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ON GENERALIZED GLOBAL FRACTIONAL-ORDER COMPOSITE DYNAMICAL SYSTEMS WITH SET-VALUED PERTURBATIONS

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Abstract. In this paper, we investigate a class of generalized global fractional-order composite dynamical systems involving set-valued perturbations in real separable Hilbert spaces. First, we prove that the solution set of the systems is nonempty and closed under some suitable conditions. Second, we show that the solution set is continuous with respect to the initial value in the sense of the Hausdorff metric. Last, an example is provided to illustrate the applicability of the main results.

Keywords. Fractional-order composite dynamical system; Set-valued perturbation; Sensitivity.

1. INTRODUCTION

In 1993, Dupuis and Nagurney [1] revealed the connection between a projective dynamical system and an associated variational inequality via the following local projective dynamical system

$$\frac{dx}{dt} = \lim_{\rho\to 0}\frac{P_K(x-\rho N(x))-x}{\rho}.$$

Subsequently, Friesze *et al.* [2] extended this type of systems to global ones and applied them to the problems in traffic network equilibrium analysis. Meanwhile, they showed that the tatonnement model of a certain traffic problem can be formulated as a simultaneous projective dynamical system

$$\begin{cases} \frac{dh(t)}{dt} = \eta\{P_K(h(t)-\rho ETC(h(t),u(t)))-h(t)\}, & \forall t\in[0,b], \\ \frac{du(t)}{dt} = \kappa\{P_K(u(t)+\lambda ETD(u(t),h(t)))-u(t)\}, & \forall t\in[0,b], \\ h(0)=h_0 \text{ and } u(0)=u_0, \end{cases}$$

where $\rho,\lambda\in\mathbb{R}^1_+=[0,\infty)$.

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Projective dynamical systems have been under the spotlight of research for the past decades. Both in-depth theories and wide applications of the projective dynamical systems have been studied extensively. For example, we refer the reader to [1–10] and the references therein.

To the best of our knowledge, many real life scientific or engineering problems are neither linear nor nonlinear, which makes it inappropriate to model them with differential systems of integer order. Instead, fractional derivatives provide a better tool than integer order ones to describe many physical processes, especially those with memory and hereditary properties;, see, e.g., [11–13]. Recently, some new results on fractional order differential systems have appeared in this field; see, e.g., [14, 15]. In particular, in 1984, Torvik and Bagley [11] showed that the fractional order model was effective in describing the behavior of real material. In 2011, Li and Zhang [12] did a survey on the stability of fractional differential equations. In 2012, Ozalp and Koca [13] introduced a fractional order dynamical model of interpersonal relationships. In 2015, Buyukkilic *et al.* [14] investigated the cumulative growth of a physical quantity via the Fibonacci method and fractional calculus.

On the other hand, Wu and Zou [16], for the first time, proposed the following global fractional-order projective dynamical system in $\mathbb{R}^n$

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = P_K(x(t) - \rho M x(t) - \rho q) - x(t), \quad t \geq 0, \\ x_i(0) = x_{i,0}, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, and M is a real $n \times n$ matrix. Then the existence and uniqueness of solutions and the existence of equilibrium points were obtained, and they showed that equilibrium points were α-exponentially stable.

Recently, Wu *et al.* [17] considered a class of fractional set-valued projected dynamical systems in $\mathbb{R}^n$

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) \in P_K(g(x(t)) - \lambda N(x(t))) - g(x(t)), \quad \text{for } t \in [0, b]; \\ x(0) = p, \; x(b) = q, \quad p, q \neq 0, \end{cases} \tag{1.2}$$

where $\alpha \in [1, 2)$, and $N : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued mapping. The nonemptiness and closedness of the solution set were obtained, and they showed that the solution set was continuous with respect to the boundary value in the sense of the Hausdorff metric.

In 2016, based on Wardropian user equilibrium tatonnement model, Wu *et al.* [18] investigated a class of fractional-order interval projective dynamical systems in $\mathbb{R}^n \times \mathbb{R}^m$

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = P_{K_1}[x(t) - \rho(Ax(t) + A^* y(t)) - \rho p] - x(t), \; t \geq 0, \\ {}_0^C D_t^{\alpha} y(t) = P_{K_2}[y(t) - \lambda(By(t) + B^* x(t)) - \lambda q] - y(t), \; t \geq 0, \\ x(0) = x_0 \text{ and } y(0) = y_0, \end{cases} \tag{1.3}$$

where $0 < \alpha \leq 1$ and

$$\begin{cases} A \in A_{\mathscr{I}} = \{(a_{i,j})_{n \times n} : \underline{A} \leq A \leq \overline{A}, \quad \text{i.e., } \underline{a}_{i,j} \leq a_{i,j} \leq \overline{a}_{i,j}\}, \\ A^* \in A^*_{\mathscr{I}} = \{(a^*_{i,j})_{n \times m} : \underline{A^*} \leq A^* \leq \overline{A^*}, \quad \text{i.e., } \underline{a^*}_{i,j} \leq a_{i,j} \leq \overline{a^*}_{i,j}\}, \\ B \in B_{\mathscr{I}} = \{(b_{i,j})_{m \times m} : \underline{B} \leq B \leq \overline{B}, \quad \text{i.e., } \underline{b}_{i,j} \leq b_{i,j} \leq \overline{b}_{i,j}\}, \\ B^* \in B^*_{\mathscr{I}} = \{(b^*_{i,j})_{m \times n} : \underline{B^*} \leq B^* \leq \overline{B^*}, \quad \text{i.e., } \underline{b^*}_{i,j} \leq b_{i,j} \leq \overline{b^*}_{i,j}\}. \end{cases}$$

They proved the existence ant uniqueness of the equilibrium point under some suitable conditions and obtained the α-exponential stability of this type of projective dynamical systems.

It is well known that, for the global fractional-order projective dynamical systems in abstract spaces, there are little few results in the existing literature. Very recently, Wu *et al.* [3] made an attempt in this direction. Moreover, the system may appear perturbation by external factor. In particular, the perturbation is a set-valued mapping. In [3], the following fractional projective dynamical system with set-valued perturbations was first introduced in real separable Hilbert spaces:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) \in P_{K_1}(x(t) - \rho M(x(t), y(t)) - \rho p) - x(t) + G_1(x(t)), & \text{for a.e. } t \in [0,b], \\ {}_0^C D_t^{\alpha} y(t) \in P_{K_2}(y(t) - \lambda N(y(t), x(t)) - \lambda q) - y(t) + G_2(y(t)), & \text{for a.e. } t \in [0,b], \\ x(0) = x_0 \text{ and } y(0) = y_0, \end{cases} \tag{1.4}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1]$, P_{K_1}, and P_{K_2} are two projection operators, K_1 and K_2 are two nonempty closed convex subsets of two separable Hilbert spaces V_1 and V_2, respectively, $\rho > 0$ and $\lambda > 0$ are two constants, $x, x_0, p \in V_1$, $y, y_0, q \in V_2$, $M : V_1 \times V_2 \to V_1$, and $N : V_2 \times V_1 \to V_2$ are two nonlinear mappings, and $G_1 : V_1 \to 2^{V_1}$ and $G_2 : V_2 \to 2^{V_2}$ are two set-valued mappings. Obviously, in Euclidean spaces, if $G_1 = G_2 = 0$, $M(x(t), y(t)) = Ax(t) + A^* y(t)$, and $N(y(t), x(t)) = By(t) + B^* x(t)$, then (1.4) reduces to (1.3). If $G_1 = G_2 = 0$, $N = 0$, $q = 0$, $y = 0$, $\alpha \in (0,1)$, and M is a real matrix, then (1.4) reduces to (1.1). In addition, if we consider the two point boundary value problem, $\alpha \in [1,2)$, M is a set-valued mapping, then (1.4) reduces to (1.2). It is worth mentioning that if $K_1 = V_1$ and $K_2 = V_2$, then (1.4) reduces to a system of two related fractional differential inclusions.

