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STABILITY ON PARAMETRIC STRONG SYMMETRIC QUASI-EQUILIBRIUM PROBLEMS VIA NONLINEAR SCALARIZATION

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Abstract. This paper focuses on the stability analysis of a class of parametric strong symmetric quasi-equilibrium problems (PSSQEP) via scalarization approaches. Based on the oriented distance function, a new nonlinear scalarization function which can separate sets $\{0\}$ and $-C\backslash\{0\}$ is presented. By virtue of the scalarization function, a new form of scalar parametric strong symmetric quasi-equilibrium problem (PSSQEP)$_\psi$ is obtained, and the union relation between the solution set of (PSSQEP) and the solution sets of a series of (PSSQEP)$_\psi$ is established. Finally, the sufficient conditions of the Berge-semicontinuity of solution mappings for (PSSQEP) are obtained via the union relation and the nonlinear scalarization technique, which is different from the scalarization method recently announced. Some interesting examples are given to illustrate the main results.

Keywords. Berge-semicontinuity; Nonlinear scalarization; Oriented distance function; Stability; Strong symmetric quasi-equilibrium problem.

1. Introduction

The stability analysis of solutions, as an important topic in optimization theory and applications, is to study the changes in the behavior of solutions with respect to changes in the data of the problems. It includes Berge-semicontinuity, Hausdorff-semicontinuity, connectedness, Painlevé-Kuratowski convergence, well-posedness and so on; see, e.g., [1, 2, 3] and the references therein. Among them, the Berge-semicontinuity plays an indispensable role, and it was widely used in the theoretical analysis and algorithm design. Recently, numerous results were devoted to this subject in optimization problems and related problems. In 2002, Li, Chen and Teo [4] discussed the Berge-upper/lower semicontinuity of solution mappings for generalized vector quasivariational inequality problems by using some suitable continuity assumptions. Without of the monotonicity assumptions, Khanh and Luu [5] studied the Berge-upper semicontinuity of solution mappings for parametric vector quasivariational inequalities. Based on the concept of convergence, Chen and Gong [6] obtained the Berge-upper semicontinuity of solution mappings for symmetric vector quasi-equilibrium problems. Recently, Li

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and Chen [7] discussed the Berge-lower semicontinuity of solution mappings for weak vector variational inequalities by using the gap function. In 2014, by using the density result of solutions and the linear scalarization method, Peng and Peng [8] obtained the Berge-upper semicontinuity of strong solution mappings for parametric generalized systems. In 2015, Gong [9] obtained some sufficient conditions of the Berge-lower semicontinuity of solution mappings for semi-infinite vector optimization problems with the perturbations of both the constraint set and the objective function under some continuity assumptions. In 2018, under weaker assumptions, Peng et al. [10] and Fan et al. [3] investigated the stability analysis (Berge-upper/lower semicontinuity, etc.) of weak efficient solutions and essential solutions for semi-infinite vector optimization problems, and generalized semi-infinite optimization problems, respectively. Applying the approach of Li and Chen [7], Wang et al. [11] derived sufficient and necessary conditions of Berge-upper/lower semicontinuity of solution mappings for generalized vector quasi-equilibrium problems via free-disposal set.

On the other hand, the symmetric quasi-equilibrium problem (SQEP) is an important part in vector optimization. The problem (SQEP) was first introduced and studied by Noor and Oettli [12] in 1994, which includes as special cases, such as variational inequalities, optimization problems, fixed pointed problems, saddle point problems and so on. As a generalization of the equilibrium problem proposed by Blum and Oettli [13], it was proved to be more suitable in several practical situations like non-cooperative games and multi-objective optimization. Thus the study of the problem (SQEP) is interesting and meaningful. In recent years, the research on symmetric equilibrium problem mainly aims to the following two aspects: a) the existence of the solutions (see, e.g., [14, 15, 16, 17, 18] and the references therein); b) the stability of solutions, including well-posedness, connectedness, and Berge-semicontinuity (see, e.g., [6, 19, 20, 21, 22] and the reference therein). It is noticeable that there are few results concerning with the stability of the solution mapping to (SQEP) with the parameter perturbation. In the papers mentioned above, we observe that most of results are applicable for the ordering relations respect to a closed, convex and pointed cone C with int$C \neq \emptyset$. However, in some cases, the ordering cone of vector optimization problems is a neither open nor closed set (that is, a closed and convex cone with the origin removed $C\backslash\{0\}$ in strong vector equilibrium problems) or the ordering cone has an empty interior (for example in l^2), which makes the study more difficult and more challenging. This arouses our interest in the parametric strong symmetric quasi-equilibrium problem (PSSQEP). Up to now, to the best of our knowledge, there is no paper dealing with the stability of solutions for (PSSQEP). In addition, we find the linear scalarization method is the major approach to study the vector optimization problem with non-open and non-closed ordering set in most papers (such as [8, 17, 23] and the reference therein). But, the linear scalarization method can be used only when the topological dual (Y^*) of the image space (Y) is nonempty. Thus it is natural to raise the following questions:

a) Whether the sufficient condition of the stability analysis for (PSSQEP) can be obtained, which involves a closed and convex cone with the origin removed?

b) Is there a new approach that can be used to study the stability of optimization and related problems with non-open and non-closed ordering set, and overcome the flaws of the linear scalarization method?

This paper aims to solve the two problems above. The structure of the paper is as follows. In Section 2, we introduce the model of (PSSQEP), and recall some concepts which will be used in the sequel. In Section 3, by a new nonlinear scalarization function, we first establish a nonlinear scalarization problem $(\text{PSSQEP})_\psi$, and discuss the relationships between $(\text{PSSQEP})_\psi$ and (PSSQEP). Then, under some suitable continuity assumptions, we discuss the Berge-upper/lower semicontinuity of solution mappings for (PSSQEP) by using the new nonlinear scalarization method. Moreover, some interesting examples are given to illustrate the results.

2. Preliminaries

Throughout this paper, unless specified otherwise, let X, Y, Λ, and M be real normed vector spaces. We denote the norm by $\|\cdot\|$ in any real normed vector space. $A \subset X$ and $B \subset Y$ are two nonempty compact convex subsets. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $C := \mathbb{R}^n_+$. Obviously, C is a closed, convex, and pointed cone of $\mathbb{R}^n$. We assume that $f : A \times B \to \mathbb{R}^n$ and $g : A \times B \to \mathbb{R}^n$ are two vector-valued mappings, and $H : A \times B \times \Lambda \rightrightarrows A$ and $T : A \times B \times M \rightrightarrows B$ are two set-valued mappings with closed valued. Let clD, D^c, $intD$, ∂D, and $B(x,\delta)$ denote the closure, the complement, the topological interior, the boundary of a set $D \subseteq \mathbb{R}^n$, and an open ball with $x \in \mathbb{R}^n$ as center and $\delta > 0$ as radius, respectively. Let $d(x,y) :=\| x - y \|$ denote the distance between the point x and y in a real normed space. Let $d_D(x) := \inf_{y \in D} \| x - y \|$ denote the distance between the point x and the set D, and $d_\emptyset(x) = +\infty$.

