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# REGULARIZATION METHODS FOR QUASI-MIXED EQUILIBRIUM PROBLEMS IN BANACH SPACES

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**Abstract.** In this paper, we establish a regularization method of a class of quasi-mixed equilibrium problems in Banach spaces. First, an existence result of Minty inequality is given by applying the Tychonov fixed point theorem. Second, we show that the existence of solutions of the quasi-mixed equilibrium problems. Finally, some assumptions are proposed to ensure the boundedness and the convergence of regularized solutions.
**Keywords.** Equilibrium problems; Maximal monotone operator; Minty inequality; Pseudomonotone operator; Regularization.

## 1. Introduction

Let $C$ be a nonempty convex and closed subset of a reflexive Banach space $X$ with its topological dual space denoted by $X^*$. The main research object of this paper is to consider the following quasi-mixed equilibrium problems (QMEP, for short): find $x \in C$ with $x \in \mathscr{K}(x)$ and $x^* \in F(x)$ satisfying

$$\langle x^*, y-x\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in \mathscr{K}(x), \tag{1.1}$$

where $F : C \rightrightarrows X^*$ is a maximal monotone mapping, $\Psi$ and $\Phi : C \times C \to \mathbb{R}$ are two real valued bifunctions, and $\mathscr{K} : C \rightrightarrows C$ is a mapping such that $\mathscr{K}(x)$ is nonempty convex and closed for every $x \in C$.

If $\mathscr{K}(x) = C$ for every $x \in C$, then (1.1) reduces to a classical mixed equilibrium problem (MEP, for short): find $x \in C$ and $x^* \in F(x)$ satisfying

$$\langle x^*, y-x\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The research in this paper is based on the theory of set-valued mappings and equilibrium problems. Equilibrium problems, which have outstanding applications in many aspects such as

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optimization, variational inequalities, economics, and transportation networks, are an important part of nonlinear functional analysis. Equilibrium problems were first considered when Fan studied minimax inequality in [1], and the equilibrium problems was formally introduced by Blum and Oettli in [2]. Involving the latest research on equilibrium problems, we refer to [3, 4] and the references therein.

In [5], Chadli, Ansari and Yao studied the following class of mixed equilibrium problems

$$F(x,y)+\Psi(x,y)+\Phi(x,y)\geq 0, \quad \forall y\in C, \tag{1.3}$$

where $F: C\times C\to\mathbb{R}$ is monotone and maximal monotone bifunction, $\Psi$ and $\Phi: C\times C\to\mathbb{R}$ are pseudo-monotone and quasi-monotone bifunctions in the topological sense, respectively. The authors of this paper proved that the solution set is nonempty and weakly compact under the coercivity assumption, and applied their results to the anti-periodic solutions of nonlinear devolution problems. By using the KKM theorem and the monotonicity of bifunctions, Liu, Migórski and Zeng obtained the existence of solutions and optimal control of mixed equilibrium problems, see [6] and the references therein. In [7], Sahu, Chadli and Mohapatra further obtained some existence results of mixed equilibrium problems under the weaker conditions. Recently, many scholars conducted in-depth research on the regularization of solutions to variational inequalities and equilibrium problems; see [8, 9, 10, 11, 12, 13] and the references therein.

It is well-known that solving the quasi-equilibrium problems is challenging and meaningful. In general, quasi-equilibrium problems are ill-posed. That is, there may either have no solution or set-valued solutions for quasi-equilibrium problems and the large errors of the solution may be caused by small errors of the original data. Using regularized solutions to approximate the original solutions is an effective method, and the object of this paper is to develop a regularization theory for quasi-mixed equilibrium problems.

The paper is arranged as follows. In the next section, Section 2, some notions and basic definitions related to equilibrium problems and nonlinear analysis are present. In Section 3, the Minty inequality of mixed equilibrium problems is given. In Section 4, we study the existence results of quasi-mixed equilibrium problems. In Section 5, we establish the existence of solutions for regularized quasi-mixed equilibrium problems. In Section 6, the last section, the boundedness and convergence of regularized quasi-mixed equilibrium problems under some conditions.

## 2. Preliminaries

Let $X$ be a reflexive Banach space and denote its topological dual space by $X^*$, and let $F: X\rightrightarrows X^*$ be a given set-valued mapping. One uses $\mathscr{D}(F):=\{x\in X|\ F(x)\neq\emptyset\}$ to stand for the domain of $F$. In the sequel, strong and weak convergence are denoted by the symbols $\to$ and $\rightharpoonup$, respectively. $\mathbb{R}_+$ and $\mathbb{R}$ stand for the set of positive real numbers and real numbers. Recall the following definitions, which can be found in [14, 15, 16].

**Definition 2.1.** A set-valued mapping $F: X\rightrightarrows X^*$ is said to be

(a) monotone if, for any $x^*\in F(x)$ and any $y^*\in F(y)$,

$$\langle y^*-x^*, y-x\rangle\geq 0.$$

(b) maximal monotone if $F$ is monotone and for any $(y,y^*) \in X \times X^*$ satisfying

$$\langle y^* - x^*, y - x\rangle \geq 0 \quad \text{for all } x^* \in F(x),$$

$y^* \in F(y)$.

(c) generalized pseudomonotone if for every $x_n^* \in F(x_n)$ with $x_n \rightharpoonup x$ and $x_n^* \rightharpoonup x^*$ satisfying $\limsup_{n\to\infty}\langle x_n^*, x_n\rangle \leq \langle x^*, x\rangle$, $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$ and $x^* \in F(x)$.

(d) lower semicontinuous (for short, l.s.c.) at $\bar{x}$ if for any given open $V \subset X^*$ such that $F(\bar{x}) \cap V \neq \emptyset$, there exists neighborhood $U$ of $\bar{x}$ satisfying $F(y) \cap V \neq \emptyset$ for any $y \in U$.

(e) lower hemicontinuous (for short, l.h.c.) if the restriction of $F$ to each line segment of $X$ is l.s.c. with respect to the weak topology of $X^*$.

(f) coercive if $\langle x^*, x\rangle \geq \mu(\|x\|)\|x\|$ for any $x^* \in F(x)$, where $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty}\mu(r) = \infty$.

(g) quasi-compact if $F(U)$ is a relatively compact subset of $X^*$ with the weak topology for every relative compact subset $U \subset X$.

**Definition 2.2.** Let $C$ be a nonempty closed convex subset of a reflexive Banach space $X$. The real-valued bifunction $\Psi : C \times C \to \mathbb{R}$ is said to be

(a) monotone if, for any $x, y \in C$, $\Psi(x,y) + \Psi(y,x) \leq 0$.

(b) pseudomonotone if for each sequence $\{x_n\}$ such that $x_n \rightharpoonup x$ in $C$ and $\liminf_{n\to\infty}\Psi(x_n, x) \geq 0$, then, for any $y \in C$, $\limsup_{n\to\infty}\Psi(x_n, y) \leq \Psi(x,y)$.

Obviously, if $\Psi$ is weakly upper semicontinuous with respect to the first variable, then it is pseudomonotone.