Let $\alpha \in (0,1]$, $J = [0,b]$ for some $0 < b < \infty$, and $V = V_1 \times V_2$, where V_i is a separable Hilbert space for $i = 1,2$. Endowed with the norm defined by $\|\mathbf{x}\|_V := \|x_1\|_{V_1} + \|x_2\|_{V_2}$ for all $\mathbf{x} = (x_1, x_2) \in V$, V is a reflexive Banach space (see e.g., [19]). Inspired by the above research works in [3], in this paper, we consider the following generalized fractional composite dynamical system with set-valued perturbations in real separable Hilbert spaces:

$$\begin{cases} {}_0^C D_t^{\alpha} x_1(t) \in P_1(x_1(t) - \rho_1 N_1(\Phi_1^1(x_1(t)), \Phi_2^1(x_2(t))) - \rho_1 q_1) - x_1(t) + G_1(x_1(t)), \text{ a.e. } t \in J, \\ {}_0^C D_t^{\alpha} x_2(t) \in P_2(x_2(t) - \rho_2 N_2(\Phi_1^2(x_1(t)), \Phi_2^2(x_2(t))) - \rho_2 q_2) - x_2(t) + G_2(x_2(t)), \text{ a.e. } t \in J, \\ x_1(0) = x_{1,0} \text{ and } x_2(0) = x_{2,0}, \end{cases} \tag{1.5}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1]$. For $i = 1,2$, $P_i : V_i \to V_i$ is nonexpansive, $\rho_i > 0$ is a constant, $x_i, x_{i,0}, q_i \in V_i$, $N_i : V_1 \times V_2 \to V_i$ is a nonlinear mapping, and $\Phi_i^1, \Phi_i^2, G_i : V_i \to 2^{V_i}$ all are set-valued mappings. In particular, if, for $i = 1,2$, $P_i = P_{K_i}$ and $\Phi_i^1 = \Phi_i^2 = I$ the identity mapping of V_i, system (1.5) actually reduces to system (1.4).

The rest of this paper is organized below. In Section 2, we begin with some basic concepts and useful lemmas. In Section 3, the nonemptiness and closedness of the solution set for system (1.5) are first shown under some suitable conditions. It is also proven that the solution set is continuous with respect to the initial value in the sense of the Hausdorff metric. An example is provided to support our main results in Section 4. Finally, the concluding remark is given in Section 5.

2. Preliminaries

In this section, we introduce some basic definitions and preliminaries which are used throughout this paper. We will denote the norm and dual space of a Banach space X by $\|\cdot\|_X$ and X^*, respectively, and the duality pairing between X^* and X by $\langle \cdot, \cdot \rangle_X$. Let $C(J, X)$ be the Banach space of all continuous functions $x(t)$ from $J=[0,b]$ into X with norm

$$\|x\|_{C(J,X)}=\max_{t\in J}\|x(t)\|_X,$$

and let $L^1(J,X)$ be the Banach space of all Bochner integrable functions $x:J\to X$ with norm

$$\|x\|_{L^1(J,X)}=\int_0^b\|x(t)\|dt.$$

Let $\mathscr{C}(X)$ denote the family of all nonempty compact subsets of X, and let $\mathscr{H}(\cdot,\cdot)$ be the Hausdorff metric $\mathscr{H}(\cdot,\cdot)$ on $\mathscr{C}(X)$ defined by

$$\mathscr{H}(A,B)=\max\{\sup_{y\in B}\inf_{x\in A}d(x,y),\sup_{x\in A}\inf_{y\in B}d(x,y)\},\quad \forall A,B\in\mathscr{C}(X),$$

where d is a metric induced by the norm $\|\cdot\|_X$ of X.

Lemma 2.1 ([20]). *Let $A,B\in\mathscr{C}(X)$. For every $x\in A$, there exists $y\in B$ such that $d(x,y)\le\mathscr{H}(A,B)$.*

We next recall the Riemann-Liouville fractional integral with order $\alpha>0$ of a suitable function x (e.g., $x\in L^1([t_0,t_1],\mathbf{R})$) formulated below (see [21])

$$I_{t_0}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau,\quad (t>t_0),$$

where $t_0\in\mathbf{R}$ and Γ is the Gamma function. It is well known that $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ $(\alpha>0)$ and $\Gamma(1)=1$. Also, for a suitable function x given on the interval $[t_0,t_1]$, we recall the Caputo fractional order derivative of order α of x formulated below (see [21])

$${}_{t_0}^{C}D_t^{\alpha}x(t)=I_{t_0}^{n-\alpha}x^{(n)}(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau, & n-1<\alpha<n,\\ x^{(n)}(t), & \alpha=n,\end{cases}$$

where $t>t_0$ and n is a positive integer.

It is remarkable that if x is an abstract function with values in a Banach space X, then the integrals which appear in the above definitions are taken in Bochner's sense. Let $T:X\to X$ be a nonlinear operator on X. T is said to be nonexpansive if $\|Tx-Ty\|_X\le\|x-y\|_X$, $\forall x,y\in X$. In particular, we provide an example of nonexpansive operators. Suppose that K is a nonempty, closed, and convex subset of a real Hilbert space H. Then the projection operator $P_K:H\to K$ is nonexpansive.

Definition 2.1. Let $N_1:V_1\times V_2\to V_1$ be a nonlinear operator and $\Phi_1^1:V_1\to\mathscr{C}(V_1)$ be a set-valued mapping. The operator $N_1(\cdot,\cdot)$ is said to be $k_{1,1}$-strongly monotone for Φ_1^1 with respect to the first argument if, for any $x_{1,1},x_{1,2}\in V_1$,

$$\langle N_1(w_{1,1},\cdot)-N_1(w_{1,2},\cdot),x_{1,1}-x_{1,2}\rangle_{V_1}\ge k_{1,1}\|x_{1,1}-x_{1,2}\|_{V_1}^2,\ \forall w_{1,1}\in\Phi_1^1(x_{1,1}),\ w_{1,2}\in\Phi_1^1(x_{1,2}).$$

Similarly, we can define the $k_{2,2}$-strong monotonicity of $N_2(\cdot,\cdot)$ for Φ_2^2 with respect to the second argument.

Definition 2.2. Let $N_1:V_1\times V_2\to V_1$ be a nonlinear operator. The operator $N_1(\cdot,\cdot)$ is said to be $c_{1,1}$-Lipschitz with respect to the first argument if, for any $x_{1,1},x_{1,2}\in V_1$,

$$\|N_1(x_{1,1},\cdot)-N_1(x_{1,2},\cdot)\|_{V_1}\le c_{1,1}\|x_{1,1}-x_{1,2}\|_{V_1}.$$

Similarly, we can define the $l_{1,2}$-Lipschitz continuity of $N_1(\cdot,\cdot)$ with respect to the second argument.

Let (Ω,d) be a metric space, and let $T:\Omega\to 2^{\Omega}$ be a set-valued mapping with closed values. We say that T is ω-Lipschitz continuous if there exists a constant $\omega>0$ such that

$$\mathscr{H}(T(x),T(y))\leq \omega d(x,y),\quad \forall x,y\in\Omega.$$

In particular, if $\omega<1$, then T is called a set-valued ω-contraction.

