ the equalities $u'(\xi-0) = \Delta u(\xi)$ and $u'(\xi+0) = \Delta u(\xi)$ hold, where $\Delta u(\xi) = u(\xi+0) - u(\xi-0)$. The latter equalities are called δ'*–interactions*.

The investigation of (1.2) for the cases of various impulse perturbations (singular potentials) is relevant. The most profound results concerning this topic associated with the works of Shkalikov and Savchuk [25]–[27], Vladikina and Shkalikov [31], Mityagin and Djakov [5], Mikhailets [17, 18], Hryniv and Mykytyuk [7]. The main research tool in these works is the theory of generalized functions (the theory of Schwarz-Sobolev distributions).

The pointwise analysis of solutions of Equation (1.1) in the case of strong singularities in the potential q, when q can be considered as a generalized derivative of a function Q of bounded variation, was proposed by Pokornyi in [19]. Developing ideas of Atkinson and Feller (see [2, 6]), Equation (1.1) with singularities in coefficients and in the right-hand side (of δ-function type) was replaced in [21] by the integro-differential equation

$$-(pu')(x) + (pu')(0) + \int_0^x u dQ = F(x) - F(0)$$

with absolutely continuous solutions whose derivatives, as well as p, Q, F are functions of bounded variation on the segment $[0,l]$; the integral is understood in the Stieltjes sense. In contrast with [5]-[8], [16]–[18], [25]–[27] and [31], the last equation is defined at each point. Due to this fact it was possible to develop qualitative methods for analyzing the classical oscillation properties of the Sturm-Liouville problem [21].

In this paper, we study a model of deformations of a discontinuous Stieltjes string with singularities of both the δ-interaction and the δ'–interaction types. It has been found by variational methods that such model can be described as the problem:

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int_0^x u d[Q] = F(x) - F(0), \\ u(0) = 0, \\ -p(l-0)u'_\mu(l-0) - \gamma u(l) \in N_{[-k,k]} u(l), \end{cases} \tag{1.3}$$

with the equation defined at each point x, where u'_μ denotes the derivative with respect to the measure, generated by a given strictly increasing function $\mu(x)$ on the segment $[0,l]$. The function $u(x)$ determines the deformation of the string, $p(x)$ characterizes the elasticity of the string, $Q(x)$ and $F(x)$ describe the elastic response of the environment and the external load, respectively. The integral $\int_0^x ud[Q]$ is understood in the extended sense according to Stieltjes. To emphasize that we are talking about such integral, we enclose the function under the differential in square brackets.

We assume that at the point $x=0$ the string is rigidly fixed, that can be expressed by the condition $u(0)=0$. At the point $x=l$ there is an elastic support (spring of elasticity γ). Additionally, at the point $x=l$ we have a limiter $[-k,k]$ on the motion of the elastically fixed (by a spring with elasticity γ) right end of the string so that $|u(l)|\le k$. Depending on the applied external load, the right end of the string either touches the boundary points of the limiter, or remains within the interval $(-k,k)$. In this paper, it is proved that the last condition can be written in the form $-p(l-0)u'_\mu(l-0)-\gamma u(l)\in N_{[-k,k]}u(l)$, where $N_{[-k,k]}u(l)$ denotes the outward normal cone at the point $u(l)$ to the segment $[-k,k]$. The model of deformations of a continuous string with boundary conditions $u(0)=0$, $-p(l)u'(l)\in N_{[-k,k]}u(l)$ was studied in [12]. In the present paper, we discuss the discontinuous case.

As in [12, 14], [19]–[24], the proposition that our problem has a physical nature will be essential for us, i.e., solutions minimize some energy functional. Notice that the term "string" has purely mathematical character, as well as in the simpler case, when our equation is equivalent to Equation (1.1) for smooth $p(x),Q(x),F(x)$ and $\mu(x)=x$, where $q=Q'$ and $f=F'$. It is worth noting that Equation (1.1) has a universal character in a wide variety of natural science problems: from the Shrödinger equation in quantum mechanics to the processes in electrical circuits, acoustic pipes, neural fibers, various waveguides and so on (see, e.g., [1, 28, 33]).

In Section 2, we collect some preliminary results and terminologies. In Section 3, we develop the mathematical model of the deformation of a discontinuous Stieltjes string, and relate the solutions with the minimizers of the potential energy functional of the system. In Section 4, the last section, we will present the complete solution to the discontinuous Stieltjes string problem.

2. Preliminaries

In this section, we recall some notions and facts which we will need in the sequel.

The space $BV[0,l]$. This space is defined as the set of functions whose variation

$$V_0^l(u)=\sup_{0\le x_0<x_1<\ldots<x_k\le l}\sum_{i=0}^{k-1}|u(x_{i+1})-u(x_i)|$$

is bounded (see [35]-[38]). Every function $u(x)$ in $BV[0,l]$ admits the Jordan decomposition $u=u_1-u_2$, where u_1 and u_2 are non-decreasing functions. In the sequel, we denote by $S(u)$ the set of discontinuity points of a function $u(x)$. For any $u\in BV[0,l]$, the set $S(u)$ is at most countable. We can also assume that $S(u)=S(u_1)\cup S(u_2)$.

Jumps of BV functions. For any $u(x)$ in $BV[0,l]$ and at any point $\xi\in(0,l]$, both the left– and right–hand limit exist, that is,

$$u(\xi-0)=\lim_{x\to\xi-0}u(x)\quad\text{and}\quad u(\xi+0)=\lim_{x\to\xi+0}u(x).$$

By a *simple jump* of $u(x)$ at a point $x = \xi$, we mean the quantity

$$\Delta u(\xi) = u(\xi + 0) - u(\xi - 0).$$

By convention, we set

$$u(0-0) = u(0) \quad \text{and} \quad u(l+0) = u(l).$$

While speaking of the simple jump $\Delta u(\xi)$ for $0 < \xi < l$, we ignore the value $u(\xi)$ of the function $u(x)$ at the point $x = \xi$, and take into account only the left-hand limit $u(\xi - 0)$ and right-hand limit $u(\xi + 0)$.

By the *left jump* $u(x)$ at the point $x = \xi$, we mean the value

$$\Delta^- u(\xi) = u(\xi) - u(\xi - 0).$$

By the *right jump* $u(x)$ at the point $x = \xi$, we mean the value

$$\Delta^+ u(\xi) = u(\xi + 0) - u(\xi).$$

Notice that both $\Delta^\pm u(\xi) = 0$ except for those ξ in the at most countable set $S(u)$ of points of discontinuity of u. We define the *jump function* $u_s(x)$ for $u \in BV[0,l]$ as

$$u_s(x) = \sum_{0<\xi\le x} \Delta^- u(\xi) + \sum_{0\le\xi<x} \Delta^+ u(\xi),$$

where $u_s(0) = 0$. Any function $u(x)$ from $BV[0,l]$ can be represented as

$$u(x) = u_c(x) + u_s(x),$$

where $u_c(x)$ is a continuous function and $u_s(x)$ is the jump function.

μ–absolute continuity. Assume that the function $\mu(x)$ strictly increases on the segment $[0,l]$. We denote by μ the Lebesgue–Stieltjes measure [35]-[38] generated by the function $\mu(x)$ on $[0,l]$.