**Definition 2.3.** Let $F : C \times C \to \mathbb{R}$ be a bifunction with $F(x,x) = 0$ for any $x \in C$. We say that $F$ is maximal monotone if, for each $x \in C$ and for each convex function $\varphi : C \to \mathbb{R}$ with $\varphi(x) = 0$,

$$F(y,x) \leq \varphi(y), \text{ for any } y \in C \quad \Rightarrow \quad 0 \leq F(x,y) + \varphi(y) \text{ for any } y \in C.$$

**Remark 2.1.** Suppose that $A : X \to X^*$ and $F_A : X \times X \to \mathbb{R}$ is defined by $F_A(x,y) = \langle Ax, y - x\rangle$. Then the following properties (see [5]) are satisfied: (i) if $F_A$ is monotone and maximal monotone, then $A$ is maximal monotone. However, the reverse is usually not true. (ii) if $A$ is monotone and hemicontinuous, then $F_A$ is monotone and maximal monotone;

**Definition 2.4.** Let $\Psi$ and $\Phi : C \times C \to \mathbb{R}$ be two real valued bifunctions. We say that $F : C \rightrightarrows X^*$ is $(\Psi,\Phi)$-pseudomonotone if, for all $x, y \in C$ and for all $x^* \in F(x)$ and $y^* \in F(y)$,

$$\langle x^*, y - x\rangle + \Psi(x,y) \geq \Phi(y,x) \quad \Rightarrow \quad \langle y^*, y - x\rangle + \Psi(x,y) \geq \Phi(y,x).$$

If $\Phi = 0$, then $F$ is said to be $\Psi$-pseudomonotone.

**Definition 2.5.** The mapping $\mathscr{K} : C \rightrightarrows C$ is said to be $M-$continuous relative to $\Psi$ and $\Phi$ if the following conditions $(M1)$ and $(M2)$ are satisfied:

(M1) for any $y_k \in \mathscr{K}(x_k)$ with $x_k \rightharpoonup x$ and $y_k \rightharpoonup y$, we have $y \in \mathscr{K}(x)$.

(M2) for any sequence $x_k \in C$ with $x_k \rightharpoonup x$ and for any $z \in \mathscr{K}(x)$, there exists $z_k \in \mathscr{K}(x_k)$ such that $z_k \to z$ and $\Psi(y_k, z_k) \to \Psi(y,z)$, $\Phi(y_k, z_k) \to \Phi(y,z)$ for any sequence $y_k \in C$ with $y_k \rightharpoonup y$.

The following two properties will be used as important tools in the following section, which can be found in [17] and [18] respectively.

**Lemma 2.1.** *([17]) Let $X$ be a reflexive Banach space and denote its topological dual space by $X^*$. Let $F : X \rightrightarrows X^*$ be a monotone mapping. Assume that, for any given $x_0 \in int(\mathscr{D}(F))$, there exists a positive constant $\rho = \rho(x_0)$ satisfying, for any $x^* \in F(x)$ and corresponding $c := sup\{\|\tilde{w}\| : \|\tilde{x} - x_0\| \leq \rho,$ and $\tilde{w} \in F(\tilde{x})\} < +\infty$,*

$$\rho\|x^*\| \leq \langle x^*, x - x_0\rangle + c(\rho + \|x - x_0\|).$$

**Theorem 2.1.** *([18]) Assume that $X$ is a reflexive Banach space, $C \subset X$ is a nonempty convex and closed set, $\Gamma : C \rightrightarrows C$ is sequentially weakly-weakly closed mapping and $\Gamma(x)$ is nonempty convex and closed for every $x \in C$. If $\Gamma(C)$ is bounded, then the mapping $\Gamma$ admits a fixed point.*

## 3. Minty Inequalities

Minty inequality is an important result of equilibrium problems, it mainly involves the following equilibrium problems:

Find $x \in C$ such that for some $x^* \in F(x)$

$$\langle x^*, y - x\rangle + \Theta(x, y) \geq 0 \quad \forall y \in C. \tag{3.1}$$

Find $x \in C$ such that

$$\langle y^*, y - x\rangle + \Theta(x, y) \geq 0 \quad \forall y \in C, \forall y^* \in F(y). \tag{3.2}$$

The mappings $F$ and $\Theta$ satisfy the following assumptions:

$(H_F)$ the mapping $F : C \rightrightarrows X^*$ is strongly-weakly closed, quasi-compact and $\Theta-$pesudomonotone.
$(H_C)$ $C \subset int(\mathscr{D}(F))$.
$(H_\Theta)$ $\Theta : C \times C \to \mathbb{R}$ is a bifunction such that
- $(i)$ $\Theta(x, x) = 0$ for every $x \in C$,
- $(ii)$ $y \mapsto \Theta(x, y)$ is convex and u.s.c. for every $x \in C$.

**Lemma 3.1.** *Let $X$ be a reflexive Banach space, and let $C \subset X$ be a nonempty convex and closed set. Let $C^* \subset X^*$ be a nonempty convex closed and bounded, and let $x \in C$ be arbitrary. Assume that $(H_\Theta)$ holds, and, for every $y \in C$, there exists some $x_y^* \in C^*$ such that*

$$\langle x_y^*, y - x\rangle + \Theta(x, y) \geq 0. \tag{3.3}$$

*Then there exists $x^* \in C^*$ satisfying*

$$\langle x^*, y - x\rangle + \Theta(x, y) \geq 0 \quad \forall y \in C.$$

*Proof.* If the conclusion is not true, then, for any $x^* \in C^*$, there exists $y \in C$ such that

$$\langle x^*, y - x\rangle + \Theta(x, y) < 0.$$

For any $y \in C$, the subset $B_y$ of $X^*$ is defined by

$$B_y = \{x^* \in C^* : \langle x^*, y - x\rangle + \Theta(x, y) < 0\}.$$

It is easy to see that $B_y$ is open with respect to the weak topology of $X^*$. Since $C^*$ is weakly compact subset of $X^*$, it follows that there exists a finite subsequence $\{y_1, y_2, \cdots, y_n\}$ of $C$ satisfying the fact that $\{B_{y_1}, B_{y_2}, \cdots, B_{y_n}\}$ constitute an open cover of $C^*$, that is, $\cup_{i=1}^n B_{y_i} \supset C^*$.

Let $\beta_1, \beta_2, \cdots, \beta_n$ be a partition of unity and let $\beta_i : C^* \to \mathbb{R}$ be a weakly-strongly continuous function for each $i = 1, \cdots, n$, that is,

$$0 \leq \beta_i \leq 1 \text{ and } \sum_{i=1}^{n} \beta_i(x^*) = 1 \quad \text{for every } x^* \in C^*.$$

The mapping related to partition of unity $\mathscr{L} : C^* \to C$ is defined as follows

$$\mathscr{L}(x^*) = \sum_{i=1}^{n} \beta_i(x^*)(y_i).$$

Since $\mathscr{L}(x^*)$ is a convex combination of $y_i$ and $\beta_i(\cdot)$ is weakly-strongly continuous, it can be concluded that $\mathscr{L}(\cdot)$ is also weakly-strongly continuous.

From the convexity of $\Theta(x, \cdot)$, we obtain

$$\begin{aligned} & \langle x^*, \mathscr{L}(x^*) - x\rangle + \Theta(x, \mathscr{L}(x^*)) \\ = \ & \langle x^*, \sum_{i=1}^{n} \beta_i(x^*) y_i - x\rangle + \Theta(x, \sum_{i=1}^{n} \beta_i(x^*) y_i) \\ \leq \ & \sum_{i=1}^{n} \beta_i(x^*)[\langle x^*, y_i - x\rangle + \Theta(x, y_i)] \\ < \ & 0. \end{aligned} \tag{3.4}$$

Next, we define $P : C \rightrightarrows C^*$ and $Q : C^* \rightrightarrows C^*$ by

$$P(y) = \{x^* \in C^* : \langle x^*, y - x\rangle + \Theta(x, y) \geq 0\}$$

and

$$Q(x^*) = P(\mathscr{L}(x^*)).$$

From the boundedness of $C^*$ and (3.3), we obtain that $P(y)$ is nonempty weakly compact for any $y \in C$.