In this paper, we consider the following parametric strong symmetric quasi-equilibrium problem (in short, PSSQEP) consisting of, for $(\lambda,u) \in \Lambda \times M$, finding $(\bar{x},\bar{y}) \in A \times B$ such that $(\bar{x},\bar{y}) \in H(\bar{x},\bar{y},\lambda) \times T(\bar{x},\bar{y},u)$ and

$$(PSSQEP) \begin{cases} f(x,\bar{y}) - f(\bar{x},\bar{y}) \notin -C \backslash \{0\}, & \forall x \in H(\bar{x},\bar{y},\lambda), \\ g(\bar{x},y) - g(\bar{x},\bar{y}) \notin -C \backslash \{0\}, & \forall y \in T(\bar{x},\bar{y},u). \end{cases}$$

For each $(\lambda,u) \in \Lambda \times M$, let $E(\lambda,u) := \{(\bar{x},\bar{y}) \in A \times B | (\bar{x},\bar{y}) \in H(\bar{x},\bar{y},\lambda) \times T(\bar{x},\bar{y},u)\}$. We denote the solution set of (PSSQEP) by $S(\lambda,u)$ corresponding to parameters λ and u, i.e.,

$$\begin{aligned} S(\lambda,u) := \{(\bar{x},\bar{y}) \in E(\lambda,u) | & f(x,\bar{y}) - f(\bar{x},\bar{y}) \notin -C\backslash\{0\}, \ \forall x \in H(\bar{x},\bar{y},\lambda), \\ & g(\bar{x},y) - g(\bar{x},\bar{y}) \notin -C\backslash\{0\}, \ \forall y \in T(\bar{x},\bar{y},u)\}. \end{aligned}$$

Next, we recall some basic definitions and lemmas, which needed in the sequel.

Definition 2.1. [24] Let K be a convex cone in a normed space X. Assume that $f : A \subseteq X \to \mathbb{R}$ is a given function on A. If, for each $x_1, x_2 \in A$, $x_1 - x_2 \in K \Rightarrow f(x_1) \geq f(x_2)$ (or $x_1 - x_2 \in K\backslash\{0_X\} \Rightarrow f(x_1) > f(x_2)$; $x_1 - x_2 \in intK \Rightarrow f(x_1) > f(x_2)$), then f is said to be K-monotone (or strongly K-monotone; strictly K-monotone) on A.

Remark 2.1. Clearly, strong K-monotonicity is stronger than K-monotonicity and strictly K-monotonicity . If f is a strong K-monotone function on A, then f is both K-monotone and strictly K-monotone on A.

Definition 2.2. [19, 25] Let X and Y be topological vector spaces, and let $F : X \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be Berge-upper semicontinuous (B-u.s.c, for short) at $x_0 \in X$, if for any open set V with $F(x_0) \subset V$, there exists a neighborhood U of x_0 in X such that $F(x) \subset V$ for all $x \in U$;

(ii) F is said to be Berge-lower semicontinuous (B-l.s.c, for short) at $x_0 \in X$ if, for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

(iii) F is said to be Berge-continuous at $x_0 \in X$ if it is both B-l.s.c and B-u.s.c at $x_0 \in X$. F is said to be B-l.s.c (resp. B-u.s.c) on X if it is B-l.s.c (resp. B-u.s.c) at each $x \in X$;

(iv) F is closed if Graph(F) is a closed set in $X \times Y$. F has compact (resp. closed) values if $F(x)$ is a compact (resp. closed) set for each $x \in X$.

In [25], Aubin and Ekeland also gave the following properties for B-u.s.c/B-l.s.c.

Lemma 2.1. *Let X and Y be topological vector spaces, and let $F : X \rightrightarrows Y$ be a set-valued mapping.*

(i) *F is B-l.s.c at $x_0 \in X$ if and only if, for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in F(x_0)$, there exists $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y_0$.*

(ii) *If F has compact values (i.e., $F(x)$ is a compact set for each $x \in X$), then F is B-u.s.c at x_0 if and only if, for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Definition 2.3. [26] Let X and Y be two topological vector spaces. Let $K \subset Y$ be a closed convex pointed cone and let E be a nonempty subset of X. Let g be a mapping from E to Y. g is said to be $K-$lower semicontinuous (resp. K-upper semicontinuous) at $x_0 \in E$ if, for any neighborhood V of 0_Y in Y, there exists a neighborhood U of x_0 such that, for each $x \in U \cap E$, $g(x) \in g(x_0) + V + K$ (*resp.* $g(x) \in g(x_0) + V - K$). g is said to be K-lower semicontinuous (resp. K-upper semicontinuous) on E if g is K-lower semicontinuous (resp. K-upper semicontinuous) at every point of E. g is said to be K-semicontinuous at $x_0 \in E$ if it is both K-upper semicontinuous and K-lower semicontinuous at $x_0 \in E$.

The oriented distance function $(\triangle)$, which is now widely used in nonlinear scalarization field, was introduced by Hiriart-Urruty [27] as follows.

Definition 2.4. [27, 28] Let W be a normed space. For a set $Q \subset W$ and any $y \in W$, let the function $\triangle_Q : W \to \mathbb{R} \cup \{\pm\infty\}$ be defined as $\triangle_Q(y) = d_Q(y) - d_{W\setminus Q}(y)$, where $d_Q(y) := \inf_{x \in Q} \| x - y \|$, $d_\emptyset(y) = +\infty$.

Lemma 2.2. [28] *If the set $Q \subset W$ is a nonempty set and $Q \neq W$, then*

(i) *$\triangle_Q$ is continuous with real valued;*

(ii) *$\triangle_Q(y) < 0$ for every $y \in intQ$, $\triangle_Q(y) = 0$ for every $y \in \partial Q$, and $\triangle_Q(y) > 0$ for every $y \in int\ Q^c$;*

(iii) *if Q is closed, then $Q = \{y : \triangle_Q(y) \leq 0\}$;*

(iv) *$\triangle_Q$ is 1-Lipschitzian;*

(v) *if Q is a closed and convex cone, then $-\Delta_Q$ is Q-monotone and strictly Q-monotone.*

Based on the oriented distance function $(\triangle)$, for any $\alpha \in \mathbb{R}_{++}$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, we now establish a new nonlinear scalarization function $\psi : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ as $\psi(y, \alpha) = \alpha\Delta_{-C}(y) + \Sigma_{i=1}^n y_i$.

Now, we can obtain some properties of the nonlinear scalarization function ψ as follows.

Lemma 2.3. *For any $\alpha \in \mathbb{R}_{++}$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, let $\psi : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ be defined as $\psi(y, \alpha) = \alpha\Delta_{-C}(y) + \Sigma_{i=1}^n y_i$. Then*

(i) *for each* $\alpha \in \mathbb{R}_{++}$, $\psi(\cdot,\alpha)$ *is continuous with real valued;*
(ii) *for each* $\alpha \in \mathbb{R}_{++}$, $\psi(\cdot,\alpha)$ *is* $(1+\alpha)$*-Lipschitzian with respect to* $||\cdot||_1$*;*
(iii) *if* $y \in -C\backslash\{0\}$, *then* $\psi(y,\alpha) < 0$ *for each* $\alpha \in \mathbb{R}_{++}$*;*
(iv) *if* $y \notin -C\backslash\{0\}$, *then there exists* $\alpha_0 \in \mathbb{R}_{++}$ *such that* $\psi(y,\alpha_0) \geq 0$*;*
(v) $\psi(\cdot,\alpha)$ *is strong C-monotone on* $\mathbb{R}^n$ *for each* $\alpha \in \mathbb{R}_{++}$.