Definition 2.3 ([22]). Let Z and Z_1 be two topological spaces. A set-valued mapping $G:Z\to 2^{Z_1}$ is said to be

- (i) upper semicontinuous (u.s.c.) if, for every open $O\subset Z_1$, $G^{-1}(O)=\{u\in Z: G(u)\subset O\}$ is open in Z;
- (ii) lower semicontinuous (l.s.c.) if, for every closed $C\subset Z_1$, $G^{-1}(C)$ is closed in Z;
- (iii) continuous if it is both u.s.c. and l.s.c.

Definition 2.4 ([22]). Let (Ω,Σ) be a measurable space, and let Ω_* be a separable metric space. A set-valued mapping $G:\Omega\to 2^{\Omega_*}$ with closed values is said to be measurable if $G^{-1}(U)\in\Sigma$ for every open $U\subset\Omega_*$.

Definition 2.5. A pair of functions $(x_1,x_2)\in C(J,V_1\times V_2)$ is a solution of system (1.5) if there exists a pair of functions $(z_1,z_2)\in L^1(J,V_1\times V_2)$ such that

$$\begin{cases} z_1(t)\in P_1(x_1(t)-\rho_1N_1(\Phi_1^1(x_1(t)),\Phi_2^1(x_2(t)))-\rho_1q_1)-x_1(t)+G_1(x_1(t)), \text{ a.e. } t\in J,\\ z_2(t)\in P_2(x_2(t)-\rho_2N_2(\Phi_1^2(x_1(t)),\Phi_2^2(x_2(t)))-\rho_2q_2)-x_2(t)+G_2(x_2(t)), \text{ a.e. } t\in J,\end{cases}$$

and

$$\begin{cases} x_1(t)=x_{1,0}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}z_1(\tau)d\tau,\\ x_2(t)=x_{2,0}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}z_2(\tau)d\tau.\end{cases}$$

Lemma 2.2 ([22]). *Let Ω_* be a separable complete metric space. Then every measurable $G:J=[0,b]\to 2^{\Omega_*}$ has a (single-valued) selection.*

Lemma 2.3 ([22]). *Let Z,Z_1 be two topological spaces, and let $T:Z\to 2^{Z_1}$ be u.s.c. with compact values. Then $T(B)=\bigcup_{x\in B}T(x)$ is compact in Z_1 for any compact subset B of Z.*

Lemma 2.4 ([3]). *Let $T_1,T_2:J\to\mathscr{C}(X)$ be two continuous mappings in the Hausdorff metric $\mathscr{H}$, and let $f:J\to X$ be a measurable selection of T_1. Then there exists a measurable selection $g:J\to X$ of T_2 such that, for all $s\in J$,*

$$\|f(s)-g(s)\|_X\leq\mathscr{H}(T_1(s),T_2(s)).$$

Lemma 2.5 ([23]). *Let X be a Banach space. The function $r:J=[0,b]\to X$ is Bochner integrable if and only if $r(\cdot)$ is strongly measurable and $\|r(\cdot)\|_X$ is Lebesgue integrable in J.*

Lemma 2.6 ([20]). *Let (Ω,d) be a complete metric space. If $T:\Omega\to 2^{\Omega}$ is a set-valued contraction mapping with closed and bounded values, then T has a fixed point in Ω.*

Lemma 2.7 ([24]). *Let (Ω,d) be a complete metric space, and let $T_1,T_2:\Omega\to 2^{\Omega}$ be two set-valued mappings with closed values. Assume that T_1,T_2 have the same contractive constant $\theta\in(0,1)$. Then*

$$\mathscr{H}(F(T_1),F(T_2))\leq\frac{1}{1-\theta}\sup_{u\in\Omega}\mathscr{H}(T_1(u),T_2(u)),$$

where $F(T_1)$ and $F(T_2)$ are fixed-point sets of T_1 and T_2, respectively.

3. Main Results

In this section, we study the existence of solutions and the sensitivity of the set of solutions for dynamical system (1.5) under some suitable conditions. Let $S(x_{1,0},x_{2,0})$ denote the set of all solutions for system (1.5) on $J=[0,b]$ with initial value $(x_{1,0},x_{2,0})$.

For a continuous mapping $A_1(x_1(\cdot)) : J \to \mathscr{C}(V_1)$ and $x_1 \in C(J,V_1)$, we define

$$S_{A_1,x_1} = \{z_1(s) \in L^1(J,V_1) : z_1(s) \in A_1(x_1(s)) \text{ for a.e. } s \in J\}.$$

Theorem 3.1. *Let $G_1 : V_1 \to \mathscr{C}(V_1)$ be a ω_1-Lipschitz continuous mapping, and let $G_2 : V_2 \to \mathscr{C}(V_2)$ be a ω_2-Lipschitz continuous mapping. Assume that*

- (i) *$N_1 : V_1 \times V_2 \to V_1$ is $k_{1,1}$-strongly monotone for Φ_1^1 with respect to the first argument, $c_{1,1}$-Lipschitz with respect to the first argument and $l_{1,2}$-Lipschitz with respect to the second argument;*
- (ii) *$\Phi_1^1 : V_1 \to \mathscr{C}(V_1)$ is $\beta_{1,1}$-Lipschitz continuous, and $\Phi_2^1 : V_2 \to \mathscr{C}(V_2)$ is $\beta_{1,2}$-Lipschitz continuous;*
- (iii) *$N_2 : V_1 \times V_2 \to V_2$ be $k_{2,2}$-strongly monotone for Φ_2^2 with respect to the second argument, $c_{2,2}$-Lipschitz with respect to the second argument, and $l_{2,1}$-Lipschitz with respect to the first argument;*
- (iv) *$\Phi_1^2 : V_1 \to \mathscr{C}(V_1)$ is $\beta_{2,1}$-Lipschitz continuous, and $\Phi_2^2 : V_2 \to \mathscr{C}(V_2)$ is $\beta_{2,2}$-Lipschitz continuous.*

If

$$\begin{cases} \theta_1 = \frac{b^\alpha}{\Gamma(\alpha+1)}(\sqrt{\xi}+\sqrt{\eta}+\omega_1+1) < 1, \\ \theta_2 = \frac{b^\alpha}{\Gamma(\alpha+1)}(\sqrt{\xi}+\sqrt{\eta}+\omega_2+1) < 1, \end{cases} \tag{3.1}$$

where

$$\begin{cases} \xi = \max\{1-2\rho_1 k_{1,1}+\rho_1^2 c_{1,1}^2 \beta_{1,1}^2, \rho_1 l_{1,2}\beta_{1,2}+\rho_1^2 c_{1,1} l_{1,2}\beta_{1,1}\beta_{1,2}, \rho_1^2 l_{1,2}^2 \beta_{1,2}^2\} > 0, \\ \eta = \max\{1-2\rho_2 k_{2,2}+\rho_2^2 c_{2,2}^2 \beta_{2,2}^2, \rho_2 l_{2,1}\beta_{2,1}+\rho_2^2 c_{2,2} l_{2,1}\beta_{2,2}\beta_{2,1}, \rho_2^2 l_{2,1}^2 \beta_{2,1}^2\} > 0, \end{cases}$$

then $S(x_{1,0},x_{2,0})$ is nonempty, closed, and continuous with respect to the initial value $(x_{1,0},x_{2,0})$ in the sense of the Hausdorff metric.