Next, we claim that $P(\cdot)$ is strongly-weakly u.s.c.. In fact, we only need to prove that, for any weakly closed set $D \subset C^*$,

$$P^{-1}(D) = \{y \in C : P(y) \bigcap D \neq \emptyset\}$$

is closed in $C$. Letting $y_n \in P^{-1}(D)$ with $y_n \to \bar{y}$, we have that $P(y_n) \bigcap D \neq \emptyset$. Let $x_n^* \in P(y_n) \bigcap D$. From the boundedness of $C^*$, we have that there exists subsequence of $\{x_n^*\}$, still denoted by $\{x_n^*\}$, satisfying $x_n^* \rightharpoonup \bar{x}^*$. Furthermore, we have $\bar{x}^* \in D$ due to the weakly closedness of $D$. On the other hand, we obtain from the definition of $P$ that

$$\langle x_n^*, y_n - x\rangle + \Theta(x, y_n) \geq 0.$$

Hence, by taking the upper limit of the above inequality, we have

$$\begin{aligned} & \limsup_{n\to\infty} \langle x_n^*, y_n - x\rangle + \limsup_{n\to\infty} \Theta(x, y_n) \geq 0 \\ \Rightarrow \ & \langle \bar{x}^*, \bar{y} - x\rangle + \Theta(x, \bar{y}) \geq 0, \end{aligned}$$

due to the upper continuity of $\Theta(x, \cdot)$. This implies that $\bar{x}^* \in P(\bar{y})$. Furthermore, $\bar{x}^* \in P(\bar{y}) \bigcap D$. Hence, $\bar{y} \in P^{-1}(D)$.

From the weakly-strongly continuity of $\mathscr{L}$, we obtain that $Q$ is weakly-weakly u.s.c. with nonempty closed and weakly compact values. Using the Tychonov fixed point Theorem, we have that there exists $x^* \in C^*$ satisfying $x^* \in Q(x^*)$, that is, $x^* \in P(\mathscr{L}(x^*))$. Therefore, there exists $x^* \in C^*$ satisfying $\langle x^*, \mathscr{L}(x^*) - x\rangle + \Theta(x, \mathscr{L}(x^*)) \geq 0$, which contradicts with (3.4). The proof is complete.

□

For the case of $\Theta(x,y) = \varphi(y) - \varphi(x)$, we generalize the Proposition 3.1 of [9] and have the following conclusion.

**Corollary 3.1.** *Let $X$ be a reflexive Banach space, and let $C \subset X$ be a nonempty convex and closed set. Let $C^* \subset X^*$ be a nonempty convex, closed and bounded, and let $x \in C$ be arbitrary. Assume that $\varphi : X \to \overline{\mathbb{R}}$ is proper convex and l.s.c. and for every $y \in C$ satisfying for some $x_y^* \in C^*$*

$$\langle x_y^*, y - x\rangle \geq \varphi(x) - \varphi(y).$$

*Then there exists $x^* \in C^*$ such that*

$$\langle x^*, y - x\rangle \geq \varphi(x) - \varphi(y) \quad \forall y \in C.$$

**Theorem 3.1.** *Assume that $(H_F)$, $(H_C)$ and $(H_\Theta)$ hold. Then the solutions of inequalities (3.1) and (3.2) are equivalent.*

*Proof.* "⇒" Assume that $x$ is a solution of (3.1). Then the $\Theta-$pesudomonotonicity of $F$ ensures that it is a solution of (3.2).

"⇐" Assume that $x$ is a solution of (3.2). For every $y \in C$ and $\alpha_n \in (0,1]$, take $y_n = (1-\alpha_n)x + \alpha_n y$ with $\alpha_n \to 0$. Since $F$ is quasi-compact and $y_n \to x$ as $\alpha_n \to 0$, it follows that there exists $x_y^* \in X^*$, and a subsequence of $\{y_n^*\}$ with $y_n^* \in F(y_n)$, still denoted by $\{y_n^*\}$, such that $y_n^* \to x_y^*$. From the strongly-weakly closedness of $F$, we get $x_y^* \in F(x)$. On the other hand, the convexity of $C$ deduces that $y_n \in C$. Hence, by taking $y = y_n$ in (3.2), we obtain

$$\langle y_n^*, y_n - x\rangle + \Theta(x, y_n) \geq 0.$$

From the convexity of $\Theta(x,\cdot)$, we obtain

$$\begin{aligned} & \alpha_n \langle y_n^*, y - x\rangle + (1-\alpha_n)\Theta(x,x) + \alpha_n \Theta(x,y) \geq 0 \\ \Rightarrow \quad & \langle x_y^*, y - x\rangle + \Theta(x,y) \geq 0. \end{aligned}$$

Therefore, we obtain that, for any $y \in C$ and for some $x_y^* \in F(x)$,

$$\langle x_y^*, y - x\rangle + \Theta(x,y) \geq 0.$$

Furthermore, from the Lemma 3.1, we have for some $x^* \in F(x)$

$$\langle x^*, y - x\rangle + \Theta(x,y) \geq 0, \quad \forall y \in C.$$

□

The main results above are obtained under the condition of the strongly-weakly closedness and quasi-compactness of $F$. We can also obtain the corresponding results of the lower hemicontinuity with the following hypothesis:

$(H'_F)$ $F : C \rightrightarrows X^*$ is l.h.c. and $\Theta$-pesudomonotone, and $F(x)$ is nonempty convex and weakly-compact for every $x \in C$.

**Corollary 3.2.** *Assume that $(H'_F)$ and $(H_\Theta)$ hold. Then the solutions of inequalities (3.1) and (3.2) are equivalent.*

*Proof.* Following the method similar to Theorem 3.1, we can obtain the conclusion immediately with the only difference on the existence of $x_y^*$. □

As a direct result of Theorem 3.1, we can also obtain the following result.

**Corollary 3.3.** *Assume that* $(H_\Theta)$ *and* $(H_C)$ *hold, and* $F : C \rightrightarrows X^*$ *is a maximal monotone mapping. Then the solutions of inequalities* (3.1) *and* (3.2) *are equivalent.*

*Proof.* Since $F$ is maximal monotone, we have that $F$ is strongly-weakly closed and strongly-weakly u.s.c.. On the other hand, $C \subset int(\mathscr{D}(F))$ and the monotonicity of $F$ infer that $F(x)$ is weakly-compact for every $x \in C$. Hence, using the Theorem 1.1.7 of [19], we have that $F$ is quasi-compact. From Theorem 3.1, we can deduce that $x$ is a solution of (3.1) if and only if it is a solution of (3.2). □

## 4. The Existence of Solutions for Quasi-mixed Equilibrium Problems

In order to solve the quasi-mixed equilibrium problems, we need to first give some results for mixed equilibrium problems and the following assumptions:

$(H_F'')$ $F : C \rightrightarrows X^*$ is a maximal monotone mapping.

$(H_\Psi)$ $\Psi : C \times C \to \mathbb{R}$ is a pseudomonotone bifunction satisfying:
- $(i)$ $\Psi(x,x) = 0$ for every $x \in C$,
- $(ii)$ $y \mapsto \Psi(x,y)$ is convex for every $x \in C$,
- $(iii)$ $x \mapsto \Psi(x,y)$ is u.s.c. on the convex hull of each finite subset of $C$ for every $y \in C$.

