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SOME QUALITATIVE PROPERTIES OF SOLUTIONS OF HIGHER-ORDER LOWER SEMICONTINUS DIFFERENTIAL INCLUSIONS

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Abstract. Let $n,k \in \mathbf{N}$, $T > 0$, and $F : [0,T] \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be a lower semicontinuos and bounded multifunction with nonempty closed values. We prove that there exists a bounded and upper semicontinuous multifunction $G : \mathbf{R} \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ with nonempty compact convex values such that every generalized solution $u : [0,T] \to \mathbf{R}^n$ of the differential inclusion $u^{(k)} \in G(t,u,u',\dots,u^{(k-1)})$ is a generalized solution to the differential inclusion $u^{(k)} \in F(t,u,u',\dots,u^{(k-1)})$. As an application, we prove an existence and qualitative result for the generalized solutions of the Cauchy problem associated to the inclusion $u^{(k)} \in F(t,u,u',\dots,u^{(k-1)})$. In particular, we prove that if F is lower semicontinuous and bounded with nonempty closed values, then the solution multifunction admits an upper semicontinuous multivalued selection with nonempty compact connected values. Finally, by applying the latter result, we prove an analogous existence and qualitative result for the generalized solutions of the Cauchy problem associated to the differential equation $g(u^{(k)}) = f(t,u,u',\dots,u^{(k-1)})$, where f is continuous. We only assume that g is continuous and locally nonconstant.

Keywords. Cauchy problem; Differential inclusions, Differential equations; Generalized solutions, Selections.

1. Introduction

Let $n,k \in \mathbf{N}$, $T > 0$, and $p \in [1,+\infty]$. As usual, we denote by $W^{k,p}([0,T],\mathbf{R}^n)$ the space of all functions $u \in C^{k-1}([0,T],\mathbf{R}^n)$ such that $u^{(k-1)}$ is absolutely continuous in $[0,T]$ and $u^{(k)} \in L^p([0,T],\mathbf{R}^n)$. Let $F : [0,T] \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be a given multifunction. It is known that a generalized solution to the differential inclusion

$$u^{(k)} \in F(t,u,u',\dots,u^{(k-1)})$$

in $[0,T]$ is a function $u \in W^{k,1}([0,T],\mathbf{R}^n)$ such that

$$u^{(k)}(t) \in F(t,u(t),u'(t),\dots,,u^{(k-1)}(t)) \quad \text{for a.e.} \quad t \in [0,T]. \tag{1.1}$$

If the multifunction F is bounded, then every generalized solution of (1.1) in $[0,T]$ belongs to $W^{k,\infty}([0,T],\mathbf{R}^n)$.

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The study of the differential inclusion (1.1) is of fundamental importance in the study of several problems arising in several different areas of mathematics (for an introduction to the theory of differential inclusions, we refer to the classical books [1, 2]). As explicitly remarked in [3], differential inclusion (1.1) was studied under two different and separate kind of assumptions, that is,

(i) F is upper semicontinuous with nonempty compact convex values;
(ii) F is lower semicontinuous (or continuous) with nonempty compact values.

In most literatures, the results and the techniques available for these two classes of differential inclusions are substantially different. In [3], Bressan proved an equivalence result for first-order differential inclusions, which allowed to treat this two kinds of differential inclusions in a unified way. The following is Bressan's result.

Theorem 1.1. (The Theorem at p.22 of [3]). *Let $F : \mathbf{R} \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be a bounded and lower semicontinuous multifunction with nonempty compact values. Then there exists an upper semicontinuous multifunction $G : \mathbf{R} \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ with nonempty compact convex values such that every generalized solution $u : [\alpha,\beta] \to \mathbf{R}^n$ of the differential inclusion $u' \in G(t,u)$ in $[\alpha,\beta]$ (with $[\alpha,\beta]$ real compact interval) is also a generalized solution to $u' \in F(t,u)$ in $[\alpha,\beta]$.*

In substance, Theorem 1.1 allowed to apply to first-order lower semicontinuous differential inclusions, and many results are valid for the upper semicontinuous convex-valued case. In particular, it is possible to obtain new existence and qualitative results for the first-order Cauchy problem associated with a lower semicontinuous multifunction F (see, e.g., [3, Theorem 4.2]).

The aim of this paper is to extend Theorem 1.1 to the higher-order differential inclusion (1.1). Such an extension is not trivial, and requires a different technical construction. In particular, we remark that the original proof of Theorem 1.1 is based on an existence result for directionally continuous selections. Our proof, conversely, is based on a recent existence result for Riemann-measurable selections (that is, the selections which are a.e. continuous), recently proved in [4]. As an application, we prove an existence and qualitative result for the generalized solutions of the Cauchy problem

$$\begin{cases} u^{(k)} \in F(t,u,u',\dots,u^{(k-1)}) & \text{in } [0,T], \\ u^{(i)}(0) = \xi_i, \qquad i = 0,1,\dots,k-1 \end{cases} \tag{1.2}$$

(where $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$ is a given vector), associated to a lower semicontinuous multifunction F. Among the others, we give the conditions under which the multifunction

$$\xi \in (\mathbf{R}^n)^k \to \left\{ u \in W^{k,1}([0,T],\mathbf{R}^n) : u \text{ is a generalized solution of (1.2)} \right\}$$

admits an upper semicontinuous (with respect to a suitable topology) multivalued selection with nonempty compact connected values.

Finally, applying the above results, we prove an existence and qualitative result for the generalized solutions of the Cauchy problem

$$\begin{cases} g(u^{(k)}) = f(t,u,u',\dots,u^{(k-1)}) & \text{in } [0,T] \\ u^{(i)}(0) = \xi_i, \qquad i = 0,1,\dots,k-1, \end{cases} \tag{1.3}$$

where $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$ is a given vector. In particular, we give the sufficient conditions under which the solution multifunction

$$\xi \in (\mathbf{R}^n)^k \to \mathscr{S}(\xi) := \{u \in W^{k,1}([0,T], \mathbf{R}^n) : u \text{ is a generalized solution of (1.3)}\}$$

admits an upper semicontinuous multivalued selection with nonempty compact connected values. As a matter of fact, we only require the continuity of f and g, and that g is locally nonconstant.

2. PRELIMINARIES

Let $n, k \in \mathbf{N}$. If $[a,b]$ is compact interval, we consider the space $W^{k,\infty}([a,b], \mathbf{R}^n)$ with the initial topology $\tau_{n,k}^{[a,b]}$ that makes the function

$$u \in W^{k,\infty}([a,b], \mathbf{R}^n) \to (u, u^{(k)}) \in C^{k-1}([a,b], \mathbf{R}^n) \times L^\infty([a,b], \mathbf{R}^n)$$

continuous, where the space $C^{k-1}([a,b], \mathbf{R}^n)$ is considered with its strong topology, and the space $L^\infty([a,b], \mathbf{R}^n)$ with its weak-star topology.