Proof. For any $(x_1,x_2) \in V_1 \times V_2$, let

$$A_1(x_1,x_2) = P_1(x_1 - \rho_1 N_1(\Phi_1^1(x_1),\Phi_2^1(x_2)) - \rho_1 q_1) - x_1 + G_1(x_1).$$

Since $N_1 : V_1 \times V_2 \to V_1$, $G_1, \Phi_1^1 : V_1 \to \mathscr{C}(V_1)$, and $\Phi_2^1 : V_2 \to \mathscr{C}(V_2)$ are continuous mappings, we know that $A_1 : V_1 \times V_2 \to \mathscr{C}(V_1)$ is also a continuous mapping. For any $(x_{1,1},x_{2,1}),(x_{1,2},x_{2,2}) \in V_1 \times V_2$ and any $u_1^1 \in A_1(x_{1,1},x_{2,1})$, there exist $(w_{1,1},w_{2,1}) \in \Phi_1^1(x_{1,1}) \times \Phi_2^1(x_{2,1})$ and $v_1^1 \in G_1(x_{1,1})$ such that

$$u_1^1 = P_1(x_{1,1} - \rho_1 N_1(w_{1,1},w_{2,1}) - \rho_1 q_1) - x_{1,1} + v_1^1. \tag{3.2}$$

Since $(w_{1,1},w_{2,1}) \in \Phi_1^1(x_{1,1}) \times \Phi_2^1(x_{2,1})$, $\Phi_1^1 : V_1 \to \mathscr{C}(V_1)$ is $\beta_{1,1}$-Lipschitz continuous, $\Phi_2^1 : V_2 \to \mathscr{C}(V_2)$ is $\beta_{1,2}$-Lipschitz continuous and $G_1 : V_1 \to \mathscr{C}(V_1)$ is ω_1-Lipschitz continuous, it follows from Lemma 2.1 that there exist $(w_{1,2},w_{2,2}) \in \Phi_1^1(x_{1,2}) \times \Phi_2^1(x_{2,2})$ and $v_2^1 \in G_2(x_{1,2})$ such that

$$\|w_{1,1}-w_{1,2}\|_{V_1} \le \mathscr{H}(\Phi_1^1(x_{1,1}),\Phi_1^1(x_{1,2})) \le \beta_{1,1}\|x_{1,1}-x_{1,2}\|_{V_1}, \tag{3.3}$$

$$\|w_{2,1}-w_{2,2}\|_{V_2} \le \mathscr{H}(\Phi_2^1(x_{2,1}),\Phi_2^1(x_{2,2})) \le \beta_{1,2}\|x_{2,1}-x_{2,2}\|_{V_2}, \tag{3.4}$$

and

$$\|v_1^1 - v_2^1\|_{V_1} \le \mathscr{H}(G_1(x_{1,1}), G_1(x_{1,2})) \le \omega_1 \|x_{1,1} - x_{1,2}\|_{V_1}. \tag{3.5}$$

Let

$$u_2^1 = P_1(x_{1,2} - \rho_1 N_1(w_{1,2}, w_{2,2}) - \rho_1 q_1) - x_{1,2} + v_2^1. \tag{3.6}$$

Then $u_2^1 \in A_1(x_{1,2}, x_{2,2})$. Moreover, in light of (3.2)-(3.6) and the definition of nonexpansive operators, one has

$$\begin{aligned}\|u_1^1 - u_2^1\|_{V_1} &= \|(P_1(x_{1,1} - \rho_1 N_1(w_{1,1}, w_{2,1}) - \rho_1 q_1) - x_{1,1} + v_1^1)\\ &\quad -(P_1(x_{1,2} - \rho_1 N_1(w_{1,2}, w_{2,2}) - \rho_1 q_1) - x_{1,2} + v_2^1)\|_{V_1}\\ &\le \|P_1(x_{1,1} - \rho_1 N_1(w_{1,1}, w_{2,1}) - \rho_1 q_1)\\ &\quad -P_1(x_{1,2} - \rho_1 N_1(w_{1,2}, w_{2,2}) - \rho_1 q_1)\|_{V_1} + \|x_{1,1} - x_{1,2}\|_{V_1} + \|v_1^1 - v_2^1\|_{V_1}\\ &\le \|(x_{1,1} - x_{1,2}) - \rho_1(N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,2}))\|_{V_1}\\ &\quad +(1+\omega_1)\|x_{1,1} - x_{1,2}\|_{V_1}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\|(x_{1,1} - x_{1,2}) - \rho_1(N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,2}))\|_{V_1}^2\\ &= \|x_{1,1} - x_{1,2}\|_{V_1}^2 - 2\rho_1 \langle N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,2}), x_{1,1} - x_{1,2}\rangle_{V_1}\\ &\quad +\rho_1^2 \|N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,2})\|_{V_1}^2.\end{aligned} \tag{3.8}$$

Since $N_1 : V_1 \times V_2 \to V_1$ is $k_{1,1}$-strongly monotone for Φ_1^1 with respect to the first argument, $c_{1,1}$-Lipschitz with respect to the first argument and $l_{1,2}$-Lipschitz with respect to the second argument, we obtain from (3.3) and (3.4) that

$$\begin{aligned}&-2\rho_1 \langle N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,2}), x_{1,1} - x_{1,2}\rangle_{V_1}\\ &= -2\rho_1 \langle N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,1}) + N_1(w_{1,2}, w_{2,1}) - N_1(w_{1,2}, w_{2,2}), x_{1,1} - x_{1,2}\rangle_{V_1}\\ &= -2\rho_1 \langle N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,1}), x_{1,1} - x_{1,2}\rangle_{V_1}\\ &\quad -2\rho_1 \langle N_1(w_{1,2}, w_{2,1}) - N_1(w_{1,2}, w_{2,2}), x_{1,1} - x_{1,2}\rangle_{V_1}\\ &\le -2\rho_1 k_{1,1} \|x_{1,1} - x_{1,2}\|_{V_1}^2 + 2\rho_1 \|N_1(w_{1,2}, w_{2,1}) - N_1(w_{1,2}, w_{2,2})\|_{V_1} \|x_{1,1} - x_{1,2}\|_{V_1}\\ &\le -2\rho_1 k_{1,1} \|x_{1,1} - x_{1,2}\|_{V_1}^2 + 2\rho_1 l_{1,2} \|x_{1,1} - x_{1,2}\|_{V_1} \|w_{2,1} - w_{2,2}\|_{V_2}\\ &\le -2\rho_1 k_{1,1} \|x_{1,1} - x_{1,2}\|_{V_1}^2 + 2\rho_1 l_{1,2} \beta_{1,2} \|x_{1,1} - x_{1,2}\|_{V_1} \|x_{2,1} - x_{2,2}\|_{V_2},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\rho_1^2 \|N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,2})\|_{V_1}^2\\ &= \rho_1^2 \|N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,1}) + N_1(w_{1,2}, w_{2,1}) - N_1(w_{1,2}, w_{2,2})\|_{V_1}^2\\ &\le \rho_1^2 \|N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,1})\|_{V_1}^2 + 2\rho_1^2 \|N_1(w_{1,1}, w_{2,1}) - N_1(w_{1,2}, w_{2,1})\|_{V_1}\\ &\quad \times \|N_1(w_{1,2}, w_{2,1}) - N_1(w_{1,2}, w_{2,2})\|_{V_1} + \rho_1^2 \|N_1(w_{1,2}, w_{2,1}) - N_1(w_{1,2}, w_{2,2})\|_{V_1}^2\\ &\le \rho_1^2 c_{1,1}^2 \|w_{1,1} - w_{1,2}\|_{V_1}^2 + 2\rho_1^2 c_{1,1} l_{1,2} \|w_{1,1} - w_{1,2}\|_{V_1} \|w_{2,1} - w_{2,2}\|_{V_2}\\ &\quad +\rho_1^2 l_{1,2}^2 \|w_{2,1} - w_{2,2}\|_{V_2}^2\\ &\le \rho_1^2 c_{1,1}^2 \beta_{1,1}^2 \|x_{1,1} - x_{1,2}\|_{V_1}^2 + 2\rho_1^2 c_{1,1} l_{1,2} \beta_{1,1} \beta_{1,2} \|x_{1,1} - x_{1,2}\|_{V_1} \|x_{2,1} - x_{2,2}\|_{V_2}\\ &\quad +\rho_1^2 l_{1,2}^2 \beta_{1,2}^2 \|x_{2,1} - x_{2,2}\|_{V_2}^2.\end{aligned} \tag{3.10}$$