According to the Radon–Nikodym theorem [35]-[38], for every signed measure ν defined on the σ–algebra Σ of μ–measurable sets, which is absolutely continuous with respect to μ, there exists a μ–integrable function f such that

$$\nu(A) = \int_A f(x)\,d\mu, \quad A \in \Sigma.$$

If the signed measure ν is generated by a function of bounded variation $u(x)$, then $u(x)$ is μ–absolutely continuous if and only if

$$u(\beta) - u(\alpha) = \int_\alpha^\beta f\,d\mu, \quad \alpha, \beta \in [0,l],$$

where the integral is understood in the Lebesgue-Stieltjes sense. The function $f(x)$ is called the μ derivative of u with respect to the measure μ and is denoted by u'_μ.

Notice that the μ–absolute continuous function $u(x)$ can be discontinuous only at the discontinuity points of $\mu(x)$. At every point ξ of discontinuity of the function μ, it holds the equality

$$u'_\mu(\xi) = \frac{u(\xi+0) - u(\xi-0)}{\mu(\xi+0) - \mu(\xi-0)}.$$

π–integral. The generalized Stieltjes integral $\int_\alpha^\beta ud[v]$ was first introduced by Pokornyi in [19]. In the sequel, we will call such integral as the π*–integral*. Following [19], the π–integral $\int_\alpha^\beta ud[v]$ for functions $u(x)$ and $v(x)$ of bounded variation can be represented as

$$\int_\alpha^\beta ud[v] = \int_\alpha^\beta udv_c + \sum_{\alpha<s\le\beta} u(s-0)\Delta^- v(s) + \sum_{\alpha\le s<\beta} u(s+0)\Delta^+ v(s),$$

where v_c is the continuous part of v and the integral $\int_\alpha^\beta udv_c$ is understood in the Lebesgue–Stieltjes sense. For the π–integral, we have

$$\int_\alpha^\beta u\,d[v] = u(\beta)v(\beta) - u(\alpha)v(\alpha) - \int_\alpha^\beta v\,du,$$

where the integral $\int_\alpha^\beta v\,du$ is understood in the Lebesgue–Stieltjes sense (see [19]).

In view of the general nature of the π–integral, the integrating function $v(x)$ in this integral defines "splitting" measures (left and right) at singular points. Notice that the square brackets in the integral mean that the integral is considered over measures "splitting" at singular points. If $u(x)$ or $v(x)$ is continuous, then the π–integral coincides with the usual Stieltjes integral.

Integro–differential equation. The central object of this work is the integro-differential equation

$$-(pu'_\mu)(x) + \int_0^x ud[Q] = F(x) - F(0) - (pu'_\mu)(0), \quad x \in \overline{[0,l]}_S. \tag{2.1}$$

We assume the existence of a strictly increasing function $\mu(x)$, which is continuous at the points $x=0$ and $x=l$, and define the measure on the segment $[0,l]$ such that the solutions of Equation (2.1) are μ–absolutely continuous. We also assume that p, F are functions of bounded variations on $[0,l]$ with $\inf_{[0,l]} p > 0$, and the function $Q(x)$ does not decrease on $[0,l]$.

In order to use the methods of classical analysis to the situation we are studying, we must replace "conflict" points by their extensions. Let $S(\mu)$ be the set of points of discontinuity of $\mu(x)$, which is at most countable. We are looking for solutions $u(x)$ in the class of μ–absolutely continuous functions on $[0,l]$, whose derivatives u'_μ have bounded variation on $[0,l]$. Thus, any solution $u(x)$ of Equation (2.1) is a function of bounded variation on $[0,l]$, which can be discontinuous only at points from $S(\mu)$. Notice that the values $u(\xi_i)$, where $\xi_i \in S(\mu)$, are undefined: for the π–integral only the limit values $u(\xi_i - 0)$, $u(\xi_i + 0)$ are considered.

Let $J_\mu = [0,l] \setminus S(\mu)$. We introduce on J_μ the metric $\rho_\mu(x,y) = |\mu(x) - \mu(y)|$. The metric space (J_μ, ρ_μ) is not complete. Let us denote by $\overline{[0,l]}_\mu$ its completion by the ρ_μ metric. Notice that the set $\overline{[0,l]}_\mu$ consists of points in J_μ and, instead of any discontinuity point ξ in $S(\mu)$ of the function $\mu(x)$ it contains a pair of elements, denoted by $\xi - 0$ and $\xi + 0$. We define

$$u(\xi - 0) = \lim_{x\to\xi-0} u(x) \quad \text{and} \quad u(\xi+0) = \lim_{x\to\xi+0} u(x).$$

Thus, any solution of Equation (2.1) is defined on $\overline{[0,l]}_\mu$.

We introduce the function

$$\sigma(x) = \frac{\mu(x-0)+\mu(x+0)}{2} + p_1(x) + p_2(x) + Q(x) + F_1(x) + F_2(x), \quad x \in [0,l],$$

where $p = p_1 - p_2$ and $F = F_1 - F_2$ are the Jordan representations of p and F, respectively. Moreover, we can assume that the function $S(\sigma)$ contains only discontinuity points of μ, p, Q, F, i.e.,

$$S(\sigma) = S(\mu) \cup S(p) \cup S(Q) \cup S(F).$$

We denote by $S = S(\sigma) \setminus S(\mu)$, the set of discontinuity points of the function $\sigma(x)$ minus those of $\mu(x)$. Let $R_\mu = \overline{[0,l]}_\mu \bigcup S(\mu)$ and $JR_\mu = R_\mu \setminus S$. We complete JR_μ by the metric $\rho(x,y) = |\sigma(x) - \sigma(y)|$, replacing any point $s \in S$ by the pair $\{s-0, s+0\}$, and denote the resulting set by $\overline{[0,l]}_S$. Notice that the set $\overline{[0,l]}_S$ together with any point ξ of discontinuity of the function $\mu(x)$ contains the pair $\{\xi - 0, \xi + 0\}$, and every point $s \in S$ is replaced by the pair $\{s-0, s+0\}$.

From (2.1) it follows that for any point x at which all the functions μ, p, Q, F are continuous, there is a derivative $u'_\mu(x)$. At all other points, there are left and right derivatives $u'_\mu(\xi - 0)$ and $u'_\mu(\xi + 0)$ coinciding with one-sided limits. From (2.1) it follows that at the points of the discontinuity ξ of the function $\mu(x)$ the equalities

$$-p(\xi)\frac{\Delta u(\xi)}{\Delta \mu(\xi)} + p(\xi - 0)u'_\mu(\xi - 0) + u(\xi - 0)\Delta^- Q(\xi) = \Delta^- F(\xi), \tag{2.2}$$

$$p(\xi)\frac{\Delta u(\xi)}{\Delta \mu(\xi)} - p(\xi + 0)u'_\mu(\xi + 0) + u(\xi + 0)\Delta^+ Q(\xi) = \Delta^+ F(\xi), \tag{2.3}$$

hold, and at the points $s \in S$ the equality

$$-p(s+0)u'_\mu(s+0) + p(s-0)u'_\mu(s-0) + u(s)\Delta Q(s) = \Delta F(s) \tag{2.4}$$

holds.

Equation (2.1) is similar in its properties to an ordinary differential equation of the second order. The proof of the main facts are based on the following results.

Theorem 2.1 (see [20]). *For any point $x_0 \in \overline{[0,l]}_S \setminus S(\mu)$ and any scalars α and β, there is a unique solution of the problem*

$$\begin{cases} -(pu'_\mu)(x) + \int\limits_0^x u d[Q] = F(x) - F(0) - (pu'_\mu)(0) \\ u(x_0) = \alpha, \quad u'_\mu(x_0) = \beta. \end{cases} \tag{2.5}$$

Consider the homogeneous equation

$$-(pu'_\mu)(x) + \int_0^x u d[Q] = -(pu'_\mu)(0). \tag{2.6}$$

The result below follows in a classical way from Theorem 2.1.