In the following, we often make the obvious identification $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$. For all $j = 0, 1, \ldots, nk$, we denote by $\Pi_j : \mathbf{R} \times \mathbf{R}^{nk} \to \mathbf{R}$ the projection over the j-th axis. That is, if $(t,x) = (t, x_1, x_2, \ldots, x_{nk}) \in \mathbf{R} \times \mathbf{R}^{nk}$, we put

$$\Pi_j(t,x) = \begin{cases} t & \text{if } j = 0, \\ x_j & \text{if } j \in \{1, 2, \ldots, nk\}. \end{cases}$$

For every $j \in \mathbf{N}$, we denote by m_j the j-dimensional Lebesgue measure in $\mathbf{R}^j$. Moreover, we denote by $\mathscr{B}$ and $\mathscr{L}([a,b])$, respectively, the Borel family of $\mathbf{R}$ and the family of all Lebesgue measurable subsets of the interval $[a,b]$.

If $m \in \mathbf{N}$, $x \in \mathbf{R}^m$, and $r > 0$, we denote by $B_m(x,r)$ (resp., $\overline{B}_m(x,r)$) the open (resp., closed) ball in $\mathbf{R}^m$, centered in x with radius r with respect to the Euclidean norm $\|\cdot\|_m$ of $\mathbf{R}^m$. Moreover, if $A \subseteq \mathbf{R}^m$, we denote by $\overline{\text{conv}}(A)$ the closed convex hull of the set A.

The following result, which concerns some qualitative properties of first-order upper semicontinuous differential inclusions, summarizes some results proved in [1, pp.103-109], and will be a central tool in the sequel.

Theorem 2.1. *Let $x^* \in \mathbf{R}^n$, and let $\Omega \subseteq \mathbf{R} \times \mathbf{R}^n$ be an open set such that $(0, x^*) \in \Omega$. Let $G : \Omega \to 2^{\mathbf{R}^n}$ be an upper semicontinuous multifunction with nonempty compact convex values. Assume that there exist $M > 0$, $b > 0$, and $T > 0$ such that*

$$Q := [0,T] \times \overline{B}_n(x^*, b + MT) \subseteq \Omega \quad \text{and} \quad G(Q) \subseteq \overline{B}_n(0, M).$$

Then,

(i) *for every $\xi \in B_n(x^*, b)$, the solution set*

$$\mathscr{S}_{[0,T]}^G(\xi) := \{u \in W^{1,1}([0,T], \mathbf{R}^n) : u(0) = \xi \text{ and } u'(t) \in G(t, u(t)) \text{ a.e. in } [0,T]\}$$

is nonempty. Moreover, the multifunction $\xi \to \mathscr{S}_{[0,T]}^G(\xi)$ is upper semicontinuous from $B_n(x^*, b)$ to $W^{1,\infty}([0,T], \mathbf{R}^n)$ with nonempty, compact and connected values;

(ii) *The multifunction $\xi \to \mathscr{A}_{[0,T]}^G(\xi) := \{u(T) : u \in \mathscr{S}_{[0,T]}^G(\xi)\}$ is upper semicontinuous from $B_n(x^*, b)$ to $\mathbf{R}^n$ with nonempty compact connected values.*

For the basic facts and definitions about multifunctions, we refer the reader to [2, 5].

3. Main Results

The following is our main result.

Theorem 3.1. *Let $n, k \in \mathbf{N}$, $T > 0$, and $F : [0,T] \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be a given multifunction. Assume that F is bounded and lower semicontinuous with nonempty and closed values. Then, there exists a bounded and upper semicontinuous multifunction $G : \mathbf{R} \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ with nonempty convex and compact values such that*

(a) $G(\mathbf{R} \times (\mathbf{R}^n)^k) \subseteq \overline{\text{conv}}\,(F([0,T] \times (\mathbf{R}^n)^k))$;

(b) *every generalized solution $u \in W^{k,\infty}([0,T], \mathbf{R}^n)$ of the differential inclusion*
$$u^{(k)} \in G(t, u, u', \ldots, u^{(k-1)})$$
in $[0,T]$ is also a generalized solution to the differential inclusion
$$u^{(k)} \in F(t, u, u', \ldots, u^{(k-1)})$$
in $[0,T]$.

Proof. Put $S := [0,T] \times (\mathbf{R}^n)^k$. Let $M > 0$ be such that $F(S) \subseteq \overline{B}_n(0,M)$. Let $y^* \in \mathbf{R}^n$ be the vector whose components are all equal to $2M$. Let $F^* : S \to 2^{\mathbf{R}^n}$ be the multifunction defined by, for each $(t,\xi) = (t, \xi_0, \xi_1, \ldots, \xi_{k-1}) \in S$,
$$F^*(t,\xi) = y^* + F(t, \xi_0 - \frac{y^*}{k!} t^k, \xi_1 - \frac{y^*}{(k-1)!} t^{k-1}, \ldots, \xi_{k-2} - \frac{y^*}{2!} t^2, \xi_{k-1} - y^* t).$$
By [5, Theorems 7.3.11 and 7.3.15], the multifunction F^* is lower semicontinuous. Moreover, one has
$$F^*(S) \subseteq \Lambda := \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : \ M \le x_i \le 3M \text{ for all } i = 1, \ldots, n\}.$$
We can consider the multifunction F^* as defined on $\mathbf{R} \times \mathbf{R}^{nk}$ by means of the obvious identification $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$. By [4, Lemma 2.4], there exist two sets $H_0, H \in \mathscr{B}$ with $H_0 \subseteq [0,T]$ and $m_1(H_0) = m_1(H) = 0$, and a function $f : S \to \mathbf{R}^n$ such that

(i) $f(t,\xi) \in F^*(t,\xi)$ for all $(t,\xi) \in S$;

(ii) f is continuous at every point
$$(t,\xi) \in ([0,T] \setminus H_0) \times (\mathbf{R} \setminus H)^{nk}.$$

Fix any point $z^* \in f(S)$, and let $f^* : \mathbf{R} \times \mathbf{R}^{nk} \to \mathbf{R}^n$ be defined by
$$f^*(t,\xi) = \begin{cases} f(t,\xi) & \text{if } (t,\xi) \in S, \\ z^* & \text{if } (t,\xi) \in (\mathbf{R} \times \mathbf{R}^{nk}) \setminus S. \end{cases}$$
Hence, $f^*(\mathbf{R} \times \mathbf{R}^{nk}) = f(S) \subseteq F^*(S) \subseteq \Lambda$. Let
$$Z := \Big[(H_0 \cup \{0,T\}) \times \mathbf{R}^{nk}\Big] \cup \Big[S \cap \bigcup_{l=1}^{nk} \Pi_l^{-1}(H)\Big].$$
Observe that $Z \subseteq S$. We claim that $f^*|_{(\mathbf{R} \times \mathbf{R}^{nk}) \setminus Z}$ is continuous. To see this, fix $(t,\xi) \in (\mathbf{R} \times \mathbf{R}^{nk}) \setminus Z$. First, assume that $(t,\xi) \in S$. Since $(t,\xi) \notin Z$, it follows that $t \in]0,T[$ and $t \notin H_0$.