Proof. In the light of Lemma 2.2, statements (i), (iii) and (iv) can be obtained directly. Next, we prove that (ii) and (v) hold. From Lemma 2.2 (iv), $\triangle_{-C}$ is 1-Lipschitzian. Then, for each $\alpha \in \mathbb{R}_{++}$, it follows that, for any $y, y' \in \mathbb{R}^n$,

$$\begin{aligned}|\psi(y,\alpha)-\psi(y',\alpha)| &= |\alpha\Delta_{-C}(y)+\Sigma_{i=1}^n y_i - \alpha\Delta_{-C}(y') - \Sigma_{i=1}^n y_i'| \\ &\leq \alpha|\Delta_{-C}(y)-\Delta_{-C}(y')| + |\Sigma_{i=1}^n y_i - \Sigma_{i=1}^n y_i'| \\ &\leq \alpha||y-y'||_1 + |\Sigma_{i=1}^n y_i - \Sigma_{i=1}^n y_i'| \\ &\leq (1+\alpha)||y-y'||_1.\end{aligned}$$

Thus, for each $\alpha \in \mathbb{R}_{++}$, $\psi(\cdot,\alpha)$ is $(1+\alpha)$-Lipschitzian with respect to $||\cdot||_1$. At the same time, from Lemma 2.2 (v), one can obtain that, for each $\alpha \in \mathbb{R}_{++}$ and $a, b \in \mathbb{R}^n$ with $a-b \in C\backslash\{0\}$,

$$\psi(a,\alpha)-\psi(b,\alpha) = \alpha(\Delta_{-C}(a)-\Delta_{-C}(b)) + \Sigma_{i=1}^n (a_i - b_i) > 0.$$

Therefore, $\psi(\cdot,\alpha)$ is strong C-monotone on $\mathbb{R}^n$ for each $\alpha \in \mathbb{R}_{++}$. Also, form Remark 2.1, it is C-monotone and strictly C-monotone. This completes the proof. □

Lemma 2.4. *If* $f : A \times B \to \mathbb{R}^n$ *is C-semicontinuous mapping on* $A \times B$, *then, for each* $\alpha \in \mathbb{R}_{++}$, *the composite mapping* $\psi(f(\cdot,\cdot),\alpha)$ *is continuous on* $A \times B$.

Proof. Let $(x_0,y_0) \in A \times B$. As f is C-semicontinuous at (x_0,y_0), for any neighborhood V of the origin 0 in $\mathbb{R}^n$, there exists a neighborhood U_0 of (x_0,y_0) such that, for all $(x,y) \in U_0$, $f(x,y) \in f(x_0,y_0)+V+C$ and $-f(x,y) \in -f(x_0,y_0)+V+C$. Thus, combining the arbitrariness of V and the continuity of $\psi(\cdot,\alpha)$, for each $\alpha \in \mathbb{R}_{++}$ and any $\varepsilon > 0$, there exist a small enough neighborhood V_1 of the origin 0 in $\mathbb{R}^n$, $z_1^* \in f(x_0,y_0)+V_1$, and $z_2^* \in f(x_0,y_0)-V_1$ such that $f(x,y) \in z_1^*+C$, $f(x,y) \in z_2^*-C$. Then one can find that $|\psi(z_1^*,\alpha)-\psi(f(x_0,y_0),\alpha)| < \varepsilon$ and $|\psi(z_2^*,\alpha)-\psi(f(x_0,y_0),\alpha)| < \varepsilon$. Hence

$$-\varepsilon+\psi(f(x_0,y_0),\alpha) < \psi(z_1^*,\alpha) < \varepsilon+\psi(f(x_0,y_0),\alpha), \tag{2.1}$$

and

$$-\varepsilon+\psi(f(x_0,y_0),\alpha) < \psi(z_2^*,\alpha) < \varepsilon+\psi(f(x_0,y_0),\alpha). \tag{2.2}$$

Invoking Lemma 2.3 (v), one sees that $\psi(\cdot,\alpha)$ is strong C-monotone on $\mathbb{R}^n$ for each $\alpha \in \mathbb{R}_{++}$. Then, $\psi(\cdot,\alpha)$ is both C-monotone and strictly C-monotone on $\mathbb{R}^n$. It follows that, for each $\alpha \in \mathbb{R}_{++}$,

$$\psi(f(x,y),\alpha)-\psi(z_1^*,\alpha) \geq 0, \tag{2.3}$$

and

$$\psi(f(x,y),\alpha)-\psi(z_2^*,\alpha) \leq 0. \tag{2.4}$$

From (2.1) (2.2), (2.3), and (2.4), we can obtain that, for all $(x,y) \in U_0$, $|\psi(f(x,y),\alpha)-\psi(f(x_0,y_0),\alpha)| < \varepsilon$. Thus, combining the arbitrariness of (x_0,y_0), we conclude that the composite mapping $\psi(f(\cdot,\cdot),\alpha)$ is continuous on $A \times B$. The proof is complete. □

3. MAIN RESULTS

In light of the existence theorems of efficient solutions for the symmetric vector set-valued quasi-equilibrium problems in [17], the existence results of solutions for (PSSQEP) can be obtained with some suitable modifications. In this section, we mainly discuss the sufficient conditions of Berge-upper/lower semicontinuity of solution mappings for (PSSQEP) by using the new nonlinear scalarization technique. We always assume that the solution sets considered in this section are nonempty.

For each $(\lambda,u,\alpha)\in\Lambda\times M\times\mathbb{R}_{++}$, we consider the following nonlinear scalar parametric symmetric quasi-equilibrium problem (PSSQEP)$_\psi$: find $(\bar{x},\bar{y})\in A\times B$ such that $(\bar{x},\bar{y})\in H(\bar{x},\bar{y},\lambda)\times T(\bar{x},\bar{y},u)$ and

$$(PSSQEP)_\psi \begin{cases}\psi(f(x,\bar{y})-f(\bar{x},\bar{y}),\alpha)\geq 0, & \forall x\in H(\bar{x},\bar{y},\lambda),\\ \psi(g(\bar{x},y)-g(\bar{x},\bar{y}),\alpha)\geq 0, & \forall y\in T(\bar{x},\bar{y},u).\end{cases}$$

We denote the solution set of (PSSQEP)$_\psi$ by $\Gamma(\lambda,u,\alpha)$, i.e.,

$$\Gamma(\lambda,u,\alpha):=\{(\bar{x},\bar{y})\in E(\lambda,u)|\psi(f(x,\bar{y})-f(\bar{x},\bar{y}),\alpha)\geq 0,\ \forall x\in H(\bar{x},\bar{y},\lambda),\\ \psi(g(\bar{x},y)-g(\bar{x},\bar{y}),\alpha)\geq 0,\ \forall y\in T(\bar{x},\bar{y},u)\}.$$

Before discussing the relationship between the solution sets $S(\lambda,u)$ and $\Gamma(\lambda,u,\alpha)$, we give the following result.

Theorem 3.1. *Let $K\subset\mathbb{R}^n$ $(n\in N)$ be a nonempty compact set with* $\Delta_{-C}(z)=||z||_1$ (or $\Delta_{-C}(z)=||z||_2$, $\Delta_{-C}(z)=||z||_\infty$) for all $z\in(B(0,\delta)\cap K)\setminus -C$, where $\delta>0$ and $K\backslash B(0,\delta)\neq\emptyset$. Then, $K\cap(-C\setminus\{0\})=\emptyset$ if and only if there exists $\alpha_0\in\mathbb{R}_{++}$, such that $\psi(y,\alpha_0)\geq 0$, $\forall y\in K$.