Theorem 2.2. *For any point $x_0 \in \overline{[0,l]}_S \setminus S(\mu)$ two initial problems*

$$u(x_0) = 0, \quad u'_\mu(x_0) = 1, \tag{2.7}$$

$$u(x_0) = 1, \quad u'_\mu(x_0) = 0 \tag{2.8}$$

correspond two linearly independent solutions of Equation (2.6). The solution space of Equation (2.6) is their linear envelope.

Let functions $\varphi_1(x)$ and $\varphi_2(x)$ be solutions of Equation (2.6). Define functions $W_1(x)$, $W_2(x)$, $W_3(x)$ as follows.

$$W_1(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ {\varphi_1}'_\mu(x) & {\varphi_2}'_\mu(x) \end{vmatrix}, \quad \text{where } x \in \overline{[0,l]}_S \setminus S(\mu),$$

$$W_2(x) = \begin{vmatrix} \varphi_1(x-0) & \varphi_2(x-0) \\ {\varphi_1}'_\mu(x) & {\varphi_2}'_\mu(x) \end{vmatrix}, \quad \text{where } x \in S(\mu),$$

$$W_3(x) = \begin{vmatrix} \varphi_1(x+0) & \varphi_2(x+0) \\ {\varphi_1}'_\mu(x) & {\varphi_2}'_\mu(x) \end{vmatrix}, \quad \text{where } x \in S(\mu).$$

Notice that $W_2(x) \equiv W_3(x)$.

On $\overline{[0,l]}_S$ we define the *Wronskian* $W(x)$ as

$$W(x) = \begin{cases} W_1(x), & x \in \overline{[0,l]}_S \setminus S(\mu); \\ W_2(x), & x \in S(\mu). \end{cases}$$

Theorem 2.3. *Let $\varphi_1(x)$ and $\varphi_2(x)$ be two arbitrary solutions of Equation (2.6). The following statements are equivalent:*

1) for any x belonging to $\overline{[0,l]}_S$, the function $W(x)$ does not equal zero;

2) there is a point x^* of the set $\overline{[0,l]}_S$, in which $W(x)$ does not equal zero;

3) the functions $\varphi_1(x)$ and $\varphi_2(x)$ are linearly independent.

The proof also follows in a classical way from Theorem 2.1.

Lemma 2.1. *Let $\varphi_1(x)$ and $\varphi_2(x)$ be solutions of the homogeneous equation (2.6). Then the function $(pW)(x)$ is constant.*

The proof follows from a direct verification of the equality $(pW)'_\mu(x) \equiv 0$.

Let $u(x)$ be a function of bounded variation on $[0,l]$, continuous at the points $x=0$ and $x=l$.

Definition 2.1. By a *zero point* of the function $u(x)$, we mean a point $s \in [0,l]$ with

$$u(s-0)u(s+0) \le 0.$$

Notice that if at a zero point ξ the function $u(x)$ is continuous, then $u(\xi)=0$. If the zero point ξ is a point of discontinuity of u, then either one of limits $u(\xi-0)$ and $u(\xi+0)$ equals zero, or the function $u(x)$ changes sign at the point ξ.

According to [20], any nontrivial solution $u(x)$ of Equation (2.6) can have only a finite number of zero points.

Definition 2.2. We say that Equation (2.6) is *non-oscillating* on $[0,l]$ if every nontrivial solution of (2.6) has at most one zero point in $[0,l]$.

Theorem 2.4. *For the non-oscillation of Equation (2.6) on $[0,l]$, it is sufficient that the function $Q(x)$ be monotonically non-decreasing on the segment $[0,l]$.*

Proof. Let $\xi_1 < \xi_2$ be two adjacent zero points of a nontrivial solution $u(x)$ of Equation (2.6). Consider the case when $u(\xi_1-0)<0$, $u(\xi_1+0)>0$, $u(\xi_2-0)>0$, $u(\xi_2+0)<0$. It follows from Equation (2.6) that

$$p(\xi_1+0)u'_\mu(\xi_1+0) = p(\xi_1)u'_\mu(\xi_1) + u(\xi_1+0)\Delta^+Q(\xi_1),$$

$$p(\xi_2-0)u'_\mu(\xi_2-0)=p(\xi_2)u'_\mu(\xi_2)-u(\xi_2-0)\Delta^- Q(\xi_2).$$

Thus, $p(\xi_1+0)u'_\mu(\xi_1+0)>0$, $p(\xi_2-0)u'_\mu(\xi_2-0)<0$. On the other hand, on $[\xi_1+0,\xi_2-0]$ we rewrite Equation (2.6) as

$$(pu'_\mu)(x)=\int_{\xi_1+0}^{x} ud[Q]+p(\xi_1+0)u'_\mu(\xi_1+0), \tag{2.9}$$

whence it follows that $p(\xi_2-0)u'_\mu(\xi_2-0)>0$. We have a contradiction. Other cases can be investigated similarly. The theorem is proved. □

Outward normal cone. Let H be a Hilbert space, and let $G\subset H$ be a nonempty closed convex set. For $x\in G$, the set

$$N_G(x)=\{\xi\in H:\langle\xi,c-x\rangle\le 0,\ \forall c\in G\}$$

denotes the *outward normal cone* to G at x.

Notice that if x is an interior point of G, then $N_G(x)=\{0\}$. If $G=[-k,k]$, where $k>0$, then $N_G(k)=[0,+\infty)$ and $N_G(-k)=(-\infty,0]$.

3. Variational Motivation of Our Approach

Let a discontinuous Stieltjes string (a chain of strings held together by springs) be located along the segment $[0,l]$ of the Ox axis. The left end of the chain is rigidly fixed. At the point $x=l$ there is an elastic support (a spring of elasticity γ). Additionally, at the point $x=l$, we have a limiter $[-k,k]$ on the motion of the elastically fixed right end of the string.

Under the influence of an external force, determined by a function $F(x)$ of bounded variation, the string moves from the equilibrium position to the position $u(x)$. Since the left end of the string is rigidly fixed, the condition $u(0)=0$ is satisfied. The condition of the presence of a limiter on the motion of the right end of the string means that $|u(l)|\le k$. Whether or not the right end of the string touches the points $\pm k$ depends on the external load.

Let us first consider the simple case when the chain consists of two strings connected by a spring with elasticity γ_1 at the point $x=\xi$. Notice that at any discontinuity point ξ the function $u(x)$ is not defined, but the limit values $u(\xi-0)$, $u(\xi+0)$ are defined and have physical meaning: they describe deviations from the equilibrium position of the corresponding ends of the strings.

Suppose that concentrated forces f_1 and f_2 act only on the fastened ends of the strings, and there are no other forces. Then

$$F(x)=\begin{cases}0, & \text{if } 0\le x<\xi,\\ f_1, & \text{if } x=\xi,\\ f_1+f_2, & \text{if } \xi<x\le l.\end{cases}$$

Then the work performed by this force is equal to

$$f_1u(\xi-0)+f_2u(\xi+0)=\int_0^l ud[F].$$

Notice that the jumps of the function F at the points of discontinuity correspond to the forces concentrated at the corresponding points.

Let the function $p(x)$ characterize the local string tension. In the sequel we will define the values of the function $p(x)$ at the points of discontinuity of $u(x)$ equal to the elasticities of the springs connecting the strings. In particular, in the example under consideration, we define $p(\xi)=\gamma_1$. Then the internal energy accumulated by our physical system due to its own elasticity is

$$\int\limits_0^{\xi-0} \frac{p(x)u_x'^2(x)}{2}\,dx+\frac{\gamma_1(\Delta u(\xi))^2}{2}+\int\limits_{\xi+0}^{l} \frac{p(x)u_x'^2(x)}{2}\,dx=\int\limits_0^{l} \frac{p(x)u_\mu'^2(x)}{2}\,d\mu(x),$$

where $\mu(x)=x+\theta(x-\xi)$, and $\theta(x)$ is the Heaviside function.