Moreover, $\xi \in (\mathbf{R} \setminus H)^{nk}$. Therefore, by (ii), f is continuous at (t, ξ). Since $f|_{]0,T[\times \mathbf{R}^{nk}} = f^*|_{]0,T[\times \mathbf{R}^{nk}}$, it follows that $f^*|_{]0,T[\times \mathbf{R}^{nk}}$ is continuous at (t, ξ). Hence, f^* is continuous at (t, ξ), and thus $f^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus Z}$ is continuous at (t, ξ), as desired.

Conversely, assume that $(t, \xi) \notin S$. Since $(\mathbf{R}\times\mathbf{R}^{nk}) \setminus S$ is open in $\mathbf{R}\times\mathbf{R}^{nk}$ and $f^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus S}$ is constant, we obtain that f^* is continuous at (t, ξ). Hence, $f^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus Z}$ is continuous at (t, ξ), as desired.

Let $D \subseteq (\mathbf{R}\times\mathbf{R}^{nk}) \setminus Z$ be a countable set, and be dense in $\mathbf{R}\times\mathbf{R}^{nk}$. It is obvious that D exists due to $m_{1+nk}(Z) = 0$. Let $G^* : \mathbf{R}\times\mathbf{R}^{nk} \to 2^{\mathbf{R}^n}$ be the multifunction defined by, for each $(t, \xi) \in \mathbf{R}\times\mathbf{R}^{nk}$,

$$G^*(t,\xi) := \bigcap_{m\in\mathbf{N}} \overline{\operatorname{conv}}\, \overline{\Big(\bigcup_{\substack{(\lambda,\eta)\in D \\ \|(\lambda,\eta)-(t,\xi)\|_{1+nk}\leq \frac{1}{m}}} \{f^*(\lambda,\eta)\} \Big)}.$$

From the definition of G^* and the above construction, we have

$$G^*(\mathbf{R}\times\mathbf{R}^{nk}) \subseteq \overline{\operatorname{conv}}\,(F^*(S)), \tag{3.1}$$

which yileds

$$G^*(\mathbf{R}\times\mathbf{R}^{nk}) \subseteq \Lambda. \tag{3.2}$$

By [6, Proposition 2.6], taking into account that $f^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus Z}$ is continuous, we have the following facts

(i)$'$ the multifunction G^* has closed graph and nonempty closed convex values (hence, it is upper semicontinuous by (3.2) and [5, Theorem 7.1.16]);

(ii)$'$ for every $(t, \xi) \in (\mathbf{R}\times\mathbf{R}^{nk}) \setminus Z$,

$$G^*(t,\xi) = \{f^*(t,\xi)\}.$$

Since $f^*|_S = f$, we have

$$G^*(t,\xi) = \{f(t,\xi)\} \quad \text{for all} \quad (t,\xi) \in S\setminus Z. \tag{3.3}$$

We can regard the multifunction G^* as defined on $\mathbf{R}\times(\mathbf{R}^n)^k$ by means of the identification $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$. Let $G : \mathbf{R}\times(\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be the multifunction defined by, for each $(t, \xi) = (t, \xi_0, \xi_1, \ldots, \xi_{k-1}) \in \mathbf{R}\times(\mathbf{R}^n)^k$,

$$G(t,\xi) = -y^* + G^*(t, \xi_0 + \frac{y^*}{k!}t^k, \xi_1 + \frac{y^*}{(k-1)!}t^{k-1}, \ldots, \xi_{k-2} + \frac{y^*}{2!}t^2, \xi_{k-1} + y^*t).$$

Let us present that the multifunction G satisfies our conclusion. It is obvious that G has nonempty compact and convex values. By (i)$'$ and [5, Theorems 7.3.11 and 7.3.15], the multifunction G is upper semicontinuous. Moreover, taking into account (3.1), we have

$$G(\mathbf{R}\times(\mathbf{R}^n)^k) \subseteq \overline{\operatorname{conv}}\, F(S).$$

Now, let $u \in W^{k,\infty}([0,T], \mathbf{R}^n)$ be a generalized solution to the differential inclusion

$$u^{(k)} \in G(t, u, u', \ldots, u^{(k-1)}) \quad \text{in } [0,T].$$

Let $v : [0,T] \to \mathbf{R}^n$ be the function defined by

$$v(t) = u(t) + \frac{y^*}{k!}t^k.$$

It is obvious that $v \in W^{k,\infty}([0,T],\mathbf{R}^n)$. It can be easily checked that v is a solution to the differential inclusion

$$v^{(k)} \in G^*(t,v,v',\ldots,v^{(k-1)}) \quad \text{in } [0,T].$$

That is, there exists a Lebesgue measurable set $U_0 \subseteq [0,T]$ with $m_1(U_0)=0$ such that

$$v^{(k)}(t) \in G^*(t,v(t),v'(t),\ldots,v^{(k-1)}(t)) \quad \text{for all} \quad t \in [0,T]\setminus U_0. \tag{3.4}$$

By (3.2), we see that

$$v^{(k)}(t) \in \Lambda \quad \text{for all} \quad t \in [0,T]\setminus U_0. \tag{3.5}$$