$(H_\Phi)$ $\Phi : C \times C \to \mathbb{R}$ is a monotone bifunction satisfying:
- $(i)$ $\Phi(x,x) = 0$ for every $x \in C$,
- $(ii)$ $y \mapsto \Phi(x,y)$ is convex and l.s.c. for every $x \in C$,
- $(iii)$ $x \mapsto \Phi(x,y)$ is continuous for every $y \in C$.

$(H_{coer})$ there exist a nonempty weakly compact subset $U$ and a weakly compact convex subset $V$ of $C$ such that, for all $x \in C \setminus U$, for some $y_0 \in V$,

$$\langle x^*, y_0 - x\rangle + \Psi(x,y_0) < \Phi(x,y_0) \quad \text{for any } x^* \in F(x).$$

**Lemma 4.1.** *Assume that* $(H_F'')$, $(H_\Psi)$, $(H_\Phi)$, $(H_C)$ *and* $(H_{coer})$ *hold. Then there exists* $x \in C$ *such that for some* $x^* \in F(x)$

$$\langle x^*, y - x\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in C.$$

*Proof.* The conditions $(H_F'')$ and $(H_C)$ deduce that $F : C \rightrightarrows X^*$ is strongly-weakly closed and quasi-compact. Therefore, as a direct result of [7, Theorem 3.1] under the weak topology of $X$, we can conclude that $x \in C$ is a solution of (1.2).

□

**Remark 4.1.** Since the monotonicity and maximal monotonicity of bifunctions is stronger than the maximal monotonicity of functions when $F$ is single valued, our conclusion generalizes the results of [5].

**Remark 4.2.** If $\Phi$ is convex and l.s.c. with respect to the second variable, and there exists $y_0 \in C$ satisfying

$$\frac{\langle x^*, y_0 - x\rangle + \Psi(x,y_0)}{\|x - y_0\|} \to -\infty, \text{ as } \|x - y_0\| \to +\infty \text{ for any } x^* \in F(x),$$

then condition $(H_{coer})$ hold.

Indeed, set $r_0 > 0$ and $B(y_0,r_0) = \{x \in X : \|x-y_0\| \le r_0\}$. Since $\Phi(y_0,\cdot)$ is weakly l.s.c., we have that there exists $\rho \in \mathbb{R}$ satisfying

$$\Phi(y_0,y) > \rho \quad \text{for all } \ y \in B(y_0,r_0).$$

For any $x \in C \setminus B(y_0,r_0)$, set

$$y = \frac{r_0}{\|x-y_0\|}x + (1-\frac{r_0}{\|x-y_0\|})y_0 \in B(y_0,r_0).$$

On account of the convexity of $\Phi(y_0,\cdot)$ and $\Phi(y_0,y_0) = 0$, we have

$$\frac{r_0}{\|x-y_0\|}\Phi(y_0,x) + (1-\frac{r_0}{\|x-y_0\|})\Phi(y_0,y_0) \ge \Phi(y_0,y)$$
$$\Rightarrow \quad \Phi(y_0,x) \ge \frac{\|x-y_0\|}{r_0}\Phi(y_0,y) \ge \frac{\rho}{r_0}\|x-y_0\|.$$

Therefore, for any $x \in C \setminus B(y_0,r_0)$ and for all $x^* \in F(x)$, we have

$$\begin{aligned}\langle x^*, y_0 - x\rangle + \Psi(x,y_0) - \Phi(y_0,x) &\le \langle x^*, y_0 - x\rangle + \Psi(x,y_0) - \frac{\rho}{r_0}\|x-y_0\| \\ &\le \|x-y_0\| \cdot [\frac{\langle x^*, y_0 - x\rangle + \Psi(x,y_0)}{\|x-y_0\|} - \frac{\rho}{r_0}]. \end{aligned} \tag{4.1}$$

Since $\frac{\langle x^*, y_0-x\rangle + \Psi(x,y_0)}{\|x-y_0\|} \to -\infty$ as $\|x-y_0\| \to +\infty$, it follows that there exists a positive constant $r_1$ such that, for any $x \in C$ satisfying $\|x-y_0\| > r_1$,

$$\frac{\langle x^*, y_0 - x\rangle + \Psi(x,y_0)}{\|x-y_0\|} - \frac{\rho}{r_0} < 0. \tag{4.2}$$

Take $U = \{x \in X : \|x-y_0\| \le r\} \cap C$ with $r = \max\{r_0, r_1\}$. On account of the relations (4.1) and (4.2), we conclude that, for every $x \in C \setminus U$,

$$\langle x^*, y_0 - x\rangle + \Psi(x,y_0) < \Phi(x,y_0) \quad \text{for every } x^* \in F(x).$$

To discuss the quasi-mixed equilibrium problems, the coercive conditions and the assumptions of $\Psi$ and $\Phi$ are strengthened as follows:

$(H'_{\Psi})$ $\Psi : C \times C \to \mathbb{R}$ is a bifunction and satisfying:
- $(i)$ $\Psi(x,x) = 0$ for every $x \in C$,
- $(ii)$ $y \mapsto \Psi(x,y)$ is convex and u.s.c. for every $x \in C$,
- $(iii)$ $x \mapsto \Psi(x,y)$ is concave and u.s.c. for every $y \in C$.

$(H'_{\Phi})$ $\Phi : C \times C \to \mathbb{R}$ is a monotone bifunction and satisfying:
- $(i)$ $\Phi(x,x) = 0$ for every $x \in C$,
- $(ii)$ $y \mapsto \Phi(x,y)$ is convex and continuous for every $x \in C$,
- $(iii)$ $x \mapsto \Phi(x,y)$ is concave and continuous for every $y \in C$.

$(H_{\mathscr{K}})$ $\mathscr{K} : C \rightrightarrows C$ is $M-$continuous relative to $\Psi$ and $\Phi$.

$(H'_{coer})$ there exists $y_0 \in \cap_{v \in C}\mathscr{K}(v)$ satisfying $\frac{\langle x^*, y_0-x\rangle + \Psi(x,y_0)}{\|x-y_0\|} \to -\infty$ as $\|x-y_0\| \to +\infty$ for any $x^* \in F(x)$.

**Remark 4.3.** Although we did not assume that $\Psi$ is pseudomonotone, the assumption condition $(H'_\Psi)(iii)$ already includes it. Indeed, since $\Psi(\cdot,y)$ is upper semicontinuous for any $y \in C$, by applying the Proposition 1.3.4 of [20], we have that the set

$$\Psi_{\lambda,y} = \{x \in C : \Psi(x,y) \geq \lambda\}$$

is closed in $C$ for any $\lambda \in \mathbb{R}$. From the concavity of $\Psi(\cdot,y)$, we easily obtain that $\Psi_{\lambda,y}$ is a convex set. Hence, $\Psi_{\lambda,y}$ is weakly closed. Using the Proposition 1.3.4 of [20] again, we obtain that $\Psi(\cdot,y)$ is weakly upper semicontinuous. Therefore, $\Psi$ is pseudomonotone bifunction.

For any given $v \in C$, we consider the following mixed equilibrium problem with parametric variables: find $x \in \mathscr{K}(v)$ and $x^* \in F(x)$ such that

$$\langle x^*, y-x\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in \mathscr{K}(v).$$

Defining a mapping $S : C \rightrightarrows C$ corresponding to the solution set of the above inequality, the $S(v)$ represents the set of solutions to the above inequality for every $v \in C$.

