J. Nonlinear Var. Anal. 6 (2022), No. 5, pp. 517-534
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.5.06

NUMERICAL ANALYSIS OF A GENERAL ELLIPTIC VARIATIONAL-HEMIVARIATIONAL INEQUALITY

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Abstract. This paper is devoted to the numerical analysis of a general elliptic variational-hemivariational inequality. After a review of a solution existence and uniqueness result, we introduce a family of Galerkin methods to solve the problem. We prove the convergence of the numerical method under the minimal solution regularity condition available from the existence result and derive a Céa's inequality for error estimation of the numerical solutions. Then, we apply the results for the numerical analysis of a variational-hemivariational inequality in the study of a static problem which models the contact of an elastic body with a reactive foundation. In particular, under appropriate solution regularity conditions, we derive an optimal order error estimate for the linear finite element solution.

Keywords. Contact problem; Error estimation; Galerkin method; Variational-hemivariational inequality.

1. Introduction

In the recent years, there has been extensive research on the studies of hemivariational and variational-hemivariational inequalities. Such inequalities are proper mathematical formulations of physical and engineering problems where non-smooth, non-monotone, and/or set-valued relations are used among different physical quantities. While variational inequalities are featured by the presence of non-smooth convex functions in their formulations, variational-hemivariational inequalities include both nonsmooth convex functions and locally Lipschitz functions that are allowed to be nonconvex. The notion of the hemivariational inequality was first introduced by Panagiotopoulous in early 1980s [1]. Comprehensive references in this area include the books [2, 3] and, more recently, [4, 5]. The variational-hemivariational inequality we study in this paper is of the following form or its variant.

Problem 1.1. *Find an element $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + \Psi^0(u,u;v-u) \geq \langle f, v-u\rangle, \quad \forall v \in K, \tag{1.1}$$

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Received May 27, 2022; Accepted June 16, 2022.

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where K is a subset of a normed space V, A is an operator mapping V to its dual V^*, $\Phi\colon V \times V \to \mathbb{R}$, $\Psi\colon V \times V \to \mathbb{R}$ is a locally Lipschitz continuous function with respect to the second argument, and $f \in V^*$.

In (1.1) and below, Ψ^0 denotes the generalized directional derivative of Ψ with respect to its second argument. Properties of Ψ^0 and the closely related generalized gradient $\partial\Psi$ can be found in various references; see, e.g., [4, 6].

The well-posedness of Problem 1.1 was studied in [7] under the assumptions on the data which will be specified in Section 2. In this paper, we focus on the numerical analysis of Problem 1.1. Numerical analysis of elliptic hemivariational inequalities has been the subject of several papers in the recent years; see, e.g., [8, 9, 10, 11]. Nevertheless, we underline that Problem 1.1 is more general than the hemivariational inequalities studied in these references. We note in passing that numerical analysis of other types of hemivariational inequalities can be found in numerous papers, for instance, [12, 13, 14] for evolutionary hemivariational inequalities, [15, 16] for history-dependent hemivariational inequalities, and [17, 18] for hemivariational inequalities arising in fluid mechanics. In addition to the finite element method, other numerical methods were also developed to solve hemivariational inequalities; see, e.g., [19, 20, 21] on virtual element methods. We also mention the survey paper [22], which provides a summary account of numerical analysis of hemivariational inequalities.

To approximate Problem 1.1, we use a family of Galerkin methods. We prove the convergence of the numerical solutions under the minimal solution regularity available through the well-posedness result and derive a Céa's inequality that is the starting point for error estimation. Then we illustrate the applications of the results in the study of the finite element method for solving a contact problem. Processes of contact between deformable bodies or between a deformable body and a rigid foundation are commonly seen in industry and everyday life. Their modeling, analysis, and numerical simulation are the topics of a large number of references that continues to grow steadily. An early comprehensive reference in the area is [23]. More recent references include [4, 5, 24]. In these references, various contact problems were studied for different types of materials, such as elastic, viscoelastic and viscoplastic materials, associated with different contact and friction boundary conditions. The problems are formulated as variational, hemivariational and variational-hemivariational inequalities, which allow well-posedness analysis with techniques from functional analysis and nonsmooth analysis.