Fix $i \in \{1,\ldots,n\}$, and let v_i be the i-th component of the function v. By (3.5), we have that $v_i^{(k)}(t) \geq M$ for all $t \in [0,T]\setminus U_0$. Therefore, the absolutely continuous function $v_i^{(k-1)}$ is strictly increasing in $[0,T]$ (with a.e. positive derivative). By [7, Theorem 2], the function $(v_i^{(k-1)})^{-1}$ is absolutely continuous. Hence, if we put

$$C_{i,k-1} := (v_i^{(k-1)})^{-1}(H) = \{t \in [0,T] : v_i^{(k-1)}(t) \in H\},$$

then we find from [8, Theorem 18.25] that $m_1(C_{i,k-1})=0$. Since $v_i^{(k-1)}$ is strictly increasing in $[0,T]$, there exists a partition

$$0 = t_{k-1,0} < \ldots < t_{k-1,j_{k-1}} = T$$

(with $j_{k-1} \leq 2$) of $[0,T]$ such that the function $v_i^{(k-1)}$ has constant sign over each interval $]t_{k-1,l-1},t_{k-1,l}[$ (in particular, $v_i^{(k-1)}(t) \neq 0$ on each interval $]t_{k-1,l-1},t_{k-1,l}[$). Therefore, for every $l = 1,\ldots,j_{k-1}$, the function $v_i^{(k-2)}|_{[t_{k-1,l-1},t_{k-1,l}]}$ is strictly monotone. Then, for each $l = 1,\ldots,j_{k-1}$, by [7, Theorem 2] the function

$$(v_i^{(k-2)}|_{[t_{k-1,l-1},t_{k-1,l}]})^{-1}$$

is absolutely continuous. Again by [8, Theorem 18.25], for every $l = 1,\ldots,j_{k-1}$, the set

$$(v_i^{(k-2)}|_{[t_{k-1,l-1},t_{k-1,l}]})^{-1}(H) = \left\{t \in [t_{k-1,l-1},t_{k-1,l}] :\ v_i^{(k-2)}(t) \in H\right\}$$

has null Lebesgue measure. Thus it follows that the set

$$C_{i,k-2} := (v_i^{(k-2)})^{-1}(H) = \{t \in [0,T] : u_i^{(k-2)}(t) \in H\}$$

has null Lebesgue measure. Again, since the function $v_i^{(k-2)}$ is strictly monotone on each interval $[t_{k-1,l-1},t_{k-1,l}]$ with $l = 1,\ldots,j_{k-1}$, there exists a partition

$$0 = t_{k-2,0} < \ldots t_{k-2,j_{k-2}} = T$$

(with $j_{k-2} \leq 4$) of $[0,T]$ such that the function $v_i^{(k-2)}$ has constant sign over each interval $]t_{k-2,l-1},t_{k-2,l}[$ (in particular, $v_i^{(k-2)}(t) \neq 0$ on each interval $]t_{k-2,l-1},t_{k-2,l}[$). It follows that, for every $l = 1,\ldots,j_{k-2}$, the function $v_i^{(k-3)}|_{[t_{k-2,l-1},t_{k-2,l}]}$ is strictly monotone. Then, by [7, Theorem 2], for each $l = 1,\ldots,j_{k-2}$, the function

$$(v_i^{(k-3)}|_{[t_{k-2,l-1},t_{k-2,l}]})^{-1}$$

is absolutely continuous. Consequently, reasoning exactly as above, by [8, Theorem 18.25], we easily obtain that the set

$$C_{i,k-3} := [v_i^{(k-3)}]^{-1}(H) = \{t \in [0,T] : v_i^{(k-3)}(t) \in H\}$$

has null Lebesgue measure. If we apply recursively the same argument, we finally obtain that, for every $j = 0, \dots, k-1$, the set

$$C_{i,j} := [v_i^{(j)}]^{-1}(H) = \{t \in [0,T] : v_i^{(j)}(t) \in H\}$$

has null Lebesgue measure. Now, put

$$C := \{0,T\} \cup H_0 \cup U_0 \cup \Big[\bigcup_{\substack{i=1,\dots,n \\ j=0,\dots,k-1}} C_{i,j} \Big].$$

It is obvious that $m_1(C) = 0$. Choose a point $t \in [0,T] \setminus C =]0,T[\setminus C$. By the definition of the sets $C_{i,j}$, we have that, for every $i = 1, \dots, n$ and every $j = 0, \dots, k-1$,

$$v_i^{(j)}(t) \notin H.$$

Hence,

$$(t, v(t), v'(t), \dots, v^{(k-1)}(t)) \in S \setminus Z.$$

By (3.3), we arrive at

$$G^*(t, u(t), u'(t), \dots, u^{(k-1)}(t)) = \{f(t, u(t), u'(t), \dots, u^{(k-1)}(t))\}.$$

Taking into account (i) and (3.4), we then have

$$v^{(k)}(t) = f(t, v(t), v'(t), \dots, v^{(k-1)}(t)) \in F^*(t, v(t), v'(t), \dots, v^{(k-1)}(t)).$$

Consequently, since $m_1(C) = 0$, the function v is a generalized solution in $[0,T]$ to the differential inclusion

$$v^{(k)} \in F^*(t, v, v', \dots, v^{(k-1)}).$$

At this point, by the definition of F^*, it is routine matter to check that the function u is a generalized solution in $[0,T]$ to the differential inclusion

$$u^{(k)} \in F(t, u, u', \dots, u^{(k-1)}).$$

The proof is now complete. □

As an application of the previous result, we now prove an existence and qualitative result for higher-order lower semicontinuous differential inclusions.

Theorem 3.2. *Let $n, k \in \mathbf{N}$, $T > 0$, and $F : [0,T] \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be a given multifunction. Assume that F is bounded and lower semicontinuous with nonempty closed values. Then, for every $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, the solution set*

$$\mathscr{S}^F_{[0,T]}(\xi) := \left\{u \in W^{k,1}([0,T], \mathbf{R}^n) : u \text{ is a generalized solution of (1.2)}\right\}$$

of problem (1.2) is nonempty. Moreover, there exists a multifunction

$$\Phi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T];\mathbf{R}^n)}$$

such that

(a) *$\Phi(\xi) \subseteq \mathscr{S}^F_{[0,T]}(\xi)$ for all $\xi \in (\mathbf{R}^n)^k$;*

(b) *Φ is upper semicontinuous (with respect to the topology $\tau^{[0,T]}_{n,k}$ of $W^{k,\infty}([0,T];\mathbf{R}^n)$), with nonempty, compact, and connected values;*

(c) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u(T) : u \in \Phi(\xi)\}$$

is upper semicontinuous with nonempty, connected, and compact values;

(d) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u^{(k)} \in L^\infty([0,T],\mathbf{R}^n) : \ u \in \Phi(\xi)\}$$

is upper semicontinuous (with respect to the weak-star topology of $L^\infty([0,T],\mathbf{R}^n)$) with compact connected values.

Proof. As before, put $S := [0,T] \times (\mathbf{R}^n)^k$. Let $M > 0$ such that $F(S) \subseteq \overline{B}_n(0,M)$. We divide the proof into two steps.