Proof. $(\Leftarrow)$ Suppose now there exists $\alpha_0\in\mathbb{R}_{++}$ such that $\psi(y,\alpha_0)\geq 0,\forall y\in K$. We prove $K\cap(-C\setminus\{0\})=\emptyset$. Assume by contradiction that this is not the case. Without loss of generality, assume that there exists $y_0\in K$ with $y_0\in -C\setminus\{0\}$. Combining Lemma 2.3 (iii), for each $\alpha\in\mathbb{R}_{++}$, one has $\psi(y_0,\alpha_0)<0$, which contradicts the fact that $\psi(y,\alpha_0)\geq 0,\forall y\in K$. Thus the sufficiency is obvious.

$(\Rightarrow)$ Now, we prove that if $K\cap(-C\setminus\{0\})=\emptyset$, then there exists $\alpha_0\in\mathbb{R}_{++}$ such that $\psi(y,\alpha_0)\geq 0$, $\forall y\in K$. Indeed, as K is compact, the set $K\setminus B(0,\delta)$ is compact. Let $M=max_{y\in K\backslash B(0,\delta)}\Sigma_{i=1}^n|y_i|$ and $m=min_{y\in K\backslash B(0,\delta)}\triangle_{-C}(y)=min_{y\in K\backslash B(0,\delta)}d_{-C}(y)$. For any $y\in K$ with $K\cap(-C\setminus\{0\})=\emptyset$, we consider the following two cases.

Case 1 If $y\in K\cap B(0,\delta)$ when $B(0,\delta)\cap K\neq\emptyset$, then, for $\alpha_1=n\in\mathbb{R}_{++}$,

$$\psi(y,\alpha_1)=\alpha_1\triangle_{-C}(y)+\Sigma_{i=1}^n y_i=n||y||_1(or\ n||y||_2,n||y||_\infty)+\Sigma_{i=1}^n y_i\geq 0.$$

Case 2 If $y\in K\setminus B(0,\delta)$, then we set $\alpha_2=\frac{M}{m}\in\mathbb{R}_{++}$. Thus

$$\psi(y,\alpha_2)=\alpha_2\triangle_{-C}(y)+\Sigma_{i=1}^n y_i=\frac{M}{m}\triangle_{-C}(y)+\Sigma_{i=1}^n y_i\geq\frac{M}{m}\cdot m-M=0.$$

Therefore, it follows from the two cases above that there exists $\alpha_0=max\{n,\frac{M}{m}\}\in\mathbb{R}_{++}$ such that $\psi(y,\alpha_0)\geq 0$ for each $y\in K$. The proof is complete. □

Remark 3.1. Combining Lemma 2.3 (iii) and Theorem 3.1, it is easy to obtain that $-C\setminus\{0\}$ and a nonconvex compact set K can be separated by the nonlinear scalarization function ψ.

Lemma 3.1. *Let f and g be C-semicontinuous on $A\times B$ with compact values for each $(\lambda,u,x,y)\in\Gamma\times M\times A\times B$. Let $K_1 := f(H(x,y,\lambda),y)-f(x,y)$ and $K_2 := g(x,T(x,y,u))-g(x,y)$ satisfy the assumptions of Theorem 3.1 (like K in Theorem 3.1). Then, for each $(\lambda,u)\in\Lambda\times M$, the following assertion is valid: $S(\lambda,u)=\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda,u,\alpha)$, where $f(H(x,y,\lambda),y)-f(x,y)=\cup_{z_1\in H(x,y,\lambda)}(f(z_1,y)-f(x,y))$ and $g(x,T(x,y,u))-g(x,y)=\cup_{z_2\in T(x,y,u)}(g(x,z_2)-g(x,y))$.*

Proof. "($\supset$)" Let $(\bar{x},\bar{y})\in\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda,u,\alpha)$. Then there exists $\alpha_0\in\mathbb{R}_{++}$ such that $(\bar{x},\bar{y})\in\Gamma(\lambda,u,\alpha_0)$. It follows that $\bar{x}\in H(\bar{x},\bar{y},\lambda)$, $\bar{y}\in T(\bar{x},\bar{y},u)$ and

$$\begin{cases}\psi(f(x,\bar{y})-f(\bar{x},\bar{y}),\alpha_0)\geq 0, & \forall x\in H(\bar{x},\bar{y},\lambda),\\ \psi(g(\bar{x},y)-g(\bar{x},\bar{y}),\alpha_0)\geq 0, & \forall y\in T(\bar{x},\bar{y},u).\end{cases}\tag{3.1}$$

If $(\bar{x},\bar{y})\notin S(\lambda,u)$, it means that there exists $x_0\in H(\bar{x},\bar{y},\lambda)$ such that $f(x_0,\bar{y})-f(\bar{x},\bar{y})\in -C\backslash\{0\}$, or there exists $y_0\in T(\bar{x},\bar{y},u)$ such that $g(\bar{x},y_0)-g(\bar{x},\bar{y})\in -C\backslash\{0\}$. Thus by virtue of Lemma 2.3 (iii), one has $\psi(f(x_0,\bar{y})-f(\bar{x},\bar{y}),\alpha_0)<0$, or $\psi(g(\bar{x},y_0)-g(\bar{x},\bar{y}),\alpha_0)<0$, which contradicts (3.1). Therefore, $(\bar{x},\bar{y})\in S(\lambda,u)$.

"($\subset$)" Now, we prove the other inclusion. If $(\bar{x},\bar{y})\in S(\lambda,u)$, then $\bar{x}\in H(\bar{x},\bar{y},\lambda)$, $\bar{y}\in T(\bar{x},\bar{y},u)$, and

$$\begin{cases}f(x,\bar{y})-f(\bar{x},\bar{y})\notin -C\backslash\{0\}, & \forall x\in H(\bar{x},\bar{y},\lambda),\\ g(\bar{x},y)-g(\bar{x},\bar{y})\notin -C\backslash\{0\}, & \forall y\in T(\bar{x},\bar{y},u).\end{cases}$$

As f,g are C-semicontinuous on $A\times B$ with compact values, $f(A,B):=\cup_{x\in A,y\in B}f(x,y)$ and $g(A,B):=\cup_{x\in A,y\in B}g(x,y)$ are two compact sets. In the light of Lemma 2.3 (iv) and Theorem 3.1, it is easy to obtain that there exists $\alpha_1\in\mathbb{R}_{++}$ such that $\psi(f(x,\bar{y})-f(\bar{x},\bar{y}),\alpha_1)\geq 0$, $\forall x\in H(\bar{x},\bar{y},\lambda)$, where $\alpha_1\geq T_1$, $T_1=\max\{n,\frac{M_1}{m_1}\}$, $M_1=\max_{z'\in K_1\backslash B(0,\delta)}||z'||_1$, and $m_1=\min_{z'\in K_1\backslash B(0,\delta)}d_{-C}(z')$. And there exists $\alpha_2\in\mathbb{R}_{++}$, such that $\psi(g(\bar{x},y)-g(\bar{x},\bar{y}),\alpha_2)\geq 0$, $\forall y\in T(\bar{x},\bar{y},u)$, where $\alpha_2\geq T_2$, $T_2=\max\{n,\frac{M_2}{m_2}\}$, $M_2=\max_{z''\in K_2\backslash B(0,\delta)}||z''||_1$ and $m_2=\min_{z''\in K_2\backslash B(0,\delta)} d_{-C}(z'')$. It follows that there exists $\alpha_3\in\mathbb{R}_{++}$ with $\alpha_3=max\{\alpha_1,\alpha_2\}$ such that

$$\begin{cases}\psi(f(x,\bar{y})-f(\bar{x},\bar{y}),\alpha_3)\geq 0, & \forall x\in H(\bar{x},\bar{y},\lambda),\\ \psi(g(\bar{x},y)-g(\bar{x},\bar{y}),\alpha_3)\geq 0, & \forall y\in T(\bar{x},\bar{y},u).\end{cases}$$

Thus $S(\lambda,u)\subseteq\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda,u,\alpha)$. The proof is complete. □

Next, we discuss the Berge-upper semicontinuity of solution mappings for $(\text{PSSQEP})_\psi$.