The rest of the paper is organized as follows. In Section 2, we review an existence and uniqueness result from [7] on Problem 1.1 and introduce a family of Galerkin methods. In Section 3, we prove the convergence of the Galerkin method under the minimal solution regularity condition available from the existence result. In Section 4, we derive a Céa's inequality that is the starting point for error estimation of the numerical solutions. In Section 5, we introduce a contact problem. In Section 6, we apply the theoretical results from Sections 3 and 4 on the numerical analysis of the contact problem.

2. Preliminaries

For the analysis of Problem 1.1, we consider the following hypotheses on the data.

$H(K)$ V is a real Hilbert space, and K is a non-empty, closed, and convex set in V.

$H(A)$ $A\colon V \to V^*$ is Lipschitz continuous and strongly monotone.

$H(\Phi)_2$ $\Phi\colon V\times V\to\mathbb{R}$, for any $u\in V$, $\Phi(u,\cdot)\colon V\to\mathbb{R}$ is convex and bounded above on a non-empty open set, and there exists a constant $\alpha_\Phi\ge 0$ such that

$$\Phi(u_1,v_2)-\Phi(u_1,v_1)+\Phi(u_2,v_1)-\Phi(u_2,v_2)\le \alpha_\Phi\|u_1-u_2\|_V\|v_1-v_2\|_V,\quad \forall\, u_1,u_2,v_1,v_2\in V. \tag{2.1}$$

$H(\Psi)_2$ $\Psi\colon V\times V\to\mathbb{R}$ is locally Lipschitz continuous with respect to its second argument, and the following inequalities hold, for some constants $\alpha_{\Psi,1},\alpha_{\Psi,2}\ge 0$, $c\ge 0$:

$$\Psi^0(w_1,v_1;v_2-v_1)+\Psi^0(w_2,v_2;v_1-v_2)\le \alpha_{\Psi,1}\|w_1-w_2\|_V\|v_1-v_2\|_V+\alpha_{\Psi,2}\|v_1-v_2\|_V^2,\quad \forall\, w_1,w_2,v_1,v_2\in V, \tag{2.2}$$

$$\left|\Psi^0(w,u;v)\right|\le c\,(1+\|w\|_V+\|u\|_V)\,\|v\|_V,\quad \forall\, w,u,v\in V. \tag{2.3}$$

$H(f)$ $f\in V^*$.

We denote by $m_A>0$ the constant in the strong monotonicity inequality of A, i.e.,

$$\langle Av_1-Av_2,v_1-v_2\rangle\ge m_A\|v_1-v_2\|_V^2,\quad \forall\, v_1,v_2\in V. \tag{2.4}$$

Moreover, we denote by $L_A>0$ the Lipschitz constant constant of A, that is,

$$\|Av_1-Av_2\|_{V^*}\le L_A\|v_1-v_2\|_V,\quad \forall\, v_1,v_2\in V. \tag{2.5}$$

The subscript 2 in $H(\Phi)_2$ reminds the reader that this is a condition for the case where Φ has two independent variables. Similarly, the subscript 2 in $H(\Psi)_2$ reminds the reader that this is a condition for the case where Ψ has two independent variables. It is known that, for a real-valued convex function on a normed space, it is locally Lipschitz continuous on the space if and only if it is bounded above on a non-empty open set in the space (see, e.g., [25, Corollary 2.4, p. 12]). Therefore, $H(\Phi)_2$ ensures that, for any $u\in V$, $\Phi(u,\cdot)$ is locally Lipschitz continuous on V.

The following well-posedness result was proved in [7].

Theorem 2.1. *Assume $H(K)$, $H(A)$, $H(\Phi)_2$, $H(\Psi)_2$, $H(f)$, and the smallness condition*

$$\alpha_\Phi+\alpha_{\Psi,1}+\alpha_{\Psi,2}<m_A. \tag{2.6}$$

Then, Problem 1.1 has a unique solution $u\in K$. Moreover, the operator $f\mapsto u=u(f)$, which maps the element $f\in V^*$ to the solution $u\in K$ of Problem 1.1 is Lipschitz continuous.