Step 1. We assume that $T < 1$. By Theorem 3.1, there exists a bounded and upper semicontinuous multifunction $G : \mathbf{R} \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ with non empty convex compact values such that

(i)$''$ $G(\mathbf{R} \times (\mathbf{R}^n)^k) \subseteq \overline{\text{conv}}\,(F(S)) \subseteq \overline{B}_n(0,M)$;

(ii)$''$ every generalized solution $u \in W^{k,\infty}([0,T],\mathbf{R}^n)$ of the differential inclusion

$$u^{(k)} \in G(s,u,u',\dots,u^{(k-1)})$$

in $[0,T]$ is also a generalized solution in $[0,T]$ to the differential inclusion

$$u^{(k)} \in F(s,u,u',\dots,u^{(k-1)}).$$

Let $\Lambda : \mathbf{R} \times (\mathbf{R}^n)^k \to 2^{(\mathbf{R}^n)^k}$ be the multifunction defined by, for every $(s,\xi) = (s,\xi_0,\xi_1,\dots,\xi_{k-1}) \in \mathbf{R} \times (\mathbf{R}^n)^k$,

$$\Lambda(s,\xi) = \Lambda(s,\xi_0,\xi_1,\dots,\xi_{k-1}) := \{\xi_1\} \times \{\xi_2\} \times \cdots \times \{\xi_{k-1}\} \times G(s,\xi).$$

By [5, Theorem 7.3.14], the multifunction Λ is upper semicontinuous. For every fixed $\xi = (\xi_0,\xi_1,\dots,\xi_{k-1}) \in (\mathbf{R}^n)^k$, let us consider the first-order Cauchy problem

$$\begin{cases} y' \in \Lambda(s,y) & \text{in } [0,T] \\ y(0) = \xi\,. \end{cases} \tag{3.6}$$

Moreover, let

$$\mathscr{S}^{\Lambda}_{[0,T]}(\xi) := \Big\{ y(s) = (y_0(s),y_1(s),\dots,y_{k-1}(s)) \in (W^{1,1}([0,T],\mathbf{R}^n))^k : \\ : y(s) \ \text{ is a generalized solution of (3.6)} \Big\}$$

be the solution set to problem (3.6), and let

$$\mathscr{A}^{\Lambda}_{[0,T]}(\xi) := \Big\{ y(T) = (y_0(T),y_1(T),\dots,y_{k-1}(T)) : \ y \in \mathscr{S}^{\Lambda}_{[0,T]}(\xi) \Big\}$$

be the attainable set to the same problem. Fix $\xi^* \in (\mathbf{R}^n)^k$ and any $b > 0$. Since $T < 1$, we have

$$\lim_{L \to +\infty} \big[L^2 - \big((b + LT + \|\xi^*\|_{nk})^2 + M^2\big) \big] = +\infty.$$

Consequently, we can choose $L^* > 0$ in such a way that

$$(b + L^*T + \|\xi^*\|_{nk})^2 + M^2 < (L^*)^2. \tag{3.7}$$

Our aim is now to demonstrate that

$$\Lambda\big([0,T]\times\overline{B}_{nk}(\xi^*,b+L^*T)\big)\subseteq\overline{B}_{nk}(0,L^*). \tag{3.8}$$

To this aim, fix $s\in[0,T]$ and $\xi=(\xi_0,\xi_1,\dots,\xi_{k-1})\in(\mathbf{R}^n)^k$ with

$$\|\xi-\xi^*\|_{nk}\le b+L^*T.$$

Let $\zeta=(\zeta_0,\zeta_1,\dots,\zeta_{k-1})\in\Lambda(s,\xi)$. By the definition of Λ, there exists $z\in G(s,\xi)$ such that $\zeta=(\xi_1,\xi_2,\dots,\xi_{k-1},z)$. Taking into account that

$$\|\xi\|_{nk}^2\le(b+L^*T+\|\xi^*\|_{nk})^2,$$

we find from (3.7) that

$$\begin{aligned}\|\zeta\|_{nk}^2&\le\|\xi_1\|_n^2+\|\xi_2\|_n^2+\dots+\|\xi_{k-1}\|_n^2+\|z\|_n^2\le\\&\le\|\xi\|_{nk}^2+M^2\le M^2+(b+L^*T+\|\xi^*\|_{nk})^2<(L^*)^2.\end{aligned}$$

Thus inclusion (3.8) is proved. Consequently, we can apply Theorem 2.1 to problem (3.6). We then have that, for every $\xi=(\xi_0,\dots,\xi_{k-1})\in(\mathbf{R}^n)^k$ with $\|\xi-\xi^*\|_{nk}<b$, the solution set $\mathscr{S}^{\Lambda}_{[0,T]}(\xi)$ is nonempty. Moreover, the multifunction $\xi\to\mathscr{S}^{\Lambda}_{[0,T]}(\xi)$ is upper semicontinuous from $B_{nk}(\xi^*,b)$ to

$$W^{1,\infty}([0,T],\mathbf{R}^{nk})=\big(W^{1,\infty}([0,T],\mathbf{R}^n)\big)^k$$

with nonempty compact connected values (it is routine matter to check that the topology $\tau^{[0,T]}_{nk,1}$ coincides with the product topology $(\tau^{[0,T]}_{n,1})^k$). Finally, the multifunction $\xi\to\mathscr{A}^{\Lambda}_{[0,T]}(\xi)$ is upper semicontinuous in $B_{nk}(\xi^*,b)$ with nonempty compact connected values. By the arbitrariness of the point $\xi^*\in(\mathbf{R}^n)^k$, we obtain that

(i)''' the set $\mathscr{S}^{\Lambda}_{[0,T]}(\xi)$ is nonempty for every $\xi\in(\mathbf{R}^n)^k$;

(ii)''' the multifunction

$$\xi\in(\mathbf{R}^n)^k\to\mathscr{S}^{\Lambda}_{[0,T]}(\xi)\subseteq[W^{1,\infty}([0,T],\mathbf{R}^n)]^k$$

is upper semicontinuous in $(\mathbf{R}^n)^k$, with nonempty compact and connected values;

(iii)''' the multifunction

$$\xi\in(\mathbf{R}^n)^k\to\mathscr{A}^{\Lambda}_{[0,T]}(\xi)\subseteq(\mathbf{R}^n)^k$$

is upper semicontinuous in $(\mathbf{R}^n)^k$, with nonempty compact connected values.