It follows from (3.8)-(3.10) that

$$\begin{aligned}
&\|(x_{1,1}-x_{1,2})-\rho_1(N_1(w_{1,1},w_{2,1})-N_1(w_{1,2},w_{2,2}))\|_{V_1}^2\\
&\le \|x_{1,1}-x_{1,2}\|_{V_1}^2-2\rho_1k_{1,1}\|x_{1,1}-x_{1,2}\|_{V_1}^2+2\rho_1 l_{1,2}\beta_{1,2}\|x_{1,1}-x_{1,2}\|_{V_1}\|x_{2,1}-x_{2,2}\|_{V_2}\\
&\ +\rho_1^2c_{1,1}^2\beta_{1,1}^2\|x_{1,1}-x_{1,2}\|_{V_1}^2+2\rho_1^2c_{1,1}l_{1,2}\beta_{1,1}\beta_{1,2}\|x_{1,1}-x_{1,2}\|_{V_1}\|x_{2,1}-x_{2,2}\|_{V_2}\\
&\ +\rho_1^2l_{1,2}^2\beta_{1,2}^2\|x_{2,1}-x_{2,2}\|_{V_2}^2\\
&=(1-2\rho_1k_{1,1}+\rho_1^2c_{1,1}^2\beta_{1,1}^2)\|x_{1,1}-x_{1,2}\|_{V_1}^2+\rho_1^2l_{1,2}^2\beta_{1,2}^2\|x_{2,1}-x_{2,2}\|_{V_2}^2\\
&\ +2(\rho_1l_{1,2}\beta_{1,2}+\rho_1^2c_{1,1}l_{1,2}\beta_{1,1}\beta_{1,2})\|x_{1,1}-x_{1,2}\|_{V_1}\|x_{2,1}-x_{2,2}\|_{V_2}\\
&\le \xi(\|x_{1,1}-x_{1,2}\|_{V_1}^2+2\|x_{1,1}-x_{1,2}\|_{V_1}\|x_{2,1}-x_{2,2}\|_{V_2}+\|x_{2,1}-x_{2,2}\|_{V_2}^2)\\
&=\xi(\|x_{1,1}-x_{1,2}\|_{V_1}+\|x_{2,1}-x_{2,2}\|_{V_2})^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\|(x_{1,1}-x_{1,2})-\rho_1(N_1(w_{1,1},w_{2,1})-N_1(w_{1,2},w_{2,2}))\|_{V_1}\\
&\le\sqrt{\xi}(\|x_{1,1}-x_{1,2}\|_{V_1}+\|x_{2,1}-x_{2,2}\|_{V_2}).
\end{aligned} \tag{3.11}$$

Combining (3.7) and (3.11), we obtain

$$\begin{aligned}
\|u_1^1-u_2^1\|_{V_1}&\le\sqrt{\xi}(\|x_{1,1}-x_{1,2}\|_{V_1}+\|x_{2,1}-x_{2,2}\|_{V_2})+(1+\omega_1)\|x_{1,1}-x_{1,2}\|_{V_1}\\
&=(\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{V_1}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{V_2}.
\end{aligned} \tag{3.12}$$

In view of the definition of metric d and (3.12), we have

$$\begin{aligned}
d(u_1^1,A_1(x_{1,2},x_{2,2}))&=\inf_{u_2^1\in A_1(x_{1,2},x_{2,2})}\|u_1^1-u_2^1\|_{V_1}\\
&\le(\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{V_1}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{V_2}.
\end{aligned}$$

Since $u_1^1\in A_1(x_{1,1},x_{2,1})$ is arbitrary, it follows that

$$\sup_{u_1^1\in A_1(x_{1,1},x_{2,1})} d(u_1^1,A_1(x_{1,2},x_{2,2}))\le(\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{V_1}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{V_2}. \tag{3.13}$$

Similarly, we can prove

$$\sup_{u_2^1\in A_1(x_{1,2},x_{2,2})} d(A_1(x_{1,1},x_{2,1}),u_2^1)\le(\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{V_1}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{V_2}. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\mathscr{H}(A_1(x_{1,1},x_{2,1}),A_1(x_{1,2},x_{2,2}))\le(\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{V_1}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{V_2}. \tag{3.15}$$

Let

$$A_2(x_1,x_2)=P_2(x_2-\rho_2N_2(\Phi_1^2(x_1),\Phi_2^2(x_2))-\rho_2q_2)-x_2+G_2(x_2).$$

Since $N_2:V_1\times V_2\to V_2$, $G_2,\Phi_2^2:V_2\to\mathscr{C}(V_2)$ and $\Phi_1^2:V_1\to\mathscr{C}(V_1)$ are continuous mappings, we know that $A_2:V_1\times V_2\to\mathscr{C}(V_2)$ is also a continuous mapping. Similarly to the proof of (3.15), we can show that

$$\mathscr{H}(A_2(x_{1,1},x_{2,1}),A_2(x_{1,2},x_{2,2}))\le(\sqrt{\eta}+\omega_2+1)\|x_{2,1}-x_{2,2}\|_{V_2}+\sqrt{\eta}\|x_{1,1}-x_{1,2}\|_{V_1}. \tag{3.16}$$

Define a mapping $W_0:C(J,V_1\times V_2)\to 2^{C(J,V_1\times V_2)}$ as follows:

$$W_0(x_1,x_2)=(W_0^1(x_1,x_2),W_0^2(x_1,x_2))=(Q_1(x_1,x_2,x_{1,0}),Q_2(x_1,x_2,x_{2,0})),$$

where

$$Q_1(x_1,x_2,x_{1,0})=\{f_1\in C(J,V_1):f_1(t)=x_{1,0}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z_1(s)ds,\ z_1(s)\in S_{A_1,x_1}\},$$

and

$$Q_2(x_1,x_2,x_{2,0}) = \{f_2 \in C(J,V_2) : f_2(t) = x_{2,0} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z_2(s)ds,\ z_2(s) \in S_{A_2,x_2}\}.$$

Then it is easy to see that the solution of system (1.5) is equivalent to the fixed point of the mapping W_0, i.e.,

$$S(x_{1,0},x_{2,0}) = F(W_0) = \{(x_1,x_2) : (x_1,x_2) \in W_0(x_1,x_2)\}.$$

Now we show that W_0 is a set-valued mapping with nonempty and compact values. We first show that $Q_1 : C(J,V_1\times V_2) \to 2^{C(J,V_1)}$ is a set-valued mapping with nonempty and compact values. In fact, for any given $(x_1,x_2) \in C(J,V_1\times V_2)$, since $A_1 : V_1 \times V_2 \to \mathscr{C}(V_1)$ is a continuous mapping, we know that $A_1(x_1(\cdot),x_2(\cdot)) : J \to \mathscr{C}(V_1)$ is continuous and it is also measurable. It follows from the separability of V_1 and Lemma 2.2 that there exists a measurable selection $z_1(s) \in A_1(x_1(s),x_2(s))$ for all $s \in J$. So, $\|z_1(s)\|_{V_1}$ is measurable. Now Lemma 2.3 implies that there exists a constant $k_1 > 0$ such that

$$\sup\{\|z_1(s)\|_{V_1} : z_1(s) \in A_1(x_1(s),x_2(s))\} \le k_1, \quad \forall s \in J.$$