Turning to numerical approximation, we let $V^h\subset V$ be a finite dimensional subspace characterized by a discretization parameter $h>0$ with the expectation of convergence when $h\to 0$. We use the notation " $\to$" and " $\rightharpoonup$" for the strong and weak convergence in various spaces that will be specified later. Let K^h be a non-empty, closed, and convex subset of V^h. Then, a Galerkin approximation of Problem 1.1 is the following.

Problem 2.1. *Find an element $u^h\in K^h$ such that*

$$\langle Au^h,v^h-u^h\rangle+\Phi(u^h,v^h)-\Phi(u^h,u^h)+\Psi^0(u^h,u^h;v^h-u^h)\ge\langle f,v^h-u^h\rangle,\quad \forall\, v^h\in K^h. \tag{2.7}$$

Under the assumptions stated in Theorem 2.1, Problem 2.1 has a unique solution u^h. We will assume that $\{K^h\}_h$ approximates K in the following sense of Mosco (see [26]):

$$v^h\in K^h \text{ and } v^h\rightharpoonup v \text{ in } V \text{ imply } v\in K; \tag{2.8}$$

$$\forall\, v\in K,\ \exists\, v^h\in K^h \text{ such that } v^h\to v \text{ in } V \text{ as } h\to 0. \tag{2.9}$$

In the following sections, we use the modified Cauchy–Schwarz inequality on various occasions:

$$ab \le \varepsilon a^2 + c b^2, \quad \forall a,b \in \mathbb{R},\ \varepsilon > 0,\ c = 1/(4\varepsilon). \tag{2.10}$$

The parameter $\varepsilon > 0$ will be chosen suitably small. Throughout the paper, we use c to denote a generic positive constant that is independent of quantities of concern, for instant, in error analysis of numerical solutions, the constant c is independent of the discretization parameter h.

3. Convergence

As an intermediate result, which is needed in convergence analysis, we first demonstrate that the numerical solutions defined by Problem 2.1 are bounded with respect to the parameter h.

Lemma 3.1. *Keep the assumptions in Theorem 2.1 and assume that (2.9) holds. Then there exists a constant $M > 0$ such that $\|u^h\|_V \le M$ for all $h > 0$.*

Proof. Since K is non-empty, there exists an element $u_0 \in K$. By (2.9), we can find $u_0^h \in K^h$ such that $u_0^h \to u_0$ in V as $h \to 0$. By (2.4),

$$m_A \|u^h - u_0^h\|_V^2 \le \langle Au^h - Au_0^h, u^h - u_0^h\rangle = \langle Au^h, u^h - u_0^h\rangle - \langle Au_0^h, u^h - u_0^h\rangle.$$

Take $v^h = u_0^h$ in (2.7) to obtain

$$\langle Au^h, u^h - u_0^h\rangle \le \Phi(u^h, u_0^h) - \Phi(u^h, u^h) + \Psi^0(u^h, u^h; u_0^h - u^h) + \langle f, u^h - u_0^h\rangle.$$

Thus

$$m_A \|u^h - u_0^h\|_V^2 \le \Phi(u^h, u_0^h) - \Phi(u^h, u^h) + \Psi^0(u^h, u^h; u_0^h - u^h) + \langle f - Au_0^h, u^h - u_0^h\rangle. \tag{3.1}$$

By (2.1),

$$\Phi(u^h, u_0^h) - \Phi(u^h, u^h) \le \Phi(u_0, u_0^h) - \Phi(u_0, u^h) + \alpha_\Phi \|u^h - u_0\|_V \|u^h - u_0^h\|_V.$$

Since $\|u^h - u_0\|_V \le \|u^h - u_0^h\|_V + \|u_0^h - u_0\|_V$, we have

$$\Phi(u^h, u_0^h) - \Phi(u^h, u^h) \le \Phi(u_0, u_0^h) - \Phi(u_0, u^h) + \alpha_\Phi \left(\|u^h - u_0^h\|_V^2 + \|u_0^h - u_0\|_V \|u^h - u_0^h\|_V\right).$$