For each $v\in W^{k,\infty}(I,\mathbf{R}^n)$, let $y_v:[0,T]\to(\mathbf{R}^n)^k$ be defined by putting, for every $s\in I$,

$$y_v(s)=(v(s),v'(s),\dots,v^{(k-1)}(s)),$$

and let

$$E:=\{\,y_v:\ v\in W^{k,\infty}([0,T],\mathbf{R}^n)\,\}.$$

Moreover, let

$$P:\big[W^{1,\infty}([0,T],\mathbf{R}^n)\big]^k\to W^{1,\infty}([0,T],\mathbf{R}^n)$$

be the first projection. It is not difficult to check that E is a closed subset of the space $[W^{1,\infty}([0,T],\mathbf{R}^n)]^k$, $P(E)=W^{k,\infty}([0,T],\mathbf{R}^n)$, and the function

$$P|_E:(E,(\sigma^{[0,T]}_{n,1})^k)\to(W^{k,\infty}([0,T],\mathbf{R}^n),\sigma^{[0,T]}_{n,k})$$

is continuous. Now, for each $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, let us consider the Cauchy problem

$$\begin{cases} u^{(k)} \in G(t, u, u', \dots, u^{(k-1)}) & \text{in } I \\ u^{(i)}(0) = \xi_i, \end{cases} \tag{3.9}$$

and let

$$\mathscr{S}^G_{[0,T]}(\xi) := \left\{u \in W^{k,1}([0,T], \mathbf{R}^n) : u \text{ is a generalized solution of } (3.9)\right\}.$$

The boundedness of G implies that $\mathscr{S}^G_{[0,T]}(\xi) \subseteq W^{k,\infty}([0,T], \mathbf{R}^n)$ for all $\xi \in (\mathbf{R}^n)^k$. Fix $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, and let

$$w(s) = (w_0(s), w_1(s), \dots, w_{k-1}(s)) \in \mathscr{S}^\Lambda_{[0,T]}(\xi) \subseteq \left[W^{1,\infty}([0,T], \mathbf{R}^n)\right]^k.$$

Hence, $w'(s) \in \Lambda(s, w(s))$ for a.e. $s \in [0,T]$. Therefore, by the definition of Λ, for a.e. $s \in I$, we have

$$w_1(s) = w_0'(s), \quad w_2(s) = w_1'(s), \quad \dots, \quad w_{k-1}(s) = w_{k-2}'(s),$$
$$w_{k-1}'(s) \in G(s, w_0(s), \dots, w_{k-1}(s)).$$

Since the functions $w_0, w_1, \dots, w_{k-1}$ are absolutely continuous, by a standard argument, it follows that $w_0 \in C^{k-1}(I, \mathbf{R}^n)$, and, for every $s \in [0,T]$,

$$w_1(s) = w_0'(s), \quad w_2(s) = w_0''(s), \quad \dots \quad w_{k-1}(s) = w_0^{(k-1)}(s).$$

Hence, $w_0 \in W^{k,\infty}([0,T], \mathbf{R}^n))$ and

$$w_0^{(k)}(s) = w_{k-1}'(s) \in G(s, w_0(s), \dots, w_0^{(k-1)}(s))$$

for a.e. $s \in [0,T]$. Moreover, one has

$$w_0^{(j)}(0) = w_j(0) = \xi_j \quad \text{for every } j = 0, \dots, k-1.$$

Hence $w_0 \in \mathscr{S}^G_{[0,T]}(\xi)$ and $w = y_{w_0} \in E$. Thus we have proved that, for every $\xi = (\xi_0, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$,

$$\mathscr{S}^\Lambda_{[0,T]}(\xi) \subseteq E, \qquad \text{and} \quad P(\mathscr{S}^\Lambda_{[0,T]}(\xi)) \subseteq \mathscr{S}^G_{[0,T]}(\xi). \tag{3.10}$$

By (i)$'''$ and (ii)$'''$, taking into account (3.10) and the continuity of the function

$$P|_E : (E, (\tau^{[0,T]}_{n,1})^k) \to (W^{k,\infty}([0,T], \mathbf{R}^n), \tau^{[0,T]}_{n,k}),$$

we obtain that the multifunction

$$\xi \in (\mathbf{R}^n)^k \to P(\mathscr{S}^\Lambda_{[0,T]}(\xi))$$

is upper semicontinuous (with respect to the topology $\tau^{[0,T]}_{n,k}$ of the space $W^{k,\infty}([0,T], \mathbf{R}^n)$), with nonempty, compact, and connected values. If we denote by $\Pi : (\mathbf{R}^n)^k \to \mathbf{R}^n$ the first projection from $(\mathbf{R}^n)^k$ to $\mathbf{R}^n$, it is easily seen by the above construction that, for every $\xi = (\xi_0, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$,

$$\{u(T) : u \in P(\mathscr{S}^\Lambda_{[0,T]}(\xi))\} = \Pi(\mathscr{A}^\Lambda_{[0,T]}(\xi)).$$

Thus by (ii)$'''$ and by the continuity of Π_1, the multifunction

$$\xi \in (\mathbf{R}^n)^k \to \{u(T) : u \in P(\mathscr{S}^\Lambda_{[0,T]}(\xi))\}$$

is upper semicontinuous in $(\mathbf{R}^n)^k$ with nonempty, compact, and connected values. At this point, taking into account (3.10) and that

$$\mathscr{S}^G_{[0,T]}(\xi) \subseteq \mathscr{S}^F_{[0,T]}(\xi)$$

for every $\xi \in (\mathbf{R}^n)^k$, it suffices to take, for every $\xi \in (\mathbf{R}^n)^k$,

$$\Phi(\xi) := P(\mathscr{S}^\Lambda_{[0,T]}(\xi)).$$

By the above construction, it follows at once that the multifunction Φ satisfies the conclusion (conclusion (d) follows at once by conclusion (b) and by the continuity of the function

$$u \in W^{k,\infty}(I, \mathbf{R}^n) \to u^{(k)} \in L^\infty(I, \mathbf{R}^n),$$

where the last space is considered with its weak-star topology). Therefore, our claim is proved for the special case $T < 1$.

Step 2. We now prove the result in its full generality.