Theorem 3.2. *Assume that*

(i) *$H: A\times B\times\Lambda\rightrightarrows A$ is Berge-continuous with compact values on $A\times B\times\Lambda$;*

(ii) *$T: A\times B\times M\rightrightarrows B$ is Berge-continuous with compact values on $A\times B\times M$;*

(iii) *$f,g: A\times B\to\mathbb{R}^n$ are C-semicontinuous.*

Then, for each $\alpha\in\mathbb{R}_{++}$, $\Gamma(\cdot,\cdot,\alpha)$ is Berge-upper semicontinuous with compact values on $\Lambda\times M$.

Proof. On the contrary, suppose that there exist $\alpha_0\in\mathbb{R}_{++}$ and $(\lambda_0,u_0)\in\Lambda\times M$ such that $\Gamma(\cdot,\cdot,\alpha_0)$ is not Berge-upper semicontinuous at (λ_0,u_0). Then there exist an open set U_0 satisfying $\Gamma(\lambda_0,u_0,\alpha_0)\subset U_0$ and a sequence $\{(\lambda_n,u_n)\}$ with $(\lambda_n,u_n)\to(\lambda_0,u_0)$ such that $\Gamma(\lambda_n,u_n,\alpha_0)$

is not the subset of U_0. It means that there exists $(x_n,y_n)\in\Gamma(\lambda_n,u_n,\alpha_0)$, i.e., $(x_n,y_n)\in H(x_n,y_n,\lambda_n)$ $\times T(x_n,y_n,u_n)$ and

$$\begin{cases} \psi(f(x,y_n)-f(x_n,y_n),\alpha_0)\geq 0,\ \forall x\in H(x_n,y_n,\lambda_n), \\ \psi(g(x_n,y)-g(x_n,y_n),\alpha_0)\geq 0,\ \forall y\in T(x_n,y_n,u_n), \end{cases} \tag{3.2}$$

but $(x_n,y_n)\notin U_0$. Since H and T are Berge-upper semicontinuous with compact values, one sees that there exists $(x_0,y_0)\in H(x_0,y_0,\lambda_0)\times T(x_0,y_0,u_0)$ such that $(x_n,y_n)\to(x_0,y_0)$.

Now, we claim that $(x_0,y_0)\in\Gamma(\lambda_0,u_0,\alpha_0)$. Indeed, if not, there exists $\bar{x}\in H(x_0,y_0,\lambda_0)$ such that $\psi(f(\bar{x},y_0)-f(x_0,y_0),\alpha_0)<0$, or there exists $\bar{y}\in T(x_0,y_0,u_0)$ such that $\psi(g(x_0,\bar{y})-g(x_0,y_0),\alpha_0)<0$. By the Berge-lower semicontinuity of H, T and $(x_n,y_n,\lambda_n,u_n)\to(x_0,y_0,\lambda_0,u_0)$, for $\bar{x}\in H(x_0,y_0,\lambda_0)$ and $\bar{y}\in T(x_0,y_0,u_0)$, there exist $\bar{x}_n\in H(x_n,y_n,\lambda_n)$ and $\bar{y}_n\in T(x_n,y_n,u_n)$ with $\bar{x}_n\to\bar{x}$ and $\bar{y}_n\to\bar{y}$, respectively. As $\psi(\cdot,\alpha_0)$ is continuous, and $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are C-semicontinuous, it follows from Lemma 2.4 that there exists $n_0\in\mathbb{N}$ such that, for any $n\geq n_0$, $\psi(f(\bar{x}_n,y_n)-f(x_n,y_n),\alpha_0)<0$, or there exists $n_1\in\mathbb{N}$ such that, for any $n\geq n_1$, $\psi(g(x_n,\bar{y}_n)-g(x_n,y_n),\alpha_0)<0$, which contradicts (3.2). So, $(x_0,y_0)\in\Gamma(\lambda_0,u_0,\alpha_0)$. Moreover, as $(x_n,y_n)\to(x_0,y_0)$ and $(x_0,y_0)\in\Gamma(\lambda_0,u_0,\alpha_0)\subset U_0$, one has $(x_n,y_n)\in U_0$ for n large enough, which contradicts the assumption $(x_n,y_n)\notin U_0$. Therefore, for each $\alpha\in\mathbb{R}_{++}$, $\Gamma(\cdot,\cdot,\alpha)$ is Berge-upper semicontinuous on $\Lambda\times M$.

Next, we prove that, for each $(\lambda,u,\alpha)\in\Lambda\times M\times\mathbb{R}_{++}$, $\Gamma(\lambda,u,\alpha)$ is compact. As A and B are two compact sets, it means that we only need to prove that $\Gamma(\lambda,u,\alpha)$ is closed. Let $\{(\widehat{x}_n,\widehat{y}_n)\}\subset$ $\Gamma(\lambda,u,\alpha)$ with $(\widehat{x}_n,\widehat{y}_n)\to(\widehat{x}_0,\widehat{y}_0)$. By the assumptions (i) and (ii), one has $(\widehat{x}_0,\widehat{y}_0)\in H(\widehat{x}_0,\widehat{y}_0,\lambda_0)$ $\times T(\widehat{x}_0,\widehat{y}_0,u)$. By using the same method as before, one can easily obtain that $(\widehat{x}_0,\widehat{y}_0)\in\Gamma(\lambda,u,\alpha)$. Thus $\Gamma(\lambda,u,\alpha)$ is closed at each $(\lambda,u,\alpha)\in\Lambda\times M\times\mathbb{R}_{++}$. The proof is complete. □

From Theorem 3.3 and Lemma 3.1, the sufficient condition of the Berge-upper semicontinuity for $S(\cdot,\cdot)$ are obtained as follows.

Theorem 3.3. (B-u.s.c for solution mappings for (PSSQEP)) *Suppose that the assumptions of Lemma 3.1 hold and*

(i) $H: A\times B\times\Lambda\rightrightarrows A$ *is Berge-continuous with compact values on* $A\times B\times\Lambda$;

(ii) $T: A\times B\times M\rightrightarrows B$ *is Berge-continuous with compact values on* $A\times B\times M$.

Then, $S(\cdot,\cdot)$ *is Berge-upper semicontinuous on* $\Lambda\times M$.