We now apply inequality (2.10) on the term $\alpha_\Phi \|u_0^h - u_0\|_V \|u^h - u_0^h\|_V$ to get

$$\Phi(u^h, u_0^h) - \Phi(u^h, u^h) \le \Phi(u_0, u_0^h) - \Phi(u_0, u^h) + (\alpha_\Phi + \varepsilon)\, \|u^h - u_0^h\|_V^2 + c\, \|u_0^h - u_0\|_V^2.$$

Moreover, since $\Phi(u_0, \cdot)$ is convex and continuous, it is bounded below by a continuous affine functional on V (see, e.g., [27, p. 433]). Thus there exist two constants c_1 and c_2, depending on Φ and u_0 and not being necessarily non-negative, such that

$$\Phi(u_0, v) \ge c_1 + c_2 \|v\|_V, \quad \forall v \in V.$$

Then, $-\Phi(u_0, u^h) \le -c_1 - c_2 \|u^h\|_V$. By (2.2),

$$\Psi^0(u^h, u^h; u_0^h - u^h) \le -\Psi^0(u_0^h, u_0^h; u^h - u_0^h) + \left(\alpha_{\Psi,1} + \alpha_{\Psi,2}\right) \|u^h - u_0^h\|_V^2$$

and, by (2.3),

$$-\Psi^0(u_0^h, u_0^h; u^h - u_0^h) \le c \left(1 + \|u_0^h\|_V\right) \|u^h - u_0^h\|_V \le \varepsilon\, \|u^h - u_0^h\|_V^2 + c \left(1 + \|u_0^h\|_V^2\right).$$

Moreover,

$$\langle f - Au_0^h, u^h - u_0^h\rangle \le \|f - Au_0^h\|_{V^*} \|u^h - u_0^h\|_V \le \varepsilon\, \|u^h - u_0^h\|_V^2 + c\, \|f - Au_0^h\|_{V^*}^2.$$

Hence, from (3.1), we derive the following inequality

$$\begin{aligned}&\left(m_A-\alpha_\Phi-\alpha_{\Psi,1}-\alpha_{\Psi,2}-4\,\varepsilon\right)\|u^h-u_0^h\|_V^2\\&\quad\le\Phi(u_0,u_0^h)+c\left(1+\|u_0^h\|_V^2+\|u_0^h-u_0\|_V^2+\|f-Au_0^h\|_{V^*}^2\right).\end{aligned}$$

Take $\varepsilon=(m_A-\alpha_\Phi-\alpha_{\Psi,1}-\alpha_{\Psi,2})/8$ in (3.2) to find that for a constant $c>0$,

$$\|u^h-u_0^h\|_V^2\le c\left(1+\Phi(u_0,u_0^h)+\|u_0^h\|_V^2+\|u_0^h-u_0\|_V^2+\|f-Au_0^h\|_{V^*}^2\right). \tag{3.2}$$

Note that, as $h\to 0$,

$$\begin{aligned}&\Phi(u_0,u_0^h)\to\Phi(u_0,u_0),\\&\|u_0^h\|_V^2\to\|u_0\|_V^2,\\&\|u_0^h-u_0\|_V^2\to 0,\\&\|f-Au_0^h\|_{V^*}^2\to\|f-Au_0\|_{V^*}^2.\end{aligned}$$

Therefore, we conclude from (3.2) the boundedness of $\{\|u^h-u_0^h\|_V\}$, and then the boundedness of $\{\|u^h\|_V\}$, with respect to h. □

We now state and prove the following convergence result.

Theorem 3.1. *Keep the assumptions in Theorem 2.1. Assume further that (2.8)–(2.9) hold and the function $(w,v)\mapsto\Psi^0(w,w;v)$ is upper semi-continuous, i.e.,*

$$w_n\to w\ \text{ and }\ v_n\to v \text{ in } V\quad\Longrightarrow\quad\Psi^0(w,w;v)\ge\limsup_{n\to\infty}\Psi^0(w_n,w_n;v_n). \tag{3.3}$$

Then, we have the convergence of the numerical method, i.e., $u^h\to u$ in V as $h\to 0$.