Let $I := [0, 1/2]$, and let $F^* : I \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be defined by putting, for each $(s, \xi_0, \xi_1, \dots, \xi_{k-1}) \in I \times (\mathbf{R}^n)^k$,

$$F^*(s, \xi_0, \xi_1, \dots, \xi_{k-1}) = 2^k T^k F(\, 2T\, s,\ \xi_0,\ \frac{1}{2T}\, \xi_1,\ \frac{1}{2^2 T^2}\, \xi_2,\ \dots,\ \frac{1}{2^{k-1} T^{k-1}}\, \xi_{k-1}\,).$$

Consider the function $f : I \times (\mathbf{R}^n)^k \to [0, T] \times (\mathbf{R}^n)^k$ defined by putting, for each $(s, \xi_0, \xi_1, \dots, \xi_{k-1}) \in I \times (\mathbf{R}^n)^k$,

$$f(s, \xi_0, \xi_1, \dots, \xi_{k-1}) = (\, 2T\, s,\ \xi_0,\ \frac{1}{2T}\, \xi_1,\ \frac{1}{2^2 T^2}\, \xi_2,\ \dots,\ \frac{1}{2^{k-1} T^{k-1}}\, \xi_{k-1}\,).$$

It is obvious that f is continuous. Moreover,

$$F^*(s, \xi_0, \xi_1, \dots, \xi_{k-1}) = 2^k T^k F(f(s, \xi_0, \xi_1, \dots, \xi_{k-1}))$$

for every $(s, \xi_0, \xi_1, \dots, \xi_{k-1}) \in I \times (\mathbf{R}^n)^k$. Hence, by [5, Theorem 7.3.11], the multifunction F^* is lower semicontinuous with nonempty closed values, and also

$$F^*(I \times (\mathbf{R}^n)^k) \subseteq \overline{B}_n(0, 2^k T^k M).$$

By Step 1, there exists a multifunction

$$\Psi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}(I;\mathbf{R}^n)}$$

such that

(a)′ $\Psi(\xi) \subseteq \mathscr{S}^{F^*}_I(\xi)$ for all $\xi \in (\mathbf{R}^n)^k$;

(b)′ Ψ is upper semicontinuous (with respect to the topology $\sigma^I_{n,k}$ of $W^{k,\infty}(I;\mathbf{R}^n)$), with nonempty, compact, and connected values;

(c)′ the multifunction

$$\xi \in (\mathbf{R}^n)^k \to \{u(1/2) : u \in \Psi(\xi)\}$$

is upper semicontinuous with nonempty connected and compact values;

(d)′ the multifunction

$$\xi \in (\mathbf{R}^n)^k \to \{u^{(k)} \in L^\infty(I, \mathbf{R}^n) :\ u \in \Psi(\xi)\}$$

is upper semicontinuous (with compact connected values) with respect to the weak-star topology of $L^\infty(I, \mathbf{R}^n)$.

Let $h : (\mathbf{R}^n)^k \to (\mathbf{R}^n)^k$ be the continuous functions defined by putting, for each $\xi = (\xi_0, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$,

$$h(\xi) = (\xi_0,\, 2T\,\xi_1,\, 2^2T^2\,\xi_2,\, \dots,\, 2^{k-1}T^{k-1}\,\xi_{k-1}).$$

Moreover, let

$$\phi : (W^{k,\infty}(I,\mathbf{R}^n), \tau^I_{n,k}) \to (W^{k,\infty}([0,T],\mathbf{R}^n), \tau^{[0,T]}_{n,k})$$

be defined by putting, for each $u \in W^{k,\infty}(I,\mathbf{R}^n)$,

$$\phi(u)(t) = u\Big(\frac{1}{2T}t\Big) \quad \text{for every} \quad t \in [0,T].$$

It is not difficult to check that ϕ is continuous. Moreover, one has

$$\phi(\mathscr{S}^{F^*}_I(h(\xi))) \subseteq \mathscr{S}^F_{[0,T]}(\xi) \quad \text{for every} \quad \xi = (\xi_0,\xi_1,\dots,\xi_{k-1}) \in (\mathbf{R}^n)^k. \tag{3.11}$$

To see this, fix $\xi = (\xi_0,\xi_1,\dots,\xi_{k-1}) \in (\mathbf{R}^n)^k$ and a function $v \in \phi(\mathscr{S}^{F^*}_I(h(\xi)))$. Therefore, there exists $u \in \mathscr{S}^{F^*}_I(h(\xi))$ such that $v = \phi(u)$. By the definition of $\phi(u)$, for every $j = 0,\dots,k-1$ we have

$$v^{(j)}(t) = \frac{1}{2^jT^j}\,u^{(j)}\Big(\frac{t}{2T}\Big) \quad \text{for all} \quad t \in [0,T].$$

Moreover, there exists a set $K \subseteq I$ with $m_1(K) = 0$ such that

$$v^{(k)}(t) = \frac{1}{2^kT^k}\,u^{(k)}\Big(\frac{t}{2T}\Big) \quad \text{for all} \quad t \in [0,T]\setminus(2TK)$$

and

$$u^{(k)}(s) \in F^*(s,u(s),u'(s),\dots,u^{(k-1)}(s)) \quad \text{for all} \quad s \in I\setminus K. \tag{3.12}$$

Consequently, for every $j = 0,\dots,k-1$, we have

$$v^{(j)}(0) = \frac{1}{2^jT^j}\,u^{(j)}(0) = \xi_j.$$

Moreover, for every $t \in [0,T]\setminus(2TK)$, taking into account (3.12), one has

$$\begin{aligned} v^{(k)}(t) &= \frac{1}{2^kT^k}\,u^{(k)}(\frac{t}{2T}) \in \\ &\in \frac{1}{2^kT^k}\,F^*(\frac{t}{2T},u(\frac{t}{2T}),u'(\frac{t}{2T}),\dots,u^{(k-1)}(\frac{t}{2T})) = \\ &= \frac{1}{2^kT^k}\,F^*(\frac{t}{2T},v(t),2T\,v'(t),\dots,2^{k-1}T^{k-1}v^{(k-1)}(t)) = \\ &= F(t,v(t),v'(t),\dots,v^{(k-1)}(t)). \end{aligned}$$

Hence $v \in \mathscr{S}^F_{[0,T]}(\xi)$, as claimed.

Now, let $\Phi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T],\mathbf{R}^n)}$ be the multifunction defined by setting, for each $\xi \in (\mathbf{R}^n)^k$,

$$\Phi(\xi) = \phi(\Psi(h(\xi))).$$

By (b)$'$ and by the continuity of ϕ and h, it follows that Φ is upper semicontinuous (with respect to the topology $\sigma^{[0,T]}_{n,k}$ of $W^{k,\infty}([0,T],\mathbf{R}^n)$) with nonempty, compact, and connected values. Moreover, by (3.11), we have that $\Phi(\xi) \subseteq \mathscr{S}^F_{[0,T]}(\xi)$ for every $\xi \in (\mathbf{R}^n)^k$.