Let the function $Q^*(x)$ characterize the elasticity of the external environment. Suppose that the springs with elasticities γ_2 and γ_3 are attached to the ends of the strings held together by a spring with elasticity γ_1, and a spring with elasticity γ is attached at the point $x=l$. Then

$$Q^*(x)=\begin{cases} 0, & \text{if } 0\le x<\xi,\\ \gamma_2, & \text{if } x=\xi,\\ \gamma_2+\gamma_3, & \text{if } \xi<x<\ell,\\ \gamma_2+\gamma_3+\gamma, & \text{if } x=\ell. \end{cases}$$

Notice that the jumps of the function $Q^*(x)$ at the points of discontinuity coincide with the elasticities of the corresponding springs. Thus, the work of the elastic force of the external environment is

$$\frac{u^2(\xi-0)}{2}\gamma_2+\frac{u^2(\xi+0)}{2}\gamma_3+\frac{u^2(\ell)}{2}\gamma=\int\limits_0^{l} \frac{u^2(x)}{2}\,d[Q^*(x)].$$

Therefore, the potential energy functional for our physical system can be represented in the form

$$\Phi(u)=\int\limits_0^{l} \frac{pu_\mu'^2}{2}d\mu+\int\limits_0^{l} \frac{u^2}{2}d[Q^*]-\int\limits_0^{l} ud[F]. \tag{3.1}$$

In the sequel we will study the general case, when the function $u(x)$ can be discontinuous in at most a countable set of points. For the general case, the potential energy functional also has Form (3.1) (see [19, 20]). We assume the existence of a strictly increasing function $\mu(x)$, defining the measure on the segment $[0,l]$ such that the functions $u(x)$ can be considered μ–absolutely continuous. The functions $p(x),F(x)$ have bounded variation on $[0,l]$ and $\inf_{[0,l]} p(x)>0$. The function $Q^*(x)$ does not decrease on the segment $[0,l]$ and is continuous at the point $x=0$; the left jump of the function Q^* at the point $x=l$ coincides with γ, i.e.,

$$\Delta^- Q^*(l)=\gamma.$$

We assume that the functions μ,F,p are continuous at the points $x=0$ and $x=l$.

In (3.1) the first integral is understood in the Lebesgue–Stieltjes sense with respect to the measure generated by the function $\mu(x)$, while the measure of any discontinuity point ξ of the function $\mu(x)$ is determined by the jump $\Delta\mu(\xi)=\mu(\xi+0)-\mu(\xi-0)$.

The second and third integrals are understood in the extended sense proposed by Pokornyi in [19], when the measure of any discontinuity point ξ, for example, of the function Q^*, "splits" into left and right, i.e., is defined using the left jump $\Delta^- Q^*(\xi) = Q^*(\xi) - Q^*(\xi - 0)$ and the right jump $\Delta^+ Q^*(\xi) = Q^*(\xi + 0) - Q^*(\xi)$.

Let

$$E = \{u : u \text{ is } \mu\text{–absolutely continuous on } [0,l] \text{ such that } u'_\mu \in BV[0,l]\}.$$

We emphasize that the function $u(x)$ considered here is a hypothetical (virtual) deformation. We will consider (3.1) on the set of functions from E, satisfying the boundary conditions

$$u(0) = 0 \quad \text{and} \quad |u(l)| \le k. \tag{3.2}$$

According to the Hamilton–Lagrange principle, the real deformation u_0 minimizes the functional Φ with conditions (3.2); in this case we write

$$u_0 \to \min_{u(0)=0, |u(l)| \le k} \Phi(u). \tag{3.3}$$

Let us define a function

$$Q(x) = \begin{cases} Q^*(x), & \text{if } 0 \le x < l, \\ Q^*(l-0), & \text{if } x = \ell. \end{cases}$$

Then

$$\int_0^l \frac{u^2}{2} d[Q^*] = \int_0^{l-0} \frac{u^2}{2} d[Q^*] + \frac{u^2(l)}{2} \Delta^- Q^*(l) = \int_0^{l-0} \frac{u^2}{2} d[Q] + \frac{u^2(l)}{2} \gamma.$$

We rewrite Functional (3.1) as

$$\Phi(u) = \int_0^l \frac{p u'^2_\mu}{2} d\mu + \int_0^{l-0} \frac{u^2}{2} d[Q] + \frac{\gamma u^2(l)}{2} - \int_0^l u d[F].$$

Consider functions $h \in E$ such that $h(0) = h(l) = 0$. Suppose $u(x) = u_0(x) + \lambda h(x)$, where λ accepts real values, i.e., $\lambda \in \mathbb{R}$. Notice that $u \in E$, $u(0) = 0$, $|u(l)| = |u_0(l)| \le k$. Since u_0 is the minimizer of the potential energy Φ, we have

$$\Phi(u_0) \le \Phi(u_0 + \lambda h).$$

Having fixed h, we consider the function $\varphi_h(\lambda)$ of the real variable λ defined as $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$. Then, for all $\lambda \in R$,

$$\varphi_h(0) \le \varphi_h(\lambda).$$

According to the Fermat theorem, we have

$$\frac{d}{d\lambda} \varphi_h(\lambda)|_{\lambda=0} = 0.$$

Due to the continuity of the functions μ, F at the points $x = 0$ and $x = l$, the last equality can be rewritten as

$$\int_0^{l-0} p u'_{0\mu} h'_\mu d\mu + \int_0^{l-0} u_0 h d[Q] - \int_0^{l-0} h d[F] = 0.$$

Let

$$g(x) = \int_0^x u_0 d[Q].$$

Taking into account that $h(0) = h(l) = 0$, we have

$$\int_0^{l-0} u_0 h d[Q] = -\int_0^{l-0} g dh \quad \text{and} \quad -\int_0^{l-0} h d[F] = \int_0^{l-0} F dh.$$

If follows

$$\int_0^{l-0} (p u'_{0\mu} - g + F) dh = 0.$$

Since the function $h(x)$ is μ–absolutely continuous, where $\mu(x)$ is a strictly increasing function on the segment $[0,l]$, continuous at the points $x = 0$ and $x = l$, the last equality can be rewritten as

$$\int_0^l (p u'_{0\mu} - g + F) h'_\mu d\mu = 0. \tag{3.4}$$

Equality (3.4) holds for all functions $h \in E$ such that $h(0) = h(l) = 0$.

Lemma 3.1. *Let $A(x)$ be a function of bounded variation on $[0,l]$. Let $\mu(x)$ be a strictly increasing function on $[0,l]$, which is continuous at the points $x = 0$ and $x = l$. Assume that, for any $h \in E$ satisfying the conditions $h(0) = h(l) = 0$,*

$$\int_0^l A h'_\mu d\mu = 0. \tag{3.5}$$

Then A is a constant function on $[0,l]$ in the following sense. For any point ξ of discontinuity of the function $\mu(x)$, the equality

$$A(\xi - 0) = A(\xi) = A(\xi + 0) = c$$

holds. For any point s of continuity of the function $\mu(x)$, the equality

$$A(s - 0) = A(s + 0) = c$$

holds.