Proof. We know from Lemma 3.1 that $\{u^h\}$ is bounded in V. Since V is reflexive, there exist a subsequence $\{u^{h'}\}\subset\{u^h\}$ and an element $w\in V$ such that

$$u^{h'}\rightharpoonup w\ \text{ in }\ V. \tag{3.4}$$

Note that the weak limit w belongs to K thanks to the assumption (2.8). Let us strengthen the weak convergence in (3.4) to strong convergence

$$u^{h'}\to w\ \text{ in }\ V. \tag{3.5}$$

By (2.9), we have a sequence $\{w^{h'}\}\subset V$ with $w^{h'}\in K^{h'}$ such that

$$w^{h'}\to w\ \text{ in }\ V. \tag{3.6}$$

By the strong monotonicity of A, $m_A\|w-u^{h'}\|_V^2\le\langle Aw-Au^{h'},w-u^{h'}\rangle$, which is rewritten as

$$m_A\|w-u^{h'}\|_V^2\le\langle Aw,w-u^{h'}\rangle-\langle Au^{h'},w^{h'}-u^{h'}\rangle-\langle Au^{h'},w-w^{h'}\rangle. \tag{3.7}$$

We take $v^{h'}=w^{h'}$ in (2.7) with $h=h'$ to obtain

$$-\langle Au^{h'},w^{h'}-u^{h'}\rangle\le\Phi(u^{h'},w^{h'})-\Phi(u^{h'},u^{h'})+\Psi^0(u^{h'},u^{h'};w^{h'}-u^{h'})-\langle f,w^{h'}-u^{h'}\rangle. \tag{3.8}$$

Apply (2.1) with $u_1=v_1=u^{h'}$, $u_2=w$ and $v_2=w^{h'}$ to find that

$$\Phi(u^{h'},w^{h'})-\Phi(u^{h'},u^{h'})\le\Phi(w,w^{h'})-\Phi(w,u^{h'})+\alpha_\Phi\|u^{h'}-w\|_V\|u^{h'}-w^{h'}\|_V. \tag{3.9}$$

Write $\|u^{h'} - w^{h'}\|_V \le \|u^{h'} - w\|_V + \|w - w^{h'}\|_V$ and apply the inequality (2.10) to the term $\alpha_\Phi \|u^{h'} - w\|_V \|w - w^{h'}\|_V$. Then, for the last term of (3.9) and a constant c depending on ε, we have

$$\alpha_\Phi \|u^{h'} - w\|_V \|u^{h'} - w^{h'}\|_V \le (\alpha_\Phi + \varepsilon) \|w - u^{h'}\|_V^2 + c \|w - w^{h'}\|_V^2. \tag{3.10}$$

By the sub-additivity property of the generalized directional derivative ([6, Section 2.3]),

$$\Psi^0(u^{h'}, u^{h'}; w^{h'} - u^{h'}) \le \Psi^0(u^{h'}, u^{h'}; w^{h'} - w) + \Psi^0(u^{h'}, u^{h'}; w - u^{h'}).$$

So

$$\begin{aligned}\Psi^0(u^{h'}, u^{h'}; w^{h'} - u^{h'}) \le \Psi^0(u^{h'}, u^{h'}; w - u^{h'}) + \Psi^0(w, w; u^{h'} - w)\\ + \Psi^0(u^{h'}, u^{h'}; w^{h'} - w) - \Psi^0(w, w; u^{h'} - w).\end{aligned} \tag{3.11}$$

By (2.2), one has

$$\Psi^0(u^{h'}, u^{h'}; w - u^{h'}) + \Psi^0(w, w; u^{h'} - w) \le \left(\alpha_{\Psi,1} + \alpha_{\Psi,2}\right) \|w - u^{h'}\|_V^2. \tag{3.12}$$

Use (3.8)–(3.12) in (3.7) to obtain

$$\begin{aligned}\left(m_A - \alpha_\Phi - \alpha_{\Psi,1} - \alpha_{\Psi,2} - \varepsilon\right) \|w - u^{h'}\|_V^2 \le \langle Aw, w - u^{h'}\rangle - \langle Au^{h'}, w - w^{h'}\rangle - \langle f, w^{h'} - u^{h'}\rangle\\ + \Phi(w, w^{h'}) - \Phi(w, u^{h'}) + c \|w - w^{h'}\|_V^2\\ + \Psi^0(u^{h'}, u^{h'}; w^{h'} - w) - \Psi^0(w, w; u^{h'} - w).\end{aligned}$$