Proof. If there exists $(\lambda_0,u_0)\in\Lambda\times M$ such that $S(\cdot,\cdot)$ is not B-u.s.c at (λ_0,u_0), then there exist an open set U satisfying $S(\lambda_0,u_0)\subset U$ and a sequence $\{(\lambda_n,u_n)\}$ with $(\lambda_n,u_n)\to(\lambda_0,u_0)$ such that there exists $(x_n,y_n)\in S(\lambda_n,u_n)$ but $(x_n,y_n)\notin U$. Thus $(x_n,y_n)\in H(x_n,y_n,\lambda_n)\times T(x_n,y_n,u_n)$ and

$$\begin{cases} f(x,y_n)-f(x_n,y_n)\notin -C\backslash\{0\},\ \forall x\in H(x_n,y_n,\lambda_n), \\ g(x_n,y)-g(x_n,y_n)\notin -C\backslash\{0\},\ \forall y\in T(x_n,y_n,u_n). \end{cases}$$

For any open neighborhood B_1, where B_1 is an open neighborhood of $0_{X\times Y}$ in $X\times Y$, there exists a balanced open neighborhood B_0 of $0_{X\times Y}$ such that $B_0+B_0\subset B_1$. From Lemma 3.1, for each $(\lambda,u)\in\Lambda\times M$, we have $S(\lambda,u)=\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda,u,\alpha)$. And since $(x_n,y_n)\in S(\lambda_n,u_n)\subset$ $\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda_n,u_n,\alpha)$, one has $((x_n,y_n)+B_1)\bigcap(\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda_n,u_n,\alpha))\neq\emptyset$. As the compactness of $\Gamma(\lambda_n,u_n,\alpha)$ and $\alpha\in\mathbb{R}_{++}$, one finds that there exists $(x_n^{'},y_n^{'})\in\bigcup_{\alpha\in\mathbb{R}_{++}}\Gamma(\lambda_n,u_n,\alpha)$ such that

$(x_n', y_n') - (x_n, y_n) \in B_0$. Thus there exists $\alpha_0 \in \mathbb{R}_{++}$ such that $(x_n', y_n') \in \Gamma(\lambda_n, u_n, \alpha_0)$. By Theorem 3.3, $\Gamma(\cdot, \cdot, \alpha_0)$ is Berge-upper semicontinuous with compact values at (λ_0, u_0). Then there exist $(x_0, y_0) \in \Gamma(\lambda_0, u_0, \alpha_0)$ and a subsequence $\{(x_{n_k}', y_{n_k}')\}$ of $\{(x_n', y_n')\}$ such that $(x_{n_k}', y_{n_k}') \to (x_0, y_0)$. Thus there exists $n_0 \in \mathbb{N}$, for any $k \geq n_0$, $(x_{n_k}', y_{n_k}') - (x_0, y_0) \in B_0$. It means that, for any $k \geq n_0$, $(x_{n_k}, y_{n_k}) - (x_0, y_0) = (x_{n_k}, y_{n_k}) - (x_{n_k}', y_{n_k}') + (x_{n_k}', y_{n_k}') - (x_0, y_0) \in B_1$. By the arbitrariness of B_1, we obtain that $(x_{n_k}, y_{n_k}) \to (x_0, y_0)$. As $(x_0, y_0) \in \Gamma(\lambda_0, u_0, \alpha_0) \subseteq S(\lambda_0, u_0) \subset U$, one has that $(x_{n_k}, y_{n_k}) \in U$ for k large enough, which contradicts the assumption $(x_{n_k}, y_{n_k}) \notin U$. Therefore, $S(\cdot, \cdot)$ is Berge-upper semicontinuous on $\Lambda \times M$. This completes the proof. □

Remark 3.2. (i) Theorem 3.1 obtains the sufficient condition of the Berge-upper semicontinuity of solution mappings for (PSSQEP), to the best of our knowledge, which is the first result about the Berge-upper semicontinuity for strong symmetric quasi-equilibrium problems where the ordering sets are neither closed nor open ($C \backslash \{0\}$).

(ii) By using the nonlinear scalarization function ξ, Gutiérrez et al. [29] discussed the vector equilibrium problems and obtained some results for weak efficient solutions. Indeed, the well known nonlinear scalarization function ξ (see [21, 30, 31]) and the oriented distance function Δ (see [11, 27]) are limited and not applicable for efficient solutions to (PSSQEP). Theorems 3.3 and 3.1 overcome the limitations of ξ and Δ, and provide a new technique to study the efficient solutions to (PSSQEP) and related problems with the ordering set $C \backslash \{0\}$.

The following example is presented to illustrate the result when X and Y are infinite dimensional spaces.

Example 3.1. Let $X = Y = l^\infty = \{z = (z_1, z_2, \ldots, z_i, \ldots)^T : \sup_{i \geq 1} |z_i| < \infty\}$, $\Lambda = [-5, 6]$, $M = [3, 7]$, and $C = \mathbb{R}_+$. Let $A = \{z = (z_1, z_2, \ldots, z_i, \ldots)^T \in l^\infty : z_1 \geq -2, z_i \geq 0, i = 2, 3, \ldots\}$ and $B = \{z = (z_1, z_2, \ldots, z_i, \cdots)^T \in l^\infty : z_i \geq 0, i = 1, 2, \ldots\}$. Assume that $H(x, y, \lambda) = \{z = (z_1, z_2, \ldots, z_i, \ldots)^T \in l^\infty : z_i \geq |\lambda|, i = 1, 2, \ldots\}$ and $T(x, y, u) = \{z = (z_1, z_2, \ldots, z_i, \ldots)^T \in l^\infty : z_1 \geq u, z_i \geq 2, i = 2, 3, \ldots\}$, where $(x, y, \lambda, u) \in A \times B \times \Lambda \times M$. Let $f : A \times B \to \mathbb{R}$ and $g : A \times B \to \mathbb{R}$ be defined by $f(x, y) = x^T y$, $\forall (x, y) \in A \times B$, and $g(x, y) = 2(x_1 + 1, x_2 + 1, \ldots, x_i + 1, \ldots) y$, $\forall (x, y) \in A \times B$. Consider (PSSQEP). It is easy to check that all the assumptions of Theorem 3.1 are satisfied. After the simply computation, for each $(\lambda, u) \in \Lambda \times M$, one has $S(\lambda, u) = (|\lambda|, |\lambda|, \ldots, |\lambda|, \ldots)^T \times (u, 2, \ldots, 2, \ldots)^T$. Obviously, $S(\cdot, \cdot)$ is Berge-upper semicontinuous on $\Lambda \times M$. Therefore, Theorem 3.1 is applicable.

Inspired by the key hypothesis of gap functions (see, e.g., [7, 11, 31]), we introduce a new assumption by means of the nonlinear scalarization function ψ.

(Hg) : Given $(\overline{\lambda}, \overline{u}) \in \Lambda \times M$, for any $\varepsilon > 0$, there exist $\beta > 0$ and $\delta > 0$ such that, for any $(\lambda, u) \in U((\overline{\lambda}, \overline{u}), \delta)$, $\alpha \in \mathbb{R}_{++}$ and $(x, y) \in E(\lambda, u) \backslash U(S(\lambda, u), \varepsilon)$, $\min_{x' \in H(x,y,\lambda)} \psi(f(x', y) - f(x, y), \alpha) \leq -\beta$ or $\min_{y' \in T(x,y,u)} \psi(g(x, y') - g(x, y), \alpha) \leq -\beta$.

We give the following example to illustrate the existence of the assumption (Hg).