Proof. Since $h(0) = h(l) = 0$, it follows from (3.5) that, for any constant c,

$$\int_0^l (A(x) - c) dh = 0. \tag{3.6}$$

Assuming in (3.6) that

$$c = \frac{\int_0^l A d\mu}{\mu(l) - \mu(0)}, \quad h(x) = \int_0^x (A(t) - c) d\mu(t),$$

we have

$$\int_0^l (A(x)-c)^2 d\mu = 0.$$

Hence, for almost all x (with respect to μ–measure) we have the equality $A(x) = c$.

Let ξ be a point of discontinuity of $\mu(x)$. Since $\{\xi\}$ has a nonzero μ–measure, the equality $A(\xi) = c$ holds. Assume that $A(\xi - 0) \neq c$, and for definiteness, $A(\xi - 0) > c$. Then there is $\delta > 0$, such that for all $x \in (\xi - \delta, \xi)$ the inequality $A(x) > c$ holds. But the μ–measure of the interval $(\xi - \delta, \xi)$ is equal to $\mu(\xi - 0) - \mu(\xi - \delta + 0)$. Since μ is strictly increasing this measure is nonzero. This leads to a contradiction with the equality $A(x) = c$ which holds almost everywhere with respect to the μ–measure. Thus, $A(\xi - 0) = c$. Similarly, we obtain that $A(\xi + 0) = c$.

Let s be a point of continuity of $\mu(x)$. Suppose the function $A(x)$ is continuous at s. Assume that $A(s) \neq c$, and for definiteness $A(s) > c$. Then there is $\varepsilon > 0$ such that $A(x) > c$ for all $x \in (\xi - \varepsilon, \xi + \varepsilon)$. But the μ-measure of the interval $(\xi - \varepsilon, \xi + \varepsilon)$ equals $\mu(\xi + \varepsilon - 0) - \mu(\xi - \varepsilon + 0)$ and does not equal zero. This contradiction forces $A(s) = c$.

Let us show that at every discontinuity point s of the function $A(x)$ we have $A(s-0) = A(s+0) = c$. Since the function $A(x)$ has bounded variation on the segment $[0,l]$, the set of its discontinuity points is at most countable. Assume that $s > 0$. Then there is a sequence of points $\{x_n\}$ such that x_n converges to s on the left, when $n \to \infty$, and at all points x_n the function $A(x)$ is continuous. But as shown above, $A(x_n) = c$. Thus, $A(s-0) = c$. Similarly, $A(s+0) = c$. The lemma is proved. $\square$

Applying this lemma to (3.4), we obtain that

$$(pu'_{0\mu})(x) - g(x) + F(x) = \text{ constant}, \quad x \in \overline{[0,l]}_S. \tag{3.7}$$

Hence,

$$-(pu'_{0\mu})(x) + (pu'_{0\mu})(0+0) + \int_0^x u d[Q] = F(x) - F(0+0).$$

Since the functions μ, p, Q, F are continuous at the point $x = 0$, we obtain

$$-(pu'_{0\mu})(x) + (pu'_{0\mu})(0) + \int_0^x u d[Q] = F(x) - F(0).$$

Let us fix any number $c \in [-k,k]$. Consider a function $h \in E$ such that $h(0) = 0$, $h(l) = c - u_0(l)$. For example,

$$h(x) = \frac{c - u_0(l)}{\mu(l) - \mu(0)}(\mu(x) - \mu(0)).$$

Let $u(x) = u_0(x) + \lambda h(x)$. Notice that $u \in E$, and $u(0) = 0$. Consider the condition at the point $x = l$. We have

$$u(l) = u_0(l) + \lambda h(l) = u_0(l) + \lambda(c - u_0(l)) = \lambda c + (1-\lambda)u_0(l) \in [-k,k], \quad \lambda \in [0,1],$$

since both $c, u_0(l) \in [-k,k]$. Due to (3.3), the inequality

$$\Phi(u_0) \leq \Phi(u_0 + \lambda h)$$

holds. Having fixed h, we introduce the function $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$, where $\lambda \in [0,1]$. Then

$$\varphi_h(0) \leq \varphi_h(\lambda).$$

Hence, the right derivative satisfies the inequality

$$\frac{d^+}{d\lambda}\varphi_h(\lambda)|_{\lambda=0} \geq 0,$$

i.e.,

$$\int\limits_0^{l-0} ((pu'_{0_\mu})(x) - \int\limits_0^x u_0 d[Q] + F(x))dh + h(l)\int\limits_0^{l-0} u_0 d[Q] - h(l)F(l) + \gamma u_0(l)h(l) \geq 0. \quad (3.8)$$

Since F is continuous at the point $x = l$, Equality (3.7) can be rewritten as

$$(pu'_{0_\mu})(x) - \int\limits_0^x u_0 d[Q] + F(x) = p(l-0)u'_{0\mu}(l-0) - \int\limits_0^{l-0} u_0 d[Q] + F(l).$$

Substituting this representation into (3.8), we obtain that

$$\Big(p(l-0)u'_{0\mu}(l-0) + \gamma u_0(l)\Big)\, h(l) \geq 0.$$

Thus, for all $c \in [-k,k]$, with respect to $h(l) = c - u_0(l)$, we have

$$(-p(l-0)u'_{0\mu}(l-0) - \gamma u_0(l))(c - u_0(l)) \leq 0.$$

In other words,

$$-p(l-0)u'_{0\mu}(l-0) - \gamma u_0(l) \in N_{[-k,k]}(u_0(l)).$$

The last condition means that if $|u_0(l)| < k$, then

$$p(l-0)u'_{0\mu}(l-0) + \gamma u_0(l) = 0.$$

If $u_0(l) = k$, then the support reaction force f additionally acts on the string from the side of the limiter so that

$$p(l-0)u'_{0\mu}(l-0) + \gamma u_0(l) = f \leq 0.$$

Similarly, if $u_0(l) = -k$, then

$$p(l-0)u'_{0\mu}(l-0) + \gamma u_0(l) = f \geq 0.$$

In summary, we have proved a theorem on the necessary condition for the extremum of the energy functional.

Theorem 3.1. *Assume that*

$$u_0 \to \min_{u(0)=0, |u(l)| \leq k} \Phi(u),$$

namely, $u_0 \in E$ is a solution of the minimization problem. Then $u_0(x)$ is a solution of the problem

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int\limits_0^x ud[Q] = F(x) - F(0), \\ \quad u(0) = 0, \\ -p(l-0)u'_\mu(l-0) - \gamma u(l) \in N_{[-k,k]}(u(l)). \end{cases} \quad (3.9)$$

4. Main Results

Assume that

- the functions $p(x)$ and $F(x)$ have bounded variation on $[0,l]$ with $\inf_{[0,l]} p > 0$,
- the function $Q(x)$ does not decrease on $[0,l]$,
- the function $\mu(x)$ strictly increases on $[0,l]$, and
- the functions μ, p, F, Q are continuous at $x=0$ and $x=l$.

Consider Problem (3.9).

Definition 4.1. By a *solution* of Problem (3.9) we mean a function $u \in E$, which satisfies Equation (2.1) for all $x \in \overline{[0,l]}_S$ and satisfies the boundary conditions

$$u(0)=0, \quad |u(l)| \le k, \quad \text{and} \quad -p(l-0)u'_\mu(l-0)-\gamma u(l) \in N_{[-k,k]}(u(l))..$$

Notice that we assume the existence of a strictly increasing function $\mu(x)$ such that the functions $u(x)$ can be considered μ–absolutely continuous.