Therefore we know that $\|z_1(s)\|_{V_1}$ is Lebesgue integrable on $J = [0,b]$. In light of Lemma 2.5, we obtain that $z_1(s)$ is Bochner integrable. It follows that $z_1(s) \in S_{A_1,x_1}$. So, $Q_1(x_1,x_2,x_{1,0})$ is nonempty. Moreover, for any $t \in J$, since

$$f_1(t) = x_{1,0} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z_1(s)ds,$$

with $z_1(s) \in S_{A_1,x_1}$, we have

$$\begin{aligned}\|f_1\|_{C(J,V_1)} &= \|x_{1,0} + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-s)^{\alpha-1} z_1(s)ds\|_{C(J,V_1)} \\ &\le \|x_{1,0}\|_{V_1} + \tfrac{k_1}{\Gamma(\alpha)}\max_{t\in J}\textstyle\int_0^t (t-s)^{\alpha-1}ds \\ &= \|x_{1,0}\|_{V_1} + \tfrac{k_1}{\Gamma(\alpha)}\max_{t\in J}\tfrac{t^\alpha}{\alpha} \\ &= \|x_{1,0}\|_{V_1} + \tfrac{k_1 b^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

This implies that $Q_1(x_1,x_2,x_{1,0})$ is uniformly bounded.

Next, we show that $Q_1(x_1,x_2,x_{1,0})$ is equicontinuous. Indeed, let $0 \le t_1 < t_2 \le b$. Since

$$f_1(t_i) = x_{1,0} + \frac{1}{\Gamma(\alpha)}\int_0^{t_i} (t_i-s)^{\alpha-1} z_1(s)ds, \quad i = 1,2,$$

with $z_1(s) \in S_{A_1,x_1}$, one has

$$f_1(t_2) - f_1(t_1) = \frac{1}{\Gamma(\alpha)}\{\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} z_1(s)ds + \int_0^{t_1}((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})z_1(s)ds\},$$

and hence

$$\begin{aligned}\|f_1(t_2) - f_1(t_1)\|_{V_1} &\le \tfrac{k_1}{\Gamma(\alpha)}\{|\textstyle\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds| + |\int_0^{t_1}((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})ds|\} \\ &= \tfrac{k_1}{\Gamma(\alpha+1)}\{(t_2-t_1)^\alpha + |(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha|\} \\ &\le \tfrac{k_1}{\Gamma(\alpha+1)}(2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha) \le \tfrac{2k_1}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\end{aligned}$$

(due to the fact that $(a_1+a_2)^\alpha \le a_1^\alpha + a_2^\alpha$ for all $a_1,a_2 \ge 0$ with $0 < \alpha \le 1$). This shows that $Q_1(x_1,x_2,x_{1,0})$ is equicontinuous. By Arzela-Ascoli Theorem, we deduce that $Q_1(x_1,x_2,x_{1,0})$ is compact.

In a similar way, we can prove that $Q_2 : C(J,V_1 \times V_2) \to 2^{C(J,V_2)}$ is a set-valued mapping with nonempty and compact values and so W_0 is a set-valued mapping with nonempty and compact values. Let

$$W_m(x_1,x_2) = (W_m^1(x_1,x_2), W_m^2(x_1,x_2)) = (Q_1(x_1,x_2,x_{1,0}^{(m)}), Q_2(x_1,x_2,x_{2,0}^{(m)})),$$

where $(x_1,x_2) \in C(J,V_1 \times V_2)$, $m = 1,2,\ldots$. Then

$$W_m(x_1,x_2) = W_0(x_1,x_2) - (x_{1,0},x_{2,0}) + (x_{1,0}^{(m)}, x_{2,0}^{(m)})$$

and $W_m : C(J,V_1 \times V_2) \to \mathscr{C}(C(J,V_1 \times V_2))$ $(m = 0,1,2,\ldots)$ is a set-valued mapping. Let $(x_{1,0}^{(m)}, x_{2,0}^{(m)}) \to (x_{1,0},x_{2,0})$ in $V = V_1 \times V_2$ as $m \to \infty$, that is,

$$\|(x_{1,0}^{(m)}, x_{2,0}^{(m)})) - (x_{1,0},x_{2,0})\|_V = \|x_{1,0}^{(m)} - x_{1,0}\|_{V_1} + \|x_{2,0}^{(m)} - x_{2,0}\|_{V_2} \to 0, \quad \text{as } m \to \infty.$$

Then

$$\mathscr{H}(W_m(x_1,x_2), W_0(x_1,x_2)) \to 0, \quad \text{as } m \to \infty. \tag{3.17}$$

Next, we prove that $W_m : C(J,V_1 \times V_2) \to \mathscr{C}(C(J,V_1 \times V_2))$, $m = 0,1,2,\ldots$, are set-valued contraction mappings. Indeed, for any $(x_{1,1},x_{2,1}),(x_{1,2},x_{2,2}) \in C(J,V_1 \times V_2)$ and any $e_1^1 \in W_m^1(x_{1,1},x_{2,1})$, there exists $r_1^1 \in S_{A_1,x_1}$ such that

$$e_1^1(t) = x_{1,0}^{(m)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r_1^1(s)ds, \quad m = 0,1,2,\ldots.$$

Since $r_1^1(s) \in A_1(x_{1,1}(s),x_{2,1}(s))$ is Bochner integrable, it follows from Lemma 2.5 that $r_1^1(s)$ is measurable. Now Lemma 2.4 implies that there exists a measurable selection $r_2^1(s) \in A_1(x_{1,2}(s), x_{2,2}(s))$ such that, for all $s \in J = [0,b]$,

$$\|r_1^1(s) - r_2^1(s)\|_{V_1} \le \mathscr{H}(A_1(x_{1,1}(s),x_{2,1}(s)), A_1(x_{1,2}(s),x_{2,2}(s))), \tag{3.18}$$

and so $\|r_2^1(s)\|_{V_1}$ $(s \in J)$ is measurable. By Lemma 2.3, we know that $\|r_2^1(s)\|_{V_1}$ $(s \in J)$ is bounded. Hence $\|r_2^1(s)\|_{V_1}$ is Lebesgue integrable on J. Thus, by Lemma 2.5, we obtain that $r_2^1(s)$ is Bochner integrable. Let

$$e_2^1(t) = x_{1,0}^{(m)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r_2^1(s)ds, \quad m = 0,1,2,\ldots.$$