Example 3.2. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [-6, 4]$, and $M = [-2, 3]$. Let $A = [-8, 9]$ and $B = [-4, 6]$. Let $H(x, y, \lambda) = [\lambda, 7]$ and $T(x, y, u) = [u, 5]$, where $(x, y, \lambda, u) \in A \times B \times \Lambda \times M$. Let $f : A \times B \to \mathbb{R}$ and $g : A \times B \to \mathbb{R}$ be defined as $f(x, y) = 3x + 2$, $\forall (x, y) \in A \times B$, and $g(x, y) = x + 3y$, $\forall (x, y) \in A \times B$, respectively. By a simple computation, we can obtain that, for each $(\lambda, u) \in \Lambda \times M$, $S(\lambda, u) = (\lambda, u)$ and $E(\lambda, u) = [\lambda, 7] \times [u, 5]$. Given $(\overline{\lambda}, \overline{u}) \in \Lambda \times M$, for any $\varepsilon > 0$, there exist $\beta = \frac{\varepsilon}{2} > 0$ and $\delta = \frac{\varepsilon}{2} > 0$ such that, for any $(\lambda, u) \in U((\overline{\lambda}, \overline{u}), \delta)$, $\alpha \in \mathbb{R}_{++}$, and $(x,y) \in E(\lambda,u) \backslash U(S(\lambda,u),\varepsilon)$, $\min_{x' \in H(x,y,\lambda)} \psi(f(x',y) - f(x,y),\alpha) \leq -\frac{\varepsilon}{2}$ and $\min_{y' \in T(x,y,u)} \psi(g(x,y') - g(x,y),\alpha) \leq -\frac{\varepsilon}{2}$. It is easy to illustrate that the assumption (Hg) holds.

Theorem 3.4. (B-l.s.c of solution mappings for (PSSQEP)) *Suppose that the assumptions of Lemma 3.1 and* (Hg) *hold, and*

(i) $E(\cdot,\cdot)$ *is Berge-continuous with compact values on* $\Lambda \times M$*;*
(ii) $T(\cdot,\cdot,\cdot)$ *is Berge-upper semicontinuous with compact values on* $A \times B \times M$*;*
(iii) $H(\cdot,\cdot,\cdot)$ *is Berge-upper semicontinuous with compact values on* $A \times B \times \Lambda$*.*

Then, $S(\cdot,\cdot)$ *is Berge-lower semicontinuous on* $\Lambda \times M$*.*

Proof. Suppose to the contrary that there exists $(\lambda_0, u_0) \in \Lambda \times M$ such that $S(\cdot,\cdot)$ is not Berge-lower semicontinuous at $(\lambda_0, u_0) \in \Lambda \times M$. Then, there exist a sequence $\{(\lambda_n, u_n)\} \subset \Lambda \times M$ with $(\lambda_n, u_n) \to (\lambda_0, u_0)$ and $(x_0, y_0) \in S(\lambda_0, u_0)$ such that, for any $(x_n, y_n) \in S(\lambda_n, u_n)$,

$$(x_n, y_n) \nrightarrow (x_0, y_0). \tag{3.3}$$

From assumption (i), $E(\cdot,\cdot)$ is Berge-lower semicontinuous at (λ_0, u_0). For any $(\lambda_k, u_k) \in \Lambda \times M$ with $(\lambda_k, u_k) \to (\lambda_0, u_0)$ and $(x_0, y_0) \in E(\lambda_0, u_0)$, there exists $(\overline{x}_k, \overline{y}_k) \in E(\lambda_k, u_k)$ such that

$$(\overline{x}_k, \overline{y}_k) \to (x_0, y_0), \tag{3.4}$$

for k large enough. Next, we claim that for any $\varepsilon > 0$, $(\overline{x}_k, \overline{y}_k) \notin U(S(\lambda_k, u_k), \varepsilon)$. Otherwise, there exists $(x_k', y_k') \in S(\lambda_k, u_k)$ such that $||(\overline{x}_k, \overline{y}_k) - (x_k', y_k')|| < \varepsilon$. This together with (3.4) implies that $||(x_k', y_k') - (x_0, y_0)|| < 2\varepsilon$, as $k \to \infty$. By the arbitrariness of ε, it follows that $(x_k', y_k') \to (x_0, y_0)$, $k \to \infty$, which contradicts (3.3). Therefore, $(\overline{x}_k, \overline{y}_k) \notin U(S(\lambda_k, u_k), \varepsilon)$ for k large enough. As $(x_0, y_0) \in S(\lambda_0, u_0)$, it follows from Lemma 3.1 that there exists $\alpha_0 \in \mathbb{R}_{++}$ such that

$$\begin{cases} x_0 \in H(x_0, y_0, \lambda_0), & \psi(f(x_0', y_0) - f(x_0, y_0), \alpha_0) \geq 0, \forall x_0' \in H(x_0, y_0, \lambda_0), \\ y_0 \in T(x_0, y_0, u_0), & \psi(g(x_0, y_0') - g(x_0, y_0), \alpha_0) \geq 0, \forall y_0' \in T(x_0, y_0, u_0). \end{cases} \tag{3.5}$$

By the assumption (Hg) and $(\overline{x}_k, \overline{y}_k) \notin U(S(\lambda_k, u_k), \varepsilon)$, there exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\min_{x' \in H(\overline{x}_k, \overline{y}_k, \lambda_k)} \psi(f(x', \overline{y}_k) - f(\overline{x}_k, \overline{y}_k), \alpha_0) \leq -\beta,$$

or

$$\min_{y' \in T(\overline{x}_k, \overline{y}_k, u_k)} \psi(g(\overline{x}_k, y') - g(\overline{x}_k, \overline{y}_k), \alpha_0) \leq -\beta.$$

It means that there exist $x_k \in H(\overline{x}_k, \overline{y}_k, \lambda_k)$ or $y_k \in T(\overline{x}_k, \overline{y}_k, u_k)$ such that $\psi(f(x_k, \overline{y}_k) - f(\overline{x}_k, \overline{y}_k), \alpha_0) \leq -\beta$, $\forall k \geq k_0$, or $\psi(g(\overline{x}_k, y_k) - g(\overline{x}_k, \overline{y}_k), \alpha_0) \leq -\beta$, $\forall k \geq k_0$, respectively. Since $H(\cdot,\cdot,\cdot)$ and $T(\cdot,\cdot,\cdot)$ are Berge-upper semicontinuous with compact values, $(\overline{x}_k, \overline{y}_k, \lambda_k, u_k) \to (x_0, y_0, \lambda_0, u_0)$, for $x_k \in H(\overline{x}_k, \overline{y}_k, \lambda_k)$ and $y_k \in T(\overline{x}_k, \overline{y}_k, u_k)$, without loss of generality, there exist $x_0^* \in H(x_0, y_0, \lambda_0)$ and $y_0^* \in T(x_0, y_0, u_0)$ such that $x_k \to x_0^*$ and $y_k \to y_0^*$. In the light of Lemma 2.4, it follows that

$$\psi(f(x_0^*, y_0) - f(x_0, y_0), \alpha_0) \leq -\beta < 0,$$

or

$$\psi(g(x_0, y_0^*) - g(x_0, y_0), \alpha_0) \leq -\beta < 0,$$

which contradicts (3.5). Thus $S(\cdot,\cdot)$ is Berge-lower semicontinuous on $\Lambda\times M$. The proof is complete. □

Remark 3.3. (i) It is known that the linear scalarization method is the major approach to study the stability of solutions for the ordering relations with respect to $C\backslash\{0\}$. Theorem 3.3 and Theorem 3.4 obtain the sufficient conditions of the Berge-upper/lower semicontinuity for solution mappings of (PSSQEP) by a new nonlinear scalarization method, which is obviously different from the linear scalarization method in [8, 23] and the references therein, and the method can be also used when the topological dual of the image space is empty.