Theorem 4.1. *If a solution of Problem (3.9) exists, then it is unique.*

Proof. Assume that $u_1(x)$ and $u_2(x)$ are solutions of Problem (3.9). Then $u(x)=u_1(x)-u_2(x)$ is a solution of homogeneous equation (2.6) and satisfies $u(0)=0$. Assume that $u(x)$ is not a zero function. According to Theorem 2.4, $u(x)$ does not have any zero point other than $x=0$. Assume that $u(x)>0$ for all $x \in (0,l]$. Then $u'_\mu(0)>0$, and from Equation (2.6), we have $p(x)u'_\mu(x)>0$. Thus,

$$p(l-0)u'_\mu(l-0)+\gamma u(l)>0. \tag{4.1}$$

On the other hand, since $-p(l-0)u'_{1\mu}(l-0)-\gamma u_1(l) \in N_{[-k,k]}(u_1(l))$ for all $c \in [-k,k]$ we have

$$(-p(l-0)u'_{1\mu}(l-0)-\gamma u_1(l))(c-u_1(l)) \le 0.$$

Putting $c=u_2(l)$, we obtain

$$(p(l-0)u'_{1\mu}(l-0)+\gamma u_1(l))(u_2(l)-u_1(l)) \ge 0.$$

Since $u(l)=u_1(l)-u_2(l)>0$, we have $p(l-0)u'_{1\mu}(l-0)+\gamma u_1(l) \le 0$. Similarly, for all $c^* \in [-k,k]$, the inequality

$$(-p(l-0)u'_{2\mu}(l-0)-\gamma u_2(l))(c^*-u_2(l)) \le 0$$

holds. Putting $c^*=u_1(l)$, we obtain

$$(-p(l-0)u'_{2\mu}(l-0)-\gamma u_2(l))(u_1(l)-u_2(l)) \le 0.$$

Hence, $p(l-0)u'_{2\mu}(l-0)+\gamma u_2(l) \ge 0$. Due to $p(l-0)u'_\mu(l-0)+\gamma u(l) \le 0$, we obtain a contradiction to (4.1).

Similarly, the case $u(x)<0$ is not possible. Thus, $u(x) \equiv 0$. The theorem is proved. □

Theorem 4.2. *Let the functions $\varphi_1(x)$ and $\varphi_2(x)$ be solutions of the homogeneous equation (2.6), and satisfy the conditions*

$$\varphi_1(0)=1, \quad \textit{and} \quad p(l-0)\varphi'_{1\mu}(l-0)+\gamma\varphi_1(l)=0;$$

$$\varphi_2(0)=0, \quad \textit{and} \quad p(l-0)\varphi'_{2\mu}(l-0)+\gamma\varphi_2(l)=1.$$

(a) If $\left|\dfrac{\varphi_1(l)\int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)}\right| < k$*, then the solution of Problem (3.9) is*

$$u(x) = \frac{\varphi_1(x)}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{p(0)\varphi'_{2\mu}(0)}\int\limits_x^l \varphi_1(s)d[F(s)].$$

(b) If $\dfrac{\varphi_1(l)\int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)} \geq k$*, then the solution of Problem (3.9) is*

$$u(x) = \frac{\varphi_2(x)k}{\varphi_2(l)} + \frac{\varphi_1(x)}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{p(0)\varphi'_{2\mu}(0)}\int\limits_x^l \varphi_1(s)d[F(s)]$$
$$- \frac{\varphi_2(x)\varphi_1(l)}{\varphi_2(l)p(0)\varphi'_{2\mu}(0)}\int\limits_0^l \varphi_2(s)d[F(s)].$$

(c) If $\dfrac{\varphi_1(l)\int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)} \leq -k$*, then the solution of Problem (3.9) is*

$$u(x) = \frac{-\varphi_2(x)k}{\varphi_2(l)} + \frac{\varphi_1(x)}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{p(0)\varphi'_{2\mu}(0)}\int\limits_x^l \varphi_1(s)d[F(s)]$$
$$- \frac{\varphi_2(x)\varphi_1(l)}{\varphi_2(l)p(0)\varphi'_{2\mu}(0)}\int\limits_0^l \varphi_2(s)d[F(s)].$$

Proof. Notice that the problem

$$\begin{cases} -(p\varphi'_{1\mu})(x) + (p\varphi'_{1\mu})(0) + \int\limits_0^x \varphi_1 d[Q] = 0, \\ \varphi_1(0) = 1, \\ p(l-0)\varphi'_{1\mu}(l-0) + \gamma\varphi_1(l) = 0 \end{cases} \tag{4.2}$$

has a unique solution. Indeed, $\varphi_1(x) = c_1u_1(x) + c_2u_2(x)$, where u_1 and u_2 are solutions of the homogeneous equation (2.6), satisfying the boundary conditions $u_1(0) = 0$, $u'_{1\mu}(0) = 1$ and $u_2(0) = 1$, $u'_{2\mu}(0) = 0$, respectively.

Since the function $Q(x)$ does not decrease on $[0,l]$ and $u_1(0) = 0$, the function $u_1(x)$ does not have any other zero point by Theorem 2.4. From the condition $u'_{1\mu}(0) = 1$, it follows that $u_1(x) > 0$ for all $x \in (0,l]$. Since

$$(pu'_{1\mu})(x) = (pu'_{1\mu})(0) + \int\limits_0^x u_1 d[Q],$$

we obtain $p(x)u'_{1\mu}(x) > 0$. In particular,

$$p(l-0)u'_{1\mu}(l-0) > 0 \quad \text{and} \quad p(l-0)u'_{1\mu}(l-0) + \gamma u_1(l) > 0.$$

Substituting the representation for $\varphi_1(x)$ into the boundary conditions of Problem (4.2), we obtain that

$$c_2 = 1 \quad \text{and} \quad c_1 = \frac{-p(l-0)u'_{2\mu}(l-0) - \gamma u_2(l)}{p(l-0)u'_{1\mu}(l-0) + \gamma u_1(l)}.$$

Similarly, there is a solution of the problem

$$\begin{cases} -(p\varphi'_{2\mu})(x) + (p\varphi'_{2\mu})(0) + \int\limits_0^x \varphi_2 d[Q] = 0, \\ \quad \varphi_2(0) = 0, \\ p(l-0)\varphi'_{2\mu}(l-0) + \gamma\varphi_2(l) = 1. \end{cases} \tag{4.3}$$

Notice that $\varphi'_{2\mu}(0) \neq 0$. Otherwise $\varphi_2(x) \equiv 0$ by Theorem 2.1, which contradicts the condition

$$p(l-0)\varphi'_{2\mu}(l-0) + \gamma\varphi_2(l) = 1.$$

Let us show that $\varphi_2(x) > 0$. Since $\varphi_2(0) = 0$, the function φ_2 does not have any other zero point by Theorem 2.4. Assume that $\varphi_2(x) < 0$ for all $x \in (0, l]$. Then $\varphi'_{2\mu}(0) < 0$, and from the equality $(p\varphi'_{2\mu})(x) = (p\varphi'_{2\mu})(0) + \int_0^x \varphi_2 d[Q]$, we have that $p(x)\varphi'_{2\mu}(x) < 0$. Hence, $p(l-0)\varphi'_{2\mu}(l-0) < 0$ and $\varphi_2(l) < 0$. But this contradicts the condition $p(l-0)\varphi'_{2\mu}(l-0) + \gamma\varphi_2(l) = 1$. Thus, $\varphi_2(x) > 0$ for all $x \in (0, l]$. Consequently, we have $\varphi'_{2\mu}(0) > 0$.