Taking $\varepsilon = (m_A - \alpha_\Phi - \alpha_{\Psi,1} - \alpha_{\Psi,2})/2$, one has

$$\begin{aligned}\frac{1}{2}\left(m_A - \alpha_\Phi - \alpha_{\Psi,1} - \alpha_{\Psi,2}\right) \|w - u^{h'}\|_V^2 \le \langle Aw, w - u^{h'}\rangle - \langle Au^{h'}, w - w^{h'}\rangle - \langle f, w^{h'} - u^{h'}\rangle\\ + \Phi(w, w^{h'}) - \Phi(w, u^{h'}) + c \|w - w^{h'}\|_V^2\\ + \Psi^0(u^{h'}, u^{h'}; w^{h'} - w) - \Psi^0(w, w; u^{h'} - w).\end{aligned} \tag{3.13}$$

Consider now the limit of each term on the right side of (3.13) as $h' \to 0$. From the boundedness of A and the weak convergence (3.4), $\langle Aw, w - u^{h'}\rangle \to 0$. From (2.5) with $v_1 = u^{h'}$ and $v_2 = 0$, we obtain

$$\|Au^{h'}\|_{V^*} \le \|A\,0\|_{V^*} + L_A \|u^{h'}\|_V.$$

Thus

$$\left|\langle Au^{h'}, w - w^{h'}\rangle\right| \le \|Au^{h'}\|_{V^*} \|w - w^{h'}\|_V \le \left(\|A\,0\|_{V^*} + L_A \|u^{h'}\|_V\right) \|w - w^{h'}\|_V.$$

From the boundedness of $\{u^{h'}\}$ and the strong convergence (3.6), we find that $\langle Au^{h'}, w - w^{h'}\rangle \to 0$. Moreover, from (3.6) and (3.4), we know that $w^{h'} - u^{h'} \rightharpoonup 0$ in V, and so $\langle f, w^{h'} - u^{h'}\rangle \to 0$. Finally, the continuity of Φ with respect to its second argument and (3.6) yield $\Phi(w, w^{h'}) \to \Phi(w, w)$. As a consequence of the well-known Mazur Lemma, the convexity and continuity of Φ with respect to its second argument imply that Φ is weakly sequentially lower semicontinuous with respect to its second argument (cf. [27, p. 136]). Hence, by (3.4),

$$\limsup_{h' \to 0} \left[-\Phi(w, u^{h'})\right] = -\liminf_{h' \to 0} \left[\Phi(w, u^{h'})\right] \le -\Phi(w, w).$$

By (2.3),

$$\Psi^0(u^{h'}, u^{h'}; w^{h'} - w) \le c\left(1 + \|u^{h'}\|_V\right) \|w^{h'} - w\|_V.$$

Since $\{u^{h'}\}$ is bounded in V, from the convergence (3.6),

$$\limsup_{h'\to 0} \Psi^0(u^{h'}, u^{h'}; w^{h'} - w) \le 0.$$

By [6, Proposition 2.1.2], $\Psi^0(w, w; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial\Psi(w, w)\}$. Hence, for any $\xi_w \in \partial\Psi(w, w)$, we have

$$-\Psi^0(w, w; u^{h'} - w) \le -\langle \xi_w, u^{h'} - w\rangle \to 0 \quad \text{as } h' \to 0.$$

Thus

$$\limsup_{h'\to 0} \left[-\Psi^0(w, w; u^{h'} - w)\right] \le 0.$$

With the above preparations, we take the upper limit of both sides of (3.13) as $h' \to 0$ to conclude that

$$\limsup_{h'\to 0} \|w - u^{h'}\|_V^2 \le 0,$$

i.e., the strong convergence (3.5) holds.