**Theorem 4.1.** *Assume that $(H''_F)$, $(H'_\Psi)$, $(H'_\Phi)$, $(H_{\mathscr{K}})$, $(H_C)$ and $(H'_{coer})$ hold. Then there exists $x \in C$ with $x \in \mathscr{K}(x)$ and $x^* \in F(x)$ such that*

$$\langle x^*, y-x\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in \mathscr{K}(x).$$

*Proof.* In order to apply Theorem 2.1, we divide the proof into the following three steps.

**Step 1**. Show that $S(v)$ is nonempty closed and convex for every $v \in C$.

First, Lemma 4.1 and Remark 4.2 ensure that $S(v)$ is nonempty. We next prove that $S(v)$ is closed set. Let $x_n \in S(v)$ with $x_n \to x$. From the definition of $S(v)$, we see that there exists $x_n^* \in F(x_n)$ such that

$$\langle x_n^*, y-x_n\rangle + \Psi(x_n,y) + \Phi(x_n,y) \geq 0, \quad \forall y \in \mathscr{K}(v).$$

On account of the maximal monotonicity of $F$, we have that a subsequence of $\{x_n^*\}$, which is still denoted as $\{x_n^*\}$, satisfies $x_n^* \rightharpoonup x^* \in X^*$, and $x^* \in F(x)$. Since $\Psi(\cdot,y)$ and $\Psi(\cdot,y)$ are upper semicontinuous, we easily conclude that

$$\langle x^*, y-x\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in \mathscr{K}(v).$$

Hence, $x \in S(v)$, and $S(v)$ is closed set.

Next, we show that the convexity of $S(v)$. Let $x_1, x_2 \in S(v)$ and $\alpha \in [0,1]$. From the concavity of $\Psi(\cdot,y)$ and $\Phi(\cdot,y)$, we easily find for any $z \in \mathscr{K}(v)$

$$\begin{aligned} & \Psi(\alpha x_1 + (1-\alpha)x_2, z) + \Phi(\alpha x_1 + (1-\alpha)x_2, z) \\ \geq\ & \alpha[\Psi(x_1,z) + \Phi(x_1,z)] + (1-\alpha)[\Psi(x_2,z) + \Phi(x_2,z)]. \end{aligned}$$

Hence, from above the inequality and Theorem 3.1, we have for any $w_z \in F(z)$

$$\begin{aligned} & \langle w_z, z-[\alpha x_1 + (1-\alpha)x_2]\rangle + \Psi(\alpha x_1 + (1-\alpha)x_2, z) + \Phi(\alpha x_1 + (1-\alpha)x_2, z) \\ \geq\ & \alpha[\langle w_z, z-x_1\rangle + \Psi(x_1,z) + \Phi(x_1,z)] + (1-\alpha)[\langle w_z, z-x_2\rangle + \Psi(x_2,z) + \Phi(x_2,z)] \\ \geq\ & 0 \end{aligned}$$

By using Theorem 3.1 again, we deduce that there exists $x^* \in F(\alpha x_1 + (1-\alpha)x_2)$ such that

$$\langle x^*, z-[\alpha x_1 + (1-\alpha)x_2]\rangle + \Psi(\alpha x_1 + (1-\alpha)x_2, z) + \Phi(\alpha x_1 + (1-\alpha)x_2, z) \geq 0,$$

which proves that $\alpha x_1 + (1-\alpha)x_2 \in S(v)$, i.e., $S(v)$ is a convex set.

**Step 2**. Show the boundedness of $S(C)$.

If $C$ is bounded, then it is obviously that $S(C)$ is also bounded. On the contrary, in the case that $S(C)$ is unbounded, we assume that there exist sequences $v_n \in C$ and $x_n \in S(v_n)$ with $x_n \in \mathscr{K}(v_n)$ such that $\|x_n\| \to +\infty$ as $n \to +\infty$. From the definition of the solution mapping $S$, there exists $x_n^* \in F(x_n)$ such that

$$\langle x_n^*, y - x_n\rangle + \Psi(x_n, y) + \Phi(x_n, y) \geq 0 \quad \forall y \in \mathscr{K}(v_n).$$

Taking $y = y_0$ in the above inequality, we obtain

$$\langle x_n^*, y_0 - x_n\rangle + \Psi(x_n, y_0) + \Phi(x_n, y_0) \geq 0. \tag{4.3}$$

Set

$$z_n = \frac{\|y_0\|}{\|x_n - y_0\|} x_n + (1 - \frac{\|y_0\|}{\|x_n - y_0\|}) y_0 \in C$$

with $\|x_n - y_0\| > \|y_0\|$ when $n$ is large enough. It follows that $\|z_n\| \leq 2\|y_0\| + 1$ when $n$ is large enough. Therefore, there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup \tilde{z}$ for some $\tilde{z} \in C$. From the concavity of $\Phi(\cdot, y_0)$, we have

$$\begin{aligned} \Phi(z_n, y_0) &\geq \frac{\|y_0\|}{\|x_n - y_0\|}\Phi(x_n, y_0) + (1 - \frac{\|y_0\|}{\|x_n - y_0\|})\Phi(y_0, y_0) \\ &= \frac{\|y_0\|}{\|x_n - y_0\|}\Phi(x_n, y_0). \end{aligned} \tag{4.4}$$

Since $\Phi(\cdot, y_0)$ is concave and u.s.c., it follows that $\Phi(\cdot, y_0)$ is weakly u.s.c. (see Remark 4.3). Therefore, for the left side of the above inequality, we can obtain

$$\limsup_{n \to +\infty} \Phi(z_n, y_0) \leq \Phi(\tilde{z}, y_0). \tag{4.5}$$

On the other hand, from $(H'_{coer})$, we have, for any $x_n^* \in F(x_n)$,

$$\frac{\langle x_n^*, y_0 - x_n\rangle + \Psi(x_n, y_0)}{\|x_n - y_0\|} \to -\infty, \quad \text{as } \|x_n - y_0\| \to +\infty. \tag{4.6}$$

Combining (4.3) and (4.6), we arrive at

$$\frac{\Phi(x_n, y_0)}{\|x_n - y_0\|} \to +\infty, \quad \text{as } \|x_n - y_0\| \to +\infty.$$

Hence, the right side of inequality (4.4) tends to positive infinity as $\|x_n - y_0\| \to +\infty$, which contradicts inequality (4.5). Therefore, $S(C)$ is also bounded.

**Step 3**. Show that $S$ is a sequentially weakly-weakly closed mapping.

Let $v_n \rightharpoonup v$ and $x_n \rightharpoonup x$ with $x_n \in S(v_n)$. We next show that $x \in S(v)$. From the definition of $S$, we obtain that, for some $x_n^* \in F(x_n)$,

$$\langle x_n^*, y - x_n\rangle + \Psi(x_n, y) + \Phi(x_n, y) \geq 0, \quad \forall y \in \mathscr{K}(v_n). \tag{4.7}$$

We first prove that $\{x_n^*\}$ is bounded in $X^*$. Lemma 2.1 implies that there exist positive constants $c$ and $\rho$ satisfying

$$\rho\|x_n^*\| \leq \langle x_n^*, x_n - x\rangle + c(\rho + \|x_n - x\|). \tag{4.8}$$

Since $\mathscr{K}$ is $M-$continuous, we have $x \in \mathscr{K}(v)$. Moveover, there exists sequence $\{z_n\}$ with $z_n \in \mathscr{K}(v_n)$ satisfying $z_n \to x$, $\Psi(x_n,z_n) \to \Psi(x,x)$ and $\Phi(x_n,z_n) \to \Phi(x,x)$. Combining (4.7) and (4.8), we conclude that

$$\begin{aligned}\rho\|x_n^*\| &\leq \langle x_n^*, x_n - z_n\rangle + \langle x_n^*, z_n - x\rangle + c(\rho + \|x_n - x\|)\\ &\leq \Psi(x_n,z_n) + \Phi(x_n,z_n) + \langle x_n^*, z_n - x\rangle + c(\rho + \|x_n - x\|).\end{aligned}$$