Then $e_2^1 \in W_m^1(x_{1,2},x_{2,2})$. Furthermore, from (3.15) and (3.18), we have

$$\begin{aligned}
&\|e_1^1 - e_2^1\|_{C(J,V_1)} \\
&= \max_{t\in J} \tfrac{1}{\Gamma(\alpha)} \| \textstyle\int_0^t (t-s)^{\alpha-1}(r_1^1(s) - r_2^1(s))ds\|_{V_1} \\
&\le \max_{t\in J} \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \|r_1^1(s) - r_2^1(s)\|_{V_1} ds \\
&\le \max_{t\in J} \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1}((\sqrt{\xi} + \omega_1 + 1)\|x_{1,1}(s) - x_{1,2}(s)\|_{V_1} + \sqrt{\xi}\|x_{2,1}(s) - x_{2,2}(s)\|_{V_2})ds \\
&\le \max_{t\in J} \tfrac{1}{\Gamma(\alpha)} ((\sqrt{\xi} + \omega_1 + 1)\|x_{1,1} - x_{1,2}\|_{C([0,t],V_1)} + \sqrt{\xi}\|x_{2,1} - x_{2,2}\|_{C([0,t],V_2)}) \textstyle\int_0^t (t-s)^{\alpha-1} ds \\
&\le \tfrac{b^\alpha}{\Gamma(\alpha+1)} ((\sqrt{\xi} + \omega_1 + 1)\|x_{1,1} - x_{1,2}\|_{C(J,V_1)} + \sqrt{\xi}\|x_{2,1} - x_{2,2}\|_{C(J,V_2)}).
\end{aligned}$$

It follows that

$$\begin{aligned}&\sup_{e_1^1\in W_m^1(x_{1,1},x_{2,1})} d(e_1^1,W_m^1(x_{1,2},x_{2,2}))\\ &=\sup_{e_1^1\in W_m^1(x_{1,1},x_{2,1})}\ \inf_{e_2^1\in W_m^1(x_{1,2},x_{2,2})}\|e_1^1-e_2^1\|_{C(J,V_1)}\\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{C(J,V_2)}).\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}&\sup_{e_2^1\in W_m^1(x_{1,2},x_{2,2})} d(W_m^1(x_{1,1},x_{2,1}),e_2^1)\\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{C(J,V_2)}).\end{aligned}$$

By the definition of the Hausdorff metric, one has

$$\begin{aligned}&\mathscr{H}(W_m^1(x_{1,1},x_{2,1}),W_m^1(x_{1,2},x_{2,2}))\\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{C(J,V_2)}).\end{aligned}\tag{3.19}$$

By the proof of (3.19), we can show that

$$\begin{aligned}&\mathscr{H}(W_m^2(x_{1,1},x_{2,1}),W_m^2(x_{1,2},x_{2,2}))\\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\eta}+\omega_2+1)\|x_{2,1}-x_{2,2}\|_{C(J,V_2)}+\sqrt{\eta}\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}).\end{aligned}\tag{3.20}$$

In terms of (3.19) and (3.20), one has

$$\begin{aligned}&\mathscr{H}(W_m(x_{1,1},x_{2,1}),W_m(x_{1,2},x_{2,2}))\\ &=\mathscr{H}(W_m^1(x_{1,1},x_{2,1}),W_m^1(x_{1,2},x_{2,2}))+\mathscr{H}(W_m^2(x_{1,1},x_{2,1}),W_m^2(x_{1,2},x_{2,2}))\\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\xi}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}+\sqrt{\xi}\|x_{2,1}-x_{2,2}\|_{C(J,V_2)})\\ &\quad+\frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\eta}+\omega_2+1)\|x_{2,1}-x_{2,2}\|_{C(J,V_2)}+\sqrt{\eta}\|x_{1,1}-x_{1,2}\|_{C(J,V_1)})\\ &=\frac{b^\alpha}{\Gamma(\alpha+1)}((\sqrt{\xi}+\sqrt{\eta}+\omega_1+1)\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}\\ &\quad+(\sqrt{\xi}+\sqrt{\eta}+\omega_2+1)\|x_{2,1}-x_{2,2}\|_{C(J,V_2)})\\ &\le \theta(\|x_{1,1}-x_{1,2}\|_{C(J,V_1)}+\|x_{2,1}-x_{2,2}\|_{C(J,V_2)}),\end{aligned}\tag{3.21}$$

where $\theta=\max\{\theta_1,\theta_2\}$ with $\theta_i=b^\alpha(\sqrt{\xi}+\sqrt{\eta}+\omega_i+1)/\Gamma(\alpha+1)$ for $i=1,2$. In terms of (3.1) and (3.21), we know that $W_m: C(J,V_1\times V_2)\to\mathscr{C}(C(J,V_1\times V_2))$, $m=0,1,2,\ldots$, are set-valued contractive mappings with the same contractive constant $\theta\in(0,1)$. From Lemma 2.6 it follows that $W_0(x_1,x_2)$ has a fixed point for each given $(x_{1,0},x_{2,0})$. So, $S(x_{1,0},x_{2,0})$ is nonempty for each given $(x_{1,0},x_{2,0})$.

In addition, we show that the solution set of system (1.5) is closed. Indeed, since the solution of system (1.5) is equivalent to the fixed point of the mapping $W_0(x_1,x_2)$, we only need to show that the fixed point set of $W_0(x_1,x_2)$ is closed. Let $\{(x_{1,n},x_{2,n})\}\subseteq F(W_0)$ with $(x_{1,n},x_{2,n})\to(x_{1,0},x_{2,0})$ $(n\to\infty)$ in $V=V_1\times V_2$. Then $(x_{1,n},x_{2,n})\in W_0(x_{1,n},x_{2,n})$. In view of (3.21), we have

$$\mathscr{H}(W_0(x_{1,n},x_{2,n}),W_0(x_{1,0},x_{2,0}))\le\theta\|(x_{1,n},x_{2,n})-(x_{1,0},x_{2,0})\|_{V_1\times V_2}$$

and

$$\begin{aligned}&d((x_{1,0},x_{2,0}),W_0(x_{1,0},x_{2,0}))\\ &\le\|(x_{1,0},x_{2,0})-(x_{1,n},x_{2,n})\|_{V_1\times V_2}\\ &\quad+d((x_{1,n},x_{2,n}),W_0(x_{1,n},x_{2,n}))+\mathscr{H}(W_0(x_{1,n},x_{2,n}),W_0(x_{1,0},x_{2,0}))\\ &\le(1+\theta)\|(x_{1,n},x_{2,n})-(x_{1,0},x_{2,0})\|_{V_1\times V_2}\to 0\quad(n\to\infty).\end{aligned}$$

It follows that $(x_{1,0},x_{2,0})\in F(W_0)$ and so $F(W_0)$ is closed.

Finally, we show that the set of solutions of (1.5) is continuous with respect to the initial value in the sense of the Hausdorff metric. Indeed, from Lemma 2.7, (3.17), and (3.21), we conclude that

$$\mathscr{H}(F(W_m),F(W_0)) \le \frac{1}{1-\theta} \sup_{(x_1,x_2)\in V_1\times V_2} \mathscr{H}(W_m(x_1,x_2),W_0(x_1,x_2)) \to 0, \quad \text{as } m\to\infty.$$

Thus, $F(W_m)\to F(W_0)$ as $m\to\infty$, which implies that $S(x_{1,0}^{(m)},x_{2,0}^{(m)})\to S(x_{1,0},x_{2,0})$ as $m\to\infty$, that is, the set of solutions of system (1.5) is continuous with respect to initial value $(x_{1,0},x_{2,0})$ in the sense of the Hausdorff metric. □

It is remarkable that Theorem 3.1 improves and extends [3, Theorem 3.1] from the fractional-order projective dynamical systems involving set-valued perturbations to generalized fractional-order composite dynamical systems involving set-valued perturbations in the same frameworks of real separable Hilbert spaces. However, [3, Theorem 3.1] improves and extends [16, Theorem 3.1] from the fractional-order projective dynamical systems in finite dimensional spaces to the fractional-order projective dynamical systems involving set-valued perturbations in real separable Hilbert spaces. In addition, [3, Theorem 3.1] can be considered as a generalization of the corresponding results of [4,25] in the sense of fractional derivative. Therefore, Theorem 3.1 also improves and extends the corresponding results of [4,16,25].