Finally, let us show that the strong limit w is the unique solution of Problem 1.1. By (2.9), for any $v \in K$, we have a sequence $\{v^{h'}\} \subset V$ with $v^{h'} \in K^{h'}$ such that $v^{h'} \to v$ in V. Using this $v^{h'}$ in (2.7) with $h = h'$, we have

$$\langle Au^{h'}, v^{h'} - u^{h'}\rangle + \Phi(u^{h'}, v^{h'}) - \Phi(u^{h'}, u^{h'}) + \Psi^0(u^{h'}, u^{h'}; v^{h'} - u^{h'}) \ge \langle f, v^{h'} - u^{h'}\rangle. \tag{3.14}$$

Since $v^{h'} \to v$ and $u^{h'} \to w$ in V, and $A\colon V \to V^*$ is Lipschitz continuous, we have

$$\langle Au^{h'}, v^{h'} - u^{h'}\rangle \to \langle Aw, v - w\rangle, \quad \langle f, v^{h'} - u^{h'}\rangle \to \langle f, v - w\rangle \quad \text{as } h' \to 0. \tag{3.15}$$

An analogue of (3.9) is

$$\Phi(u^{h'}, v^{h'}) - \Phi(u^{h'}, u^{h'}) \le \Phi(w, v^{h'}) - \Phi(w, u^{h'}) + \alpha_\Phi \|u^{h'} - w\|_V \|u^{h'} - v^{h'}\|_V. \tag{3.16}$$

Since $\|u^{h'} - w\|_V \to 0$ and $\|u^{h'} - v^{h'}\|_V$ is bounded,

$$\|u^{h'} - w\|_V \|u^{h'} - v^{h'}\|_V \to 0. \tag{3.17}$$

By the assumption (3.3),

$$\Psi^0(w, w; v - w) \ge \limsup_{h'\to 0} \Psi^0(u^{h'}, u^{h'}; v^{h'} - u^{h'}). \tag{3.18}$$

We now take the upper limit as $h' \to 0$ in (3.14) and make use of the relations (3.15)–(3.18) to obtain

$$\langle Aw, v - w\rangle + \Phi(w, v) - \Phi(w, w) + \Psi^0(w, w; v - w) \ge \langle f, v - u\rangle,$$

which holds for any $v \in K$. Thus w is a solution of Problem 1.1. Since a solution of Problem 1.1 is unique, we have $w = u$. Moreover, since the limit u does not depend on the subsequence, the entire family of the numerical solutions converge, that is, $\|u - u^h\|_V \to 0$ as $h \to 0$. □

4. Error Estimation

To accommodate a more precise error estimation, we need information for a finer structure on the functional Ψ. For this purpose and with the applications of hemivariational inequalities in mind, we will replace the term $\Psi^0(w,u;v)$ by

$$I_\Delta(\psi^0(\gamma_1 w, \gamma_2 u; \gamma_2 v)),$$

where I_Δ stands for the integration operator over a measurable set Δ with $\Delta \subset \Omega$ or $\Delta \subset \partial\Omega$, and Ω is the spatial domain of the problem under consideration. For $i = 1,2$, let m_i be a positive integer, and let $\gamma_i \in L(V, L^2(\Delta;\mathbb{R}^{m_i}))$. We modify the condition $H(\Psi)_2$ to $H(\psi)_2'$:

$H(\psi)_2'$ m_1 and m_2 are positive integers, $\psi\colon \Delta \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$,
for any $y \in \mathbb{R}^{m_1}$ and any $z \in \mathbb{R}^{m_2}$, $\psi(\cdot, y, z)$ is measurable on Δ,
for any $y \in L^2(\Delta;\mathbb{R}^{m_1})$ and any $z \in L^2(\Delta;\mathbb{R}^{m_2})$, $\psi(\cdot, y(\cdot), z(\cdot)) \in L^1(\Delta)$,
for any $x \in \Delta$ and any $y \in \mathbb{R}^{m_1}$, $\psi(x,y,z)$ is locally Lipschitz continuous with respect to $z \in \mathbb{R}^{m_2}$,
for two constants $\alpha'_{\psi,1}, \alpha'_{\psi,2} \ge 0$,