In order to prove conclusion (c), we observe that, for every $\xi \in (\mathbf{R}^n)^k$,

$$\{v(T) : v \in \Phi(\xi)\} = \{u(1/2) : u \in \Psi(h(\xi))\}.$$

Consequently, conclusion (c) follows by the continuity of h and by (c)$'$. For conclusion (d), it follows by conclusion (b) and (taking into account the definition of the topology $\tau_{n,k}^{[0,T]}$) by the continuity of the function

$$v \in W^{k,\infty}([0,T],\mathbf{R}^n) \to v^{(k)} \in L^\infty([0,T],\mathbf{R}^n)$$

(where, as before, $L^\infty([0,T],\mathbf{R}^n)$ is considered with its weak-star topology). The proof is now complete. □

4. An Application

As announced in the first section, we now apply the above results to obtain the qualitative result for the generalized solutions of the Cauchy problem associated with an implicit ordinary differential equation.

Let $n,k \in \mathbf{N}$, $T > 0$, and $Y \subseteq \mathbf{R}^n$ be a nonempty set. Let $g : Y \to \mathbf{R}$ and $f : [0,T] \times (\mathbf{R}^n)^k \to \mathbf{R}$ be two given functions, and let $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$. It is known that a generalized solution to the Cauchy problem

$$\begin{cases} g(u^{(k)}) = f(t,u,u',\dots,u^{(k-1)}) & \text{in } [0,T], \\ u^{(i)}(0) = \xi_i, \qquad i = 0,1,\dots,k-1, \end{cases} \tag{4.1}$$

is a function $u \in W^{k,1}([0,T],\mathbf{R}^n)$ such that

$$u^{(k)}(t) \in Y \quad \text{and} \quad g(u^{(k)}(t)) = f(t,u(t),u'(t),\dots,u^{(k-1)}(t)) \quad \text{for a.e.} \quad t \in [0,T],$$

and $u^{(i)}(0) = \xi_i$ for every $i = 0,1,\dots,k-1$. For each fixed $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, we denote by

$$\mathscr{S}(\xi) = \Big\{ u \in W^{k,1}([0,T],\mathbf{R}^n) : u \text{ is a generalized solution of (4.1)} \Big\}$$

the solution set of problem (4.1). If Y is bounded, then every generalized solution of problem (4.1) belongs to $W^{k,\infty}([0,T],\mathbf{R}^n)$.

The following is our result.

Theorem 4.1. *Let $n,k \in \mathbf{N}$ and $T > 0$. Let Y a nonempty, compact, connected, and locally connected subset of $\mathbf{R}^n$. Let $g : Y \to \mathbf{R}$ and $f : [0,T] \times (\mathbf{R}^n)^k \to \mathbf{R}$ be two continuous functions such that*

(i) *$f([0,T] \times (\mathbf{R}^n)^k) \subseteq g(Y)$;*
(ii) *for every $r \in]\inf g(Y), \sup g(Y)[$, one has $\mathrm{int}_Y(g^{-1}(r)) = \emptyset$.*

Then, for every $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, the solution set $\mathscr{S}(\xi)$ of problem (4.1) is nonempty. Moreover, there exists a multifunction

$$\Phi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T];\mathbf{R}^n)}$$

such that

(a) *$\Phi(\xi) \subseteq \mathscr{S}(\xi)$ for all $\xi \in (\mathbf{R}^n)^k$;*
(b) *Φ is upper semicontinuous (with respect to the topology $\tau_{n,k}^{[0,T]}$ of $W^{k,\infty}([0,T];\mathbf{R}^n)$), with nonempty, compact, and connected values;*

(c) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u(T) : u \in \Phi(\xi)\}$$

is upper semicontinuous with nonempty, connected, and compact values;

(d) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u^{(k)} \in L^\infty([0,T],\mathbf{R}^n) : \ u \in \Phi(\xi)\}$$

is upper semicontinuous (with compact connected values), with respect to the weak-star topology of $L^\infty([0,T],\mathbf{R}^n)$.

Proof. Put $S := [0,T] \times (\mathbf{R}^n)^k$. By assumption (ii) and by [9, Theorem 2.4], the function g is inductively open on Y. That is, there exists a set $Y^* \subseteq Y$ such that $g(Y^*) = g(Y)$ and the function $g|_{Y^*} : Y^* \to g(Y)$ is open. Let $F : S \to 2^Y$ be the multifunction defined by setting, for each $(t,\xi) = (t,\xi_0,\xi_1,\dots,\xi_{k-1}) \in S$,

$$F(t,\xi_0,\xi_1,\dots,\xi_{k-1}) = \overline{g^{-1}(f(t,\xi_0,\xi_1,\dots,\xi_{k-1})) \cap Y^*}.$$

By the above construction, by assumptions (i) and [5, Proposition 7.3.3], the multifunction F is lower semicontinuous with nonempty closed values. Moreover, the set $F(S) \subseteq Y$ is bounded by the compactness of Y. Therefore, the multifunction F satisfies all the assumptions of Theorem 3.2. Consequently, there exists a multifunction $\Phi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T],\mathbf{R}^n)}$ satisfying the conclusion of Theorem 3.2.

Now, we claim that $\Phi(\xi) \subseteq \mathscr{S}(\xi)$ for every $\xi \in (\mathbf{R}^n)^k$. To this aim, fix $\xi \in (\mathbf{R}^n)^k$ and $u \in \Phi(\xi)$. Hence, there exists a Lebesgue measurable set $K \subseteq [0,T]$ with $m_1(K) = 0$ such that

$$u^{(k)}(t) \in F(t,u(t),u'(t),\dots,u^{(k-1)}(t)) \quad \text{for all} \quad t \in [0,T] \setminus K.$$

Hence,

$$u^{(k)}(t) \in Y \quad \text{for all} \quad t \in [0,T] \setminus K.$$

Consequently, for every $t \in [0,T] \setminus K$, taking into account the definition of F, the continuity of g and the closedness of Y, we obtain

$$\begin{aligned} u^{(k)}(t) &\in \overline{g^{-1}(f(t,u(t),u'(t),\dots,u^{(k-1)}(t))) \cap Y^*} \subseteq \\ &\subseteq \overline{g^{-1}(f(t,u(t),u'(t),\dots,u^{(k-1)}(t)))} = \\ &= g^{-1}(f(t,u(t),u'(t),\dots,u^{(k-1)}(t))). \end{aligned}$$

Hence,

$$g(u^{(k)}(t)) = f(t,u(t),u'(t),\dots,u^{(k-1)}(t)) \quad \text{for all} \quad t \in [0,T] \setminus K.$$

Thus $u \in \mathscr{S}(\xi)$, as claimed. This completes the proof. □

For other existence results concerning generalized solutions of implicit differential equations, we refer to [10]-[25] and to the reference therein.

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