(a) Assume that $\left| \dfrac{\varphi_1(l) \int\limits_0^l \varphi_2(s) d[F(s)]}{p(0)\varphi'_{2\mu}(0)} \right| < k$. Let us show that the function

$$u(x) = \frac{\varphi_1(x)}{p(0)\varphi'_{2\mu}(0)} \int\limits_0^x \varphi_2(s) d[F(s)] + \frac{\varphi_2(x)}{p(0)\varphi'_{2\mu}(0)} \int\limits_x^l \varphi_1(s) d[F(s)]$$

is the solution of Problem (3.9).

We need to verify that $u \in E$. To this end, we assume that $\alpha \leq \beta$. It follows from

$$\begin{aligned} &u(\beta) - u(\alpha) \\ &= \frac{1}{p(0)\varphi'_{2\mu}(0)} \left((\varphi_1(\beta) - \varphi_1(\alpha)) \int\limits_0^\beta \varphi_2 d[F] + (\varphi_2(\beta) - \varphi_2(\alpha)) \int\limits_\beta^l \varphi_1 d[F] \right) \\ &\quad + \frac{1}{p(0)\varphi'_{2\mu}(0)} \int\limits_\alpha^\beta ((\varphi_1(\alpha) - \varphi_1(s))\varphi_2(s) + (\varphi_2(s) - \varphi_2(\alpha))\varphi_1(s)) d[F(s)], \end{aligned}$$

the function $u(x)$ is μ-absolutely continuous.

We claim that

$$u'_\mu(x) = \frac{\varphi_1{}'_\mu(x)\int\limits_0^x \varphi_2 d[F]}{p(0)\varphi'_{2\mu}(0)} + \frac{\varphi_2{}'_\mu(x)\int\limits_x^l \varphi_1 d[F]}{p(0)\varphi'_{2\mu}(0)}. \tag{4.4}$$

Denote by

$$\Delta_\varepsilon z(x) = z(x+\varepsilon) - z(x+0),$$

where $\varepsilon > 0$, and x is a point of continuity of $\mu(x)$. Let us prove the statement for the right derivative (for the left derivative the proof is similar). We have

$$\begin{aligned}\frac{\Delta_\varepsilon u}{\Delta_\varepsilon \mu} &= \frac{1}{p(0)\varphi'_{2\mu}(0)}\frac{\Delta_\varepsilon \varphi_1}{\Delta_\varepsilon \mu}\int\limits_0^{x+\varepsilon} \varphi_2 d[F] + \frac{1}{p(0)\varphi'_{2\mu}(0)}\frac{\Delta_\varepsilon \varphi_2}{\Delta_\varepsilon \mu}\int\limits_{x+\varepsilon}^{l} \varphi_1 d[F] \\ &+ \frac{1}{p(0)\varphi'_{2\mu}(0)}\int\limits_{x+0}^{x+\varepsilon} \frac{(\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s))}{\Delta_\varepsilon \mu} d[F(s)].\end{aligned}$$

We want to see that

$$\lim_{\varepsilon\to 0^+}\left(\frac{\int\limits_{x+0}^{x+\varepsilon}(\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s))\, d[F(s)]}{\Delta_\varepsilon \mu(x)}\right) = 0. \tag{4.5}$$

In fact,

$$\begin{aligned}&\left|\frac{1}{\Delta_\varepsilon \mu(x)}\int\limits_{x+0}^{x+\varepsilon}(\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s))\, d[F(s)]\right| \\ \le\ & \frac{\sup\limits_{x+0\le s\le x+\varepsilon}|\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)|}{\Delta_\varepsilon \mu(x)} V_{x+0}^{x+\varepsilon}(F).\end{aligned}$$

Assume that τ is a point of $\overline{[0,l]}_\mu$, at which the μ–continuous function $\left|\varphi_1(x+0)\varphi_2(s) - \varphi_2(s)\varphi_1(x+0)\right|$ attains its maximum on the compact set $[x+0, x+\varepsilon]$. Then the inequality

$$\begin{aligned}&\frac{\sup\limits_{x+0\le s\le x+\varepsilon}|\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)|}{\Delta_\varepsilon \mu(x)} \\ \le\ & |\varphi_2(\tau)|\left|\frac{\varphi_1(x+0)-\varphi_1(\tau)}{\Delta_\varepsilon \mu(x)}\right| + |\varphi_1(\tau)|\left|\frac{\varphi_2(x+0)-\varphi_2(\tau)}{\Delta_\varepsilon \mu(x)}\right|\end{aligned}$$

holds, and thus the fraction

$$\frac{\sup\limits_{x+0\le s\le x+\varepsilon}|\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)|}{\Delta_\varepsilon \mu(x)}$$

is bounded. Since $V_{x+0}^{x+\varepsilon}(F) \to 0$ when $\varepsilon \to 0^+$, Equality (4.5) is proved. Thus, Equality (4.4) is proved when μ is continuous at x.

Let us show that Equality (4.4) also holds at a discontinuity point ξ of the function $\mu(x)$. We have

$$u'_\mu(\xi)=\frac{\Delta u(\xi)}{\Delta\mu(\xi)}=\frac{1}{\Delta\mu(\xi)}\left(\frac{\varphi_1(\xi+0)\int\limits_0^{\xi+0}\varphi_2 d[F]}{p(0)\varphi'_{2\mu}(0)}+\frac{\varphi_2(\xi+0)\int\limits_{\xi+0}^{l}\varphi_1 d[F]}{p(0)\varphi'_{2\mu}(0))}\right.$$
$$\left.-\frac{\varphi_1(\xi-0)\int\limits_0^{\xi-0}\varphi_2 d[F]}{p(0)\varphi'_{2\mu}(0)}-\frac{\varphi_2(\xi-0)\int\limits_{\xi-0}^{l}\varphi_1 d[F]}{p(0)\varphi'_{2\mu}(0)}\right).$$

Notice that

$$\int\limits_0^{\xi+0}\varphi_2 d[F]=\int\limits_0^{\xi}\varphi_2 d[F]+\varphi_2(\xi+0)\Delta^+F(\xi),$$
$$\int\limits_{\xi+0}^{l}\varphi_1 d[F]=\int\limits_{\xi}^{l}\varphi_1 d[F]-\varphi_1(\xi+0)\Delta^+F(\xi),$$
$$\int\limits_0^{\xi-0}\varphi_2 d[F]=\int\limits_0^{\xi}\varphi_2 d[F]-\varphi_2(\xi-0)\Delta^-F(\xi),$$
$$\int\limits_{\xi-0}^{l}\varphi_1 d[F]=\int\limits_{\xi}^{l}\varphi_1 d[F]+\varphi_1(\xi-0)\Delta^-F(\xi).$$

Thus, Representation (4.4) is true for all x in $\overline{[0,l]}_S$. From (4.4) it follows that $u'_\mu\in BV[0,l]$. Hence, $u\in E$.