$$\psi^0(y_1,z_1;z_2-z_1) + \psi^0(y_2,z_2;z_1-z_2) \le \alpha'_{\psi,1}|y_1-y_2|_{\mathbb{R}^{m_1}}|z_1-z_2|_{\mathbb{R}^{m_2}} + \alpha'_{\psi,2}|z_1-z_2|^2_{\mathbb{R}^{m_2}} \quad \forall y_1,y_2 \in \mathbb{R}^{m_1},\ \forall z_1,z_2 \in \mathbb{R}^{m_2}, \tag{4.1}$$

and for a constant $c > 0$,

$$\left|\psi^0(y,z_1;z_2)\right| \le c\,(1 + |y|_{\mathbb{R}^{m_1}} + |z_1|_{\mathbb{R}^{m_2}})\,|z_2|_{\mathbb{R}^{m_2}}, \quad \forall y \in \mathbb{R}^{m_1},\ \forall z_1,z_2 \in \mathbb{R}^{m_2}. \tag{4.2}$$

The smallness condition (2.6) is modified to

$$\alpha_\Phi + \alpha'_{\psi,1}\|\gamma_1\|\,\|\gamma_2\| + \alpha'_{\psi,2}\|\gamma_2\|^2 < m_A, \tag{4.3}$$

where $\|\gamma_i\|$ stands for the operator norm of $\gamma_i \in L(V, L^2(\Delta;\mathbb{R}^{m_i}))$, $i = 1,2$.

Then, the form of Problem 1.1 and that of its approximations, Problem 2.1, are modified to the following.

Problem 4.1. *Find an element $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v - \gamma_2 u)) \ge \langle f, v-u\rangle, \quad \forall v \in K. \tag{4.4}$$

Problem 4.2. *Find an element $u^h \in K^h$ such that*

$$\langle Au^h, v^h-u^h\rangle + \Phi(u^h,v^h) - \Phi(u^h,u^h) + I_\Delta(\psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 v^h - \gamma_2 u^h)) \ge \langle f, v^h-u^h\rangle, \quad \forall v^h \in K^h. \tag{4.5}$$

Similar to [28, Theorem 4.10], we can prove the following analogue of Theorem 2.1 for Problem 4.1 based on the result stated in Theorem 2.1.

Theorem 4.1. *Assume $H(K)$, $H(A)$, $H(\Phi)_2$, $H(\psi)_2'$, $H(f)$, and the smallness condition (4.3). Then, Problem 4.1 has a unique solution $u \in K$. Moreover, the operator $f \mapsto u = u(f)$ which maps the element $f \in V^*$ to the solution $u \in K$ of Problem 4.1 is Lipschitz continuous.*

Counterparts of Lemma 3.1 and Theorem 3.1 for the numerical solutions defined by Problem 4.2 are as follows.

Lemma 4.1. *Keep the assumptions in Theorem 4.1. Assume further that (2.9) holds. Then there exists a constant $M>0$ such that $\|u^h\|_V \le M$ for all $h>0$.*

Theorem 4.2. *Keep the assumptions in Theorem 4.1. Assume further that (2.8)–(2.9) hold and*

$$w_n \to w \ \text{ and } \ v_n \to v \text{ in } V \implies \psi^0(\gamma_1 w, \gamma_2 w; \gamma_2 v) \ge \limsup_{n\to\infty} \psi^0(\gamma_1 w_n, \gamma_2 w_n; \gamma_2 v_n) \text{ a.e. on } \Delta. \tag{4.6}$$

Then, we have the convergence of the numerical method, i.e., $u^h \to u$ in V as $h \to 0$.

Let us derive a Cea's type inequality that is the basis to bound the error $u-u^h$. Let $v\in K$ and $v^h \in K^h$ be arbitrary. From (2.4), $m_A\|u-u^h\|_V^2 \le \langle Au-Au^h, u-u^h\rangle$, which is rewritten as

$$\begin{aligned} m_A\|u-u^h\|_V^2 \le{}& \langle Au-Au^h, u-v^h\rangle + \langle Au, v^h-u\rangle + \langle Au, v-u^h\rangle \\ &+ \langle Au, u-v\rangle + \langle Au^h, u^h-v^h\rangle. \end{aligned} \tag{4.7}$$

From the defining inequality (4.4) for the solution u,

$