(ii) On the ordering relation with respect to $C\backslash\{0\}$, the obtained results also work in the space $\mathbb{R}^1$, while the corresponding results of [21, 31] (see, [21, Theorem 3.9] and [31, Theorem 3.2]) may not applicable.

Next, we give an example to illustrate Theorem 3.4.

Example 3.3. Let $X=Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, $\Lambda=[-1,4]$, and $M=[-3,8]$. Let $A=[-5,+\infty)\times[-2,+\infty)$, $B=[-7,+\infty)\times[-5,+\infty)$. For each $(x,y,\lambda,u)\in X\times Y\times\Lambda\times M$, let $H(x,y,\lambda)=[\lambda,+\infty)\times[\lambda,+\infty)$ and $T(x,y,u)=[u,+\infty)\times[u,+\infty)$. Define $f:A\times B\to\mathbb{R}^2$ and $g:A\times B\to\mathbb{R}^2$ as $f(x,y)=2(x_1^2+x_2^2,0)-(cosx_1+cosx_2,0)$, and $g(x,y)=3(0,y_1^2+y_2^2)+(0,(siny_1)^2+(siny_2)^2)$, where $x=(x_1,x_2)\in A,\ y=(y_1,y_2)\in B$ (see the following figures).

With a simple calculation, one can obtain that, for each $(\lambda,u)\in\Lambda\times M$,

$$S(\lambda,u)=\begin{cases}(0,0)\times(0,0), & \lambda\le 0,u\le 0,\\ (\lambda,\lambda)\times(0,0), & \lambda>0,u\le 0,\\ (0,0)\times(u,u), & \lambda\le 0,u>0,\\ (\lambda,\lambda)\times(u,u), & \lambda>0,u>0,\end{cases}$$

and $E(\lambda,u)=\{[\lambda,+\infty)\times[\lambda,+\infty)\}\times[u,+\infty)\times[u,+\infty)$. Given $(\overline{\lambda},\overline{u})\in\Lambda\times M$, for any $\varepsilon>0$, there exist $\beta=\varepsilon^2>0$ and $\delta=\varepsilon>0$ such that, for any $(\lambda,u)\in U((\overline{\lambda},\overline{u}),\delta)$, $\alpha\in\mathbb{R}_{++}$, and $(x,y)\in E(\lambda,u)\backslash U(S(\lambda,u),\varepsilon)$, $\min_{x'\in H(x,y,\lambda)}\psi(f(x',y)-f(x,y),\alpha)\le-\varepsilon^2$ and $\min_{y'\in T(x,y,u)}\psi(g(x,y')-g(x,y),\alpha)\le-\varepsilon^2$. It shows that assumption (Hg) holds and all conditions of the theorem are satisfied. By virtue of the value of $S(\lambda,u)$, we conclude that $S(\cdot,\cdot)$ is Berge-lower semicontinuous on $\Lambda\times M$. Therefore, Theorem 3.4 is applicable.

Finally, the following example is given to show that the assumption (Hg) in Theorem 3.4 is essential.

Example 3.4. Let $X=Y=\mathbb{R}$, $\Lambda=[-1,0]$, $M=[0,1]$, $C=\mathbb{R}_+$, and $A=B=[-1,-1]$. Let $H(x,y,\lambda)=[-1,\lambda]$ and $T(x,y,u)=[u,1]$, where $(x,y,\lambda,u)\in A\times B\times\Lambda\times M$. Let $f:A\times B\to\mathbb{R}$ and $g:A\times B\to\mathbb{R}$ be defined by $f(x,y)=x^2+x^3$, $\forall(x,y)\in A\times B$, and $g(x,y)=y^2-y^3$, $\forall(x,y)\in A\times B$. It follows from the direct computations that $E(\lambda,u)=[-1,\lambda]\times[u,1]$ and

$$S(\lambda,u)=\begin{cases}\{-1,0\}\times\{0,1\}, & \lambda=0,u=0,\\ \{-1\}\times\{0,1\}, & \lambda\ne 0,u=0,\\ \{-1,0\}\times\{1\}, & \lambda=0,u\ne 0,\\ \{-1\}\times\{1\}. & \lambda\ne 0,u\ne 0\end{cases}$$

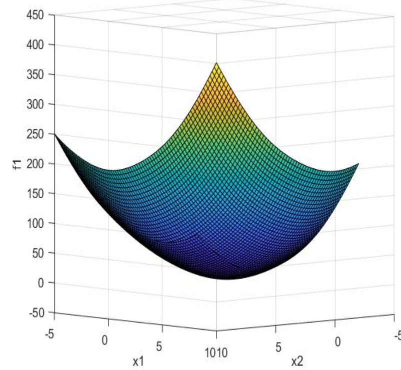

FIGURE 1.
The projection of f(x,y) on the first-dimension coordinate

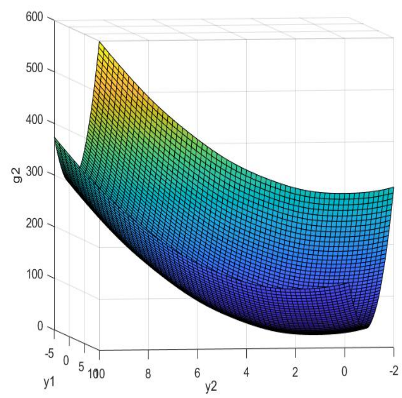

FIGURE 2.
The projection of g(x,y) on the second-dimension coordinate

By virtue of the assumption (Hg), for given $(\lambda,u)=(0,0)\in\Lambda\times M$ and any $\varepsilon>0$, $\beta>0$ and $\delta>0$, take $\{(\lambda_k,u_k)\}$, $(\lambda_k,u_k)\to(0,0)$ with $0<|\lambda_k|<2\beta$, $0<u_k<2\beta$, and $(x_k,y_k)=(\lambda_k,u_k)\in E(\lambda_k,u_k)\backslash U((\lambda_k,u_k),\varepsilon)$. We have $\min_{x'\in H(x_k,y_k,\lambda_k)}\psi(f(x',y_k)-f(x_k,y_k),\alpha)=-(\lambda_k^2+\lambda_k^3)>-\beta$ and $\min_{y'\in T(x_k,y_k,u_k)}\psi(g(x_k,y')-g(x_k,y_k),\alpha)=-(u_k^2+u_k^3)>-\beta$. Hence, (Hg) does not hold at $(0,0)$. Assumptions (i), (ii) and (iii) in Theorem 3.4 are satisfied, but $S(\lambda,u)$ is not Berge-lower semicontinuous on $\Lambda\times M$. It shows that the assumption (Hg) in Theorem 3.4 is essential.

4. CONCLUSIONS

In this paper, we considered a parametric strong symmetric quasi-equilibrium problem (PSS QEP) in a finite dimensional space. Based on the oriented distance function, we gave a new nonlinear scalarization function ψ. By using the nonlinear scalarization method, the Berge-semicontinuity of solution mappings for (PSSQEP) was obtained. This may be the first result on the Berge-semicontinuity of solutions for (PSSQEP), where the ordering set is neither closed nor open set $C\backslash\{0\}$.

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