Consequently,

$$[\rho - \|z_n - x\|]\ \|x_n^*\| \leq \Psi(x_n,z_n) + \Phi(x_n,z_n) + c(\rho + \|x_n - x\|).$$

The right side is bounded by taking the limit of the above inequality, which prove that the boundedness of $\{x_n^*\}$. For any $y \in \mathscr{K}(v)$, there exists $\{z_n\}$ with $z_n \in \mathscr{K}(v_n)$ such that $z_n \to y$, $\Psi(x_n,z_n) \to \Psi(x,y)$ and $\Phi(x_n,z_n) \to \Phi(x,y)$. Therefore, for any $w_y \in F(y)$, we have

$$\begin{aligned}\langle w_y, x_n - y\rangle &= \langle w_y - x_n^*, x_n - y\rangle + \langle x_n^*, x_n - z_n\rangle + \langle x_n^*, z_n - y\rangle\\ &\leq \Psi(x_n,z_n) + \Phi(x_n,z_n) + \langle x_n^*, z_n - y\rangle,\end{aligned}$$

where the above inequality uses the monotonicity of $F$. Taking the limit of the above inequality yields

$$\langle w_y, x - y\rangle \leq \Psi(x,y) + \Phi(x,y).$$

Using the Minty inequality again, we obtain that

$$\langle x^*, x - y\rangle \leq \Psi(x,y) + \Phi(x,y), \quad \forall y \in \mathscr{K}(v).$$

Therefore, $x \in S(v)$, and the mapping $S$ is sequentially weakly-weakly closed.

From the above steps, we can easily obtain that $S : C \rightrightarrows C$ satisfies the all conditions of Theorem 2.1. Hence, by using Theorem 2.1, we infer that $S$ admits a fixed point, that is, there exists $x \in \mathscr{K}(x)$ and $x^* \in F(x)$ such that

$$\langle x^*, x - y\rangle + \Psi(x,y) + \Phi(x,y) \geq 0, \quad \forall y \in \mathscr{K}(x).$$

This completes the proof. □

## 5. Regularization

In this section, by using the results obtained previously, our goal is to ensure the solvability of the regularization for the quasi-mixed equilibrium problems. We consider the following regularized quasi-mixed equilibrium problems (RQMEP, for short): find $x_n \in \mathscr{K}(x_n)$ and $x_n^* \in F_n(x_n)$ such that

$$\langle x_n^* + \varepsilon_n \mathscr{R}(x_n), y - x_n\rangle + \Psi_n(x_n,y) + \Phi_n(x_n,y) \geq 0, \quad \forall y \in \mathscr{K}(x_n), \tag{5.1}$$

where $F_n : C \rightrightarrows X^*$, $\mathscr{R} : X \to X^*$, $\Psi_n : C \times C \to \mathbb{R}$, $\Phi_n : C \times C \to \mathbb{R}$ and $\{\varepsilon_n\}$ is a positive real valued sequence.

In the sequel, the set of all solutions of (5.1) is represented by $S_n(RQMEP)$.

**Theorem 5.1.** *Assume that there exists a sequence $\{y_n^0\} \subset C$ satisfying $y_n^0 \in \bigcap_{v \in C} \mathscr{K}(v)$ for each $n \in \mathbb{N}$, and the following conditions are satisfied:*

$(H_{F_n})$ *$F_n : C \rightrightarrows X^*$ is a sequence maximal monotone mapping;*
$(H_{\Psi_n})$ *$\Psi_n$ satisfying all the conditions of $(H'_\Psi)$;*
$(H_{\Phi_n})$ *$\Phi_n$ satisfying all the conditions of $(H'_\Phi)$;*

$(H_C)$ *$C \subset int(\mathscr{D}(F_n))$;*
$(H_{\mathscr{K}})$ *$\mathscr{K} : C \rightrightarrows C$ is $M-$continuous relative to $\Psi_n$ and $\Phi_n$;*
$(H_{\mathscr{R}})$ *$\mathscr{R} : X \to X^*$ is maximal monotone;*
$(H''_{coer})$ *$\mathscr{R} : X \to X^*$ is a coercive map such that there exist positive constants $a$ and $b$ satisfying $\|\mathscr{R}(x)\| \le a\|x\| + b$.*

*Then regularized quasi-mixed equilibrium problems (5.1) has a solution.*

*Proof.* Note that $F_n$ and $\mathscr{R}$ are maximal monotone and $C \subset (int(\mathscr{D}(F_n)) \cap \mathscr{D}(\mathscr{R}))$. Using the Theorem 2.6 of [21], we obtain that $F_n + \varepsilon_n \mathscr{R}(\cdot)$ is also a maximal monotone mapping.

Next, we show that the following coercivity condition is satisfied:

$$\lim_{x\in C,\ \|x\|\to\infty} \frac{\langle x^* + \varepsilon_n \mathscr{R}(x), y_n^0 - x\rangle + \Psi_n(x, y_n^0)}{\|x - y_n^0\|} = -\infty, \text{ for any } x^* \in F_n(x).$$

Setting $r_1 > 0$ and $B(y_n^0, r_1) = \{x \in X : \|x - y_n^0\| \le r_1\}$, we claim that there exists constants $r_2 \in \mathbb{R}$ such that, for every $x \in C \setminus B(y_n^0, r_1)$,

$$\Psi_n(x, y_n^0) \le \frac{r_2}{r_1}\|x - y_n^0\|.$$

Since $\Psi_n(\cdot, y_n^0)$ is concave and u.s.c., it follows that $\Psi_n(\cdot, y_n^0)$ is weakly upper semicontinuous (see Remark 4.3). Hence, on account of the weakly compactness of $B(y_n^0, r_1)$, we have that there exists $r_2 \in \mathbb{R}$ such that

$$\Psi_n(z, y_n^0) \le r_2, \quad \forall z \in B(y_n^0, r_1). \tag{5.2}$$

Let $x \in C \setminus B(y_n^0, r_1)$ and set

$$z = \frac{r_1}{\|x - y_n^0\|}x + (1 - \frac{r_1}{\|x - y_n^0\|})y_n^0 \in B(y_n^0, r_1).$$

Using the concavity of $\Psi_n(\cdot, x_n^0)$ and $\Psi_n(y_n^0, y_n^0) = 0$, we arrive at

$$\begin{aligned} \Psi_n(z, y_n^0) &\ge \frac{r_1}{\|x - y_n^0\|}\Psi_n(x, y_n^0) + (1 - \frac{r_1}{\|x - y_n^0\|})\Psi_n(y_n^0, y_n^0) \\ &= \frac{r_1}{\|x - y_n^0\|}\Psi_n(x, y_n^0). \end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3), we infer that, for every $x \in C \setminus B(y_n^0, r_1)$,

$$\Psi_n(x, y_n^0) \le \frac{r_2}{r_1}\|x - y_n^0\|.$$

For any $x^* \in F_n(x)$ and $w_n^0 \in F_n(y_n^0)$, we obtain that

$$\begin{aligned} & \langle x^* + \varepsilon_n \mathscr{R}(x), y_n^0 - x\rangle + \Psi_n(x, y_n^0) \\ = \ & \langle x^*, y_n^0 - x\rangle + \langle \varepsilon_n \mathscr{R}(x), y_n^0\rangle - \langle \varepsilon_n \mathscr{R}(x), x\rangle + \Psi_n(x, y_n^0) \\ \le \ & \langle w_n^0, y_n^0 - x\rangle + \varepsilon_n \|\mathscr{R}(x)\|\|y_n^0\| - \varepsilon_n \mu(\|x\|)\|x\| + \frac{r_2}{r_1}\|x - y_n^0\| \\ \le \ & \|w_n^0\|\|y_n^0 - x\| + \varepsilon_n(a\|x\| + b)\|y_n^0\| - \varepsilon_n \mu(\|x\|)\|x\| + \frac{r_2}{r_1}\|x - y_n^0\|. \end{aligned}$$

Since $\mu(\|x\|) \to +\infty$ as $\|x\| \to +\infty$, it follows that the coercivity condition is satisfied. Therefore, Theorem 4.1 guarantees that the existence of regularized solution. □

## 6. Convergence and Boundedness of Regularized Solutions

On the basis of the previous sections, we turn to the boundedness and convergence of regularized solutions for quasi-mixed equilibrium problems. Now, we propose some conditions, which are closely related to the perturbed data and exact data.

$(H_1)$ There exists a function $k:\mathbb{R}_+\to\mathbb{R}_+$ with $k(r)\le c_1r+c_2,\ c_1>0,\ c_2>0$ such that, for every $x\in C$ and for every $x^*\in F(x)$ ( $x^*\in F_n(x)$ respectively), there exists $w\in F_n(x)$ ( $w\in F(x)$ respectively) with

$$\|w-x^*\|\le \alpha_n k(\|x\|),\quad \alpha_n>0.$$

$(H_2)$ There exists a function $l:\mathbb{R}_+\to\mathbb{R}_+$ with $l(r)\le c_3r+c_4,\ c_3>0,\ c_4>0$ such that, for every $x,y\in C$,

$$|\Psi_n(x,y)-\Psi(x,y)|\le \beta_n\, l(\|x\|)\|x-y\|,\quad \beta_n>0.$$

$(H_3)$ There exists a function $m:\mathbb{R}_+\to\mathbb{R}_+$ with $m(r)\le c_5r+c_6,\ c_5>0,\ c_6>0$ such that, for every $x,y\in C$,

$$|\Phi_n(x,y)-\Phi(x,y)|\le \gamma_n\, m(\|x\|)\|x-y\|,\quad \gamma_n>0.$$

$(H_4)$ For the sequences of real numbers defined above,

$$\{\alpha_n,\beta_n,\gamma_n,\varepsilon_n\}\to 0,\quad as\ n\to\infty.$$

**Remark 6.1.** Denote the Hausdorff distance between set $A$ and set $B$ by $d_H(A,B)$. If, for any $x\in C$, the following inequality is satisfied

$$d_H(F(x),F_n(x))\le \alpha_n k(\|x\|),$$

then $(H_1)$ holds.

**Theorem 6.1.** *Assume that $(H_1)-(H_4)$ hold, $S_n(RQMEP)\neq\emptyset$ for each $n\in\mathbb{N}$, and $\bigcup_{n\in\mathbb{N}}S_n(RQMEP)$ is bounded. Furthermore, suppose that the following conditions are satisfied:*

$(H_F'')$ *$F:C\rightrightarrows X^*$ is a maximal monotone mapping;*
$(H_{\mathscr{K}})$ *$\mathscr{K}:C\rightrightarrows C$ is $M-$continuous relative to $\Psi$ and $\Phi$;*
$(H_{\mathscr{R}})$ *$\mathscr{R}:X\to X^*$ is maximal monotone;*
$(H_{coer}'')$ *$\mathscr{R}:X\to X^*$ is a coercive map such that there exist positive constants $a$ and $b$ satisfying $\|\mathscr{R}(x)\|\le a\|x\|+b$.*

*Then, for each sequence $\{x_n\}$ with $x_n\in S_n(RQMEP)$, there exists a subsequence weakly converges to some point, which is the solution of (1.1).*

*Proof.* Since $S_n(RQMEP)$ is bounded, one has that $\{x_n\}$ is also bounded. Therefore, there exists a subsequence of $\{x_n\}$, still denoted as $\{x_n\}$, weakly converging to some $x\in X$ due to the reflexivity of $X$. By the definition of $x_n$, we deduce that $x_n\in\mathscr{K}(x_n)$ and $x_n^*\in F_n(x_n)$ satisfy

$$\langle x_n^*+\varepsilon_n\mathscr{R}(x_n),y-x_n\rangle+\Psi_n(x_n,y)+\Phi_n(x_n,y)\ge 0,\quad \forall y\in\mathscr{K}(x_n). \tag{6.1}$$

We claim that $\{x_n^*\}$ is bounded in $X^*$. In fact, Lemma 2.1 implies that there exist constants $c>0$ and $\rho>0$ satisfying

$$\rho\|x_n^*\|\le\langle x_n^*,x_n-x\rangle+c(\rho+\|x_n-x\|). \tag{6.2}$$

Since the weakly closedness of $\mathscr{K}$, we obtain $x \in \mathscr{K}(x)$. Moreover, there exists $z_n \in \mathscr{K}(x_n)$ satisfying $z_n \to x$, $\Psi(x_n, z_n) \to \Psi(x,x)$ and $\Phi(x_n, z_n) \to \Phi(x,x)$. Choosing $y = z_n$ in (6.1), we obtain

$$\langle x_n^* + \varepsilon_n \mathscr{R}(x_n), z_n - x_n \rangle + \Psi_n(x_n, z_n) + \Phi_n(x_n, z_n) \geq 0. \tag{6.3}$$

Combining with (6.2) and (6.3) obtains

$$\begin{aligned}
\rho \|x_n^*\| &\leq \langle x_n^*, x_n - z_n \rangle + \langle x_n^*, z_n - x \rangle + c(\rho + \|x_n - x\|) \\
&\leq \langle \varepsilon_n \mathscr{R}(x_n), z_n - x_n \rangle + \Psi_n(x_n, z_n) + \Phi_n(x_n, z_n) + \langle x_n^*, z_n - x \rangle + c(\rho + \|x_n - x\|) \\
&\leq \langle \varepsilon_n \mathscr{R}(x_n), z_n - x_n \rangle + \Psi_n(x_n, z_n) - \Psi(x_n, z_n) + \Psi(x_n, z_n) + \Phi_n(x_n, z_n) - \Phi(x_n, z_n) \\
&\quad + \Phi(x_n, z_n) + \langle x_n^*, z_n - x \rangle + c(\rho + \|x_n - x\|) \\
&\leq [\varepsilon_n(a\|x_n\| + b) + \beta_n l(\|x_n\|) + \gamma_n m(\|x_n\|)]\|z_n - x_n\| + \Psi(x_n, z_n) + \Phi(x_n, z_n) \\
&\quad + \langle x_n^*, z_n - x \rangle + c(\rho + \|x_n - x\|).
\end{aligned}$$

Consequently,

$$\begin{aligned}
[\rho - \|z_n - x\|]\, \|x_n^*\| &\leq [\varepsilon_n(a\|x_n\| + b) + \beta_n l(\|x_n\|) + \gamma_n m(\|x_n\|)]\|z_n - x_n\| + \Psi(x_n, z_n) \\
&\quad + \Phi(x_n, z_n) + c(\rho + \|x_n - x\|).
\end{aligned}$$

The right side of the above inequality is bounded as $n \to \infty$, which proves that the boundedness of $\{x_n^*\}$. From $(H_1)$, we conclude that there exists $w_n \in F(x_n)$ satisfying $\|x_n^* - w_n\| \leq \alpha_n k(\|x_n\|)$, which implies that $\{w_n\}$ is bounded due to the boundedness of $\{x_n^*\}$. Furthermore, we have

$$\begin{aligned}
&\langle x_n^* - w_n + w_n + \varepsilon_n \mathscr{R}(x_n), z_n - x_n \rangle + \Psi_n(x_n, z_n) - \Psi(x_n, z_n) + \Phi_n(x_n, z_n) \\
-\; &\Phi(x_n, z_n) + \Psi(x_n, z_n) + \Phi(x_n, z_n) \geq 0.
\end{aligned}$$

This, by rearrangements of terms in above inequality, yields

$$\begin{aligned}
&\langle w_n, x_n - z_n \rangle \\
\leq\; &\langle x_n^* - w_n, z_n - x_n \rangle + \langle \varepsilon_n \mathscr{R}(x_n), z_n - x_n \rangle + \Psi_n(x_n, z_n) - \Psi(x_n, z_n) + \Phi_n(x_n, z_n) - \Phi(x_n, z_n) \\
&+ \Psi(x_n, z_n) + \Phi(x_n, z_n) \\
\leq\; &[\alpha_n k(\|x_n\|) + \varepsilon_n(a\|x_n\| + b) + \beta_n l(\|x_n\|) + \gamma_n m(\|x_n\|)]\|z_n - x_n\| + \Psi(x_n, z_n) + \Phi(x_n, z_n).
\end{aligned}$$

Thus, by using the fact that $\{\alpha_n, \beta_n, \gamma_n, \varepsilon_n\} \to 0$ as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \langle w_n, x_n - x \rangle = \limsup_{n\to\infty} \langle w_n, x_n - z_n \rangle \leq 0.$$

From the maximal monotonicity of $F$, it follows that $F$ is generalized pseudomonotone. On the other hand, there exists a point $x^* \in X^*$ such that $w_n \rightharpoonup x^*$, which is due to the boundedness of $\{w_n\}$. Hence, $\lim_{n\to\infty} \langle w_n, x_n \rangle = \langle x^*, x \rangle$ and $x^* \in F(x)$.

Next, we prove that, for any $z \in K(x)$,

$$\langle x^*, z - x \rangle + \Psi(x,z) + \Phi(x,z) \geq 0.$$

For any $z \in \mathscr{K}(x)$, there exists $z_n \in \mathscr{K}(x_n)$ satisfying $z_n \to z$, $\Psi(x_n, z_n) \to \Psi(x,z)$ and $\Phi(x_n, z_n) \to \Phi(x,z)$. Taking $y = z_n$ in formula (6.1), we have

$$\langle x_n^* + \varepsilon_n \mathscr{R}(x_n), z_n - x_n \rangle + \Psi_n(x_n, z_n) + \Phi_n(x_n, z_n) \geq 0.$$

Therefore,

$$\begin{aligned}
&\langle w_n, x_n - z_n\rangle - \Psi(x_n,z_n) - \Phi(x_n,z_n)\\
= \;& \langle w_n - x_n^*, x_n - z_n\rangle + \langle x_n^*, x_n - z_n\rangle - \Psi(x_n,z_n) - \Phi(x_n,z_n)\\
\leq \;& \langle w_n - x_n^*, x_n - z_n\rangle + \langle \varepsilon_n \mathscr{R}(x_n), z_n - x_n\rangle + \Psi_n(x_n,z_n) - \Psi(x_n,z_n) + \Phi_n(x_n,z_n) - \Phi(x_n,z_n)\\
\leq \;& [\alpha_n k(\|x_n\|) + \varepsilon_n(a\|x_n\| + b) + \beta_n l(\|x_n\|) + \gamma_n m(\|x_n\|)]\|z_n - x_n\|.
\end{aligned}$$

Using again the fact $\{\alpha_n, \beta_n, \gamma_n, \varepsilon_n\} \to 0$ as $n \to \infty$, and taking the limit of the above inequality, we have

$$\langle x^*, x - z\rangle - \Psi(x,z) - \Phi(x,z) \leq 0.$$

Hence, we can conclude that $x$ is the solution of (1.1). □

**Remark 6.2.** It can be seen from the proof of Theorem 6.1 that if $F : C \rightrightarrows X^*$ is generalized pseudomonotone and bounded (i.e., $F(U)$ is bounded in $X^*$ for any bounded subset $U$ of $C$ ), and the other conditions of Theorem 6.1 except $(H_F'')$ remain unchanged, then the conclusion is also valid.

**Remark 6.3.** The boundedness of regularized solutions is a prerequisite for the convergence. Assume that $S_n(RQMEP) \neq \emptyset$ and for any given sequence $\{x_n\} \subset C$, there exists a bounded sequence $\{z_n\}$ satisfying $z_n \in \mathscr{K}(x_n)$ and $\limsup_{n\to\infty} \frac{\Psi_n(x_n,z_n)+\Phi_n(x_n,z_n)}{\|x_n\|} < +\infty$. Furthermore, if $(H_1) - (H_4)$ and the following condition are satisfied:

$(H)$ for each sequence $\{(x_n^*, x_n)\}$ with $x_n^* \in F_n(x_n)$, we have

$$\liminf_{n\to\infty} \frac{\langle x_n^*, x_n - z_n\rangle}{\|x_n\|} = +\infty, \quad \text{as } \|x_n\| \to \infty.$$

Then $\{x_n\} \subset C$ with $x_n \in S_n(RQMEP)$ is bounded.

On the contrary, let the sequence $\{x_n\} \subset C$ be unbounded and $\|x_n\| \to \infty$. By the definition of $\{x_n\}$, there exists some $x_n^* \in F_n(x_n)$ such that

$$\langle x_n^* + \varepsilon_n \mathscr{R}(x_n), z - x_n\rangle + \Psi_n(x_n, z) + \Phi_n(x_n, z) \geq 0, \quad \forall z \in \mathscr{K}(x_n).$$

By taking $z = z_n$ in the above inequality and rearranging it, one has

$$\begin{aligned}
\langle x_n^*, x_n - z_n\rangle &\leq \langle \varepsilon_n \mathscr{R}(x_n), z_n - x_n\rangle + \Psi_n(x_n,z_n) + \Phi_n(x_n,z_n)\\
&\leq \langle \varepsilon_n \mathscr{R}(z_n), z_n - x_n\rangle + \Psi_n(x_n,z_n) + \Phi_n(x_n,z_n)\\
&\leq \varepsilon_n \|\mathscr{R}(z_n)\| \ (\|z_n\| + \|x_n\|) + \Psi_n(x_n,z_n) + \Phi_n(x_n,z_n),
\end{aligned}$$

where the monotonicity of $\mathscr{R}$ is used in the second inequality above. Furthermore, we have

$$\frac{\langle x_n^*, x_n - z_n\rangle}{\|x_n\|} \leq \varepsilon_n \|\mathscr{R}(z_n)\| \ (1 + \frac{\|z_n\|}{\|x_n\|}) + \frac{\Psi_n(x_n,z_n) + \Phi_n(x_n,z_n)}{\|x_n\|}.$$

From the assumptions, we infer that the right side of the above inequality is bounded as $\|x_n\| \to \infty$, which contradicts condition $(H)$. Hence, the sequence $\{x_n\}$ of regularized solutions is bounded.

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