Now we show that the function $u(x)$ is a solution of Equation (2.1). Notice that

$$\int\limits_0^x u(s)d[Q(s)]=\frac{1}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x\varphi_1(s)\int\limits_0^s\varphi_2(t)d[F(t)]d[Q(s)]$$
$$+\frac{1}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x\varphi_2(s)\int\limits_s^l\varphi_1(t)d[F(t)]d[Q(s)].$$

Changing the limits of integration in the first term, we obtain from (2.6) that

$$\frac{1}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x\varphi_1(s)\int\limits_0^s\varphi_2(t)d[F(t)]d[Q(s)]$$
$$=\frac{1}{p(0)\varphi'_{2\mu}(0)}\int\limits_0^x\varphi_2(t)((p{\varphi_1}'_\mu)(x)-(p{\varphi_1}'_\mu)(t))d[F(t)].$$

Changing the limits of integration in the second term, we obtain from (2.6) that

$$\begin{aligned} &\frac{1}{p(0)\varphi_{2\mu}'(0)}\int_0^x \varphi_2(s)\int_s^l \varphi_1(t)d[F(t)]d[Q(s)] \\ = \quad &\frac{1}{p(0)\varphi_{2\mu}'(0)}\int_0^x \varphi_1(t)(p(t)\varphi_2{}'_\mu(t)-p(0)\varphi_2{}'_\mu(0))d[F(t)] \\ &+\frac{1}{p(0)\varphi_{2\mu}'(0)}\int_x^l \varphi_1(t)(p(x)\varphi_2{}'_\mu(x)-p(0)\varphi_{2\mu}'(0))d[F(t)]. \end{aligned}$$

Substituting the resulting representation for $\int_0^x ud[Q]$ into (2.1), and taking Lemma 2.1 into account, we have $p(t)(\varphi_1(t)\varphi_2{}'_\mu(t)-\varphi_2(t)\varphi_1{}'_\mu(t)) = p(0)\varphi_{2\mu}'(0)$, which yields (2.1). In view of $u(0)=0$, for $u(l)$, we have

$$u(l)=\frac{\varphi_1(l)\int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi_{2\mu}'(0)}.$$

Since we are considering the case

$$\left|\frac{\varphi_1(l)\int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi_{2\mu}'(0)}\right|<k,$$

we have $|u(l)|<k$.

Finally, we show that the equality $p(l-0)u_\mu'(l-0)+\gamma u(l)=0$ holds. From (4.4), taking into account the continuity of F at the point $x=l$, we obtain

$$p(l-0)u_\mu'(l-0)+\gamma u(l)=\frac{\int_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi_{2\mu}'(0)}\Big(p(l-0)\varphi_{1\mu}'(l-0)+\gamma\varphi_1(l)\Big)=0.$$

(b) Assume that $\dfrac{\varphi_1(l)\int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi_{2\mu}'(0)}\geq k$. We claim that the function

$$\begin{aligned} u(x)=\frac{\varphi_2(x)k}{\varphi_2(l)}+\frac{\varphi_1(x)}{p(0)\varphi_{2\mu}'(0)}\int_0^x \varphi_2(s)d[F(s)]+\frac{\varphi_2(x)}{p(0)\varphi_{2\mu}'(0)}\int_x^l \varphi_1(s)d[F(s)] \\ -\frac{\varphi_2(x)\varphi_1(l)}{\varphi_2(l)p(0)\varphi_{2\mu}'(0)}\int_0^l \varphi_2(s)d[F(s)] \end{aligned}$$

is a solution of Problem (3.9).

Taking into account what we have proved above, and $\varphi_1\in E$, $\varphi_2\in E$, it follows that $u\in E$. Notice that $u(0)=0$ and $u(l)=k$. Let us prove that $p(l-0)u_\mu'(l-0)+\gamma u(l)\leq 0$. Using (4.4) and the conditions on the functions φ_1 and φ_2, we obtain

$$p(l-0)u'_\mu(l-0)+\gamma u(l)=\frac{k}{\varphi_2(l)}-\frac{\varphi_1(l)\int\limits_0^l\varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)\varphi_2(l)}.$$

Since $\dfrac{\varphi_1(l)\int\limits_0^l\varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)}\geq k$ and $\varphi_2(l)>0$, we have $p(l-0)u'_\mu(l-0)+\gamma u(l)\leq 0$.

The integral equality (2.1) can be proved in a similar way.

(c) The case $\dfrac{\varphi_1(l)\int\limits_0^l\varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)}\leq -k$ can be considered similarly. The theorem is proved. □

Theorem 4.3. *Assume that $u_0(x)$ is a solution of Problem (3.9). Then*

$$u_0\to\min_{u(0)=0,|u(l)|\leq k}\Phi(u).$$

Proof. Let us prove that, for any function $u\in E$ with $u(0)=0$ and $|u(l)|\leq k$, the inequality $\Phi(u)-\Phi(u_0)\geq 0$ holds.

Represent the function $u(x)$ as $u(x)=u_0(x)+h(x)$, where $h(x)=u(x)-u_0(x)$. In view of $h(0)=0$, we have

$$\Phi(u_0+h)-\Phi(u_0)=\int\limits_0^l\frac{ph'^2_\mu}{2}d\mu+\int\limits_0^l\frac{h^2}{2}d[Q]+\int\limits_0^{l-0}(pu'_{0\mu}-\int\limits_0^x u_0d[Q]+F(x))dh$$
$$+h(l)\int\limits_0^{l-0}u_0d[Q]-h(l)F(l)+\gamma h(l)u_0(l).$$

It follows that

$$(pu'_{0\mu})(x)-\int\limits_0^x ud[Q]+F(x)=(pu'_{0\mu})(l-0)-\int\limits_0^{l-0}u_0d[Q]+F(l).$$

Since $h(l)=u(l)-u_0(l)$ and $u(l)\in[-k,k]$, we obtain

$$\Phi(u_0+h)-\Phi(u_0)=\int\limits_0^l\frac{ph'^2_\mu}{2}d\mu+\int\limits_0^l\frac{h^2}{2}d[Q]+((pu'_{0\mu})(l-0)+\gamma u_0(l))h(l)\geq 0.$$

The theorem is proved. □

Theorem 4.4. *If $k\to 0$, then the solution of Problem (3.9) tends to a solution of the problem*

$$\begin{cases}-(pu'_\mu)(x)+(pu'_\mu)(0)+\int\limits_0^x ud[Q]=F(x)-F(0),\\ u(0)=0,\quad u(l)=0,\end{cases}$$

uniformly on the set $\overline{[0,l]}_\mu$.

Proof. We will use the formulas from Theorem 4.2 for the representation of the solution $u_k(x)$ of Problem (3.9). Since $k \to 0$, we have

$$\left| \frac{\varphi_1(l) \int\limits_0^l \varphi_2(s)d[F(s)]}{p(0)\varphi'_{2\mu}(0)} \right| \geq k.$$

It follows from $\varphi_2 \in E$ that $\left|\dfrac{\varphi_2(x)}{\varphi_2(l)}\right| \leq c$. Then

$$\left| u_k(x) - \frac{\varphi_1(x)}{p(0)\varphi'_{2\mu}(0)} \int\limits_0^x \varphi_2(s)d[F(s)] - \frac{\varphi_2(x)}{p(0)\varphi'_{2\mu}(0)} \int\limits_x^l \varphi_1(s)d[F(s)] \right.$$
$$\left. + \frac{\varphi_2(x)\varphi_1(l)}{\varphi_2(l)p(0)\varphi'_{2\mu}(0)} \int\limits_0^l \varphi_2(s)d[F(s)] \right| = \left| \frac{k\varphi_2(x)}{\varphi_2(l)} \right| \leq c|k| \to 0.$$

Thus, $u_k(x)$ uniformly converges to

$$u(x) = \frac{\varphi_1(x)}{p(0)\varphi'_{2\mu}(0)} \int\limits_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{p(0)\varphi'_{2\mu}(0)} \int\limits_x^l \varphi_1(s)d[F(s)]$$
$$- \frac{\varphi_2(x)\varphi_1(l)}{\varphi_2(l)p(0)\varphi'_{2\mu}(0)} \int\limits_0^l \varphi_2(s)d[F(s)].$$

Similarly to Theorem 4.2, it can be verified that the function $u(x)$ is a solution of the problem

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int\limits_0^x u d[Q] = F(x) - F(0), \\ u(0) = 0, \quad u(l) = 0. \end{cases}$$

The theorem is proved. □

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