4. An Example

In this section, we provide an example to illustrate that all the hypotheses in Theorem 3.1 can be satisfied.

Example 4.1. Suppose that $V_1=V_2=\mathbf{R}$, $q_1=-0.7$, $q_2=2$, $x_{1,0}=-1.2$, $x_{2,0}=3.4$ and $P_1=P_{K_1}$, $P_2=P_{K_2}$ with $K_1=[-1,2]$, $K_2=[3,5]$. We define $N_1(x,y)=2x+y$ and $N_2(x,y)=x+2y$ for all $(x,y)\in V_1\times V_2$. Let

$$G_1(x)=\begin{cases} [0,0.02x], & x\ge 0,\\ 0, & x<0,\end{cases} \quad \text{and} \quad G_2(y)=\begin{cases} 0, & y>0,\\ [0,-0.03y], & y\le 0.\end{cases}$$

Moreover, we define $\Phi_1^1(x)=-0.015\cos x+1.64x$, $\Phi_2^1(y)=0.8\sin y$,

$$\Phi_1^2(x)=-0.3\sin x \quad \text{and} \quad \Phi_2^2(y)=-0.1\arctan y+2.05y,$$

for all $(x,y)\in V_1\times V_2$. Then it is easy to see that

$$\begin{aligned}
&|N_1(x^1,\cdot)-N_1(x^2,\cdot)|\le 2|x^1-x^2|=c_{1,1}|x^1-x^2|,\\
&|N_1(\cdot,y^1)-N_1(\cdot,y^2)|\le |y^1-y^2|=l_{1,2}|y^1-y^2|,\\
&(N_1(\Phi_1^1(x^1),\cdot)-N_1(\Phi_1^1(x^2),\cdot))(x^1-x^2)\ge 3.25(x^1-x^2)^2=k_{1,1}(x^1-x^2)^2,\\
&|\Phi_1^1(x^1)-\Phi_1^1(x^2)|\le 1.655|x^1-x^2|=\beta_{1,1}|x^1-x^2|,\\
&|\Phi_2^1(y^1)-\Phi_2^1(y^2)|\le 0.8|y^1-y^2|=\beta_{1,2}|y^1-y^2|,\\
&|N_2(\cdot,y^1)-N_2(\cdot,y^2)|\le 2|y^1-y^2|=c_{2,2}|y^1-y^2|,\\
&N_2(x^1,\cdot)-N_2(x^2,\cdot)|\le |x^1-x^2|=l_{2,1}|x^1-x^2|,\\
&(N_2(\cdot,\Phi_2^2(y^1))-N_2(\cdot,\Phi_2^2(y^2)))(y^1-y^2)\ge 3.9(y^1-y^2)^2=k_{2,2}(y^1-y^2)^2,\\
&|\Phi_1^2(x^1)-\Phi_1^2(x^2)|\le 0.3|x^1-x^2|=\beta_{2,1}|x^1-x^2|,\\
&|\Phi_2^2(y^1)-\Phi_2^2(y^2)|\le 2.15|y^1-y^2|=\beta_{2,2}|y^1-y^2|,\\
&\mathscr{H}(G_1(x^1),G_1(x^2))\le 0.02|x^1-x^2|=\omega_1|x^1-x^2|,\\
&\mathscr{H}(G_2(y^1),G_2(y^2))\le 0.03|y^1-y^2|=\omega_2|y^1-y^2|,
\end{aligned}$$

for all $x^1,x^2\in V_1, y^1,y^2\in V_2$. If $\rho_1=0.16$ and $\rho_2=0.2$, then

$$\begin{aligned}\xi &= \max\{1-2\rho_1 k_{1,1}+\rho_1^2 c_{1,1}^2\beta_{1,1}^2, \rho_1 l_{1,2}\beta_{1,2}+\rho_1^2 c_{1,1} l_{1,2}\beta_{1,1}\beta_{1,2}, \rho_1^2 l_{1,2}^2\beta_{1,2}^2\}\\ &= \max\{0.2405, 0.1958, 0.0164\},\\ &= 0.2405\end{aligned}$$

and

$$\begin{aligned}\eta &= \max\{1-2\rho_2 k_{2,2}+\rho_2^2 c_{2,2}^2\beta_{2,2}^2, \rho_2 l_{2,1}\beta_{2,1}+\rho_2^2 c_{2,2} l_{2,1}\beta_{2,2}\beta_{2,1}, \rho_2^2 l_{2,1}^2\beta_{2,1}^2\}\\ &= \max\{0.1796, 0.1116, 0.0036\}\\ &= 0.1796.\end{aligned}$$

Moreover, if $b=0.35,\ \alpha=0.8$, then

$$\begin{aligned}\theta_1 &= \tfrac{b^\alpha}{\Gamma(\alpha+1)}(\sqrt{\xi}+\sqrt{\eta}+\omega_1+1)\\ &= \tfrac{0.4318}{0.9314}(0.4904+0.4238+0.02+1)\\ &= 0.4636\times 1.9342 = 0.8967 < 1\end{aligned}$$

and

$$\begin{aligned}\theta_2 &= \tfrac{b^\alpha}{\Gamma(\alpha+1)}(\sqrt{\xi}+\sqrt{\eta}+\omega_2+1)\\ &= \tfrac{0.4318}{0.9314}(0.4904+0.4238+0.03+1)\\ &= 0.4636\times 1.9442 = 0.9013 < 1.\end{aligned}$$

Therefore, it is easy to see that (3.1) holds and so the solution set of this system is nonempty, closed, and continuous with respect to initial value $(x_{1,0}, x_{2,0})$ in the sense of the Hausdorff metric.

5. The Concluding Remark

In this paper, we studied, for the first time, the generalized fractional composite dynamical system with set-valued perturbations (i.e., (1.5)) in real separable Hilbert spaces. It is worth mentioning that if $P_i = I$ the identity mapping of V_i for $i=1,2$, then (1.5) reduces to the system of two related generalized fractional differential inclusions with set-valued perturbations in abstract spaces. Model (1.5) captures the desired features of both composite dynamical systems and generalized fractional differential inclusions with set-valued perturbations within the same frameworks. Therefore, the generalized fractional composite dynamical system with set-valued perturbations is important and interesting in theory and practice. And also it is necessary to study the property of the solution set. Based on the fixed point theorem for set-valued contractive mappings, selection theorem, etc., we established a sufficient condition for the nonemptiness and closedness of the solutions set of system (1.5). Furthermore, we showed that the set of solutions is continuous with respect to initial value in the sense of the Hausdorff metric. In addition, a numerical example was provided to illustrate the main results obtained in this paper. Compared with the existing model of fractional projective dynamical systems with set-valued perturbations, model (1.5) is more general and more advantageous since it captures the desired features of both composite dynamical systems and generalized fractional differential inclusions with set-valued perturbations within the same framework. It is worth to point out that Riemann-Liouville type fractional derivative has nice and useful mathematical properties. It was known in [26] that, under zeros initial conditions, the equations with Riemann-Liouville operators are equivalent to those with Caputo operators. So Theorem 3.1 holds under Riemann-Liouville type fractional derivative with zeros initial conditions. In general, does the Theorem 3.1 still hold under Riemann-Liouville type fractional derivative? We will consider such a problem in our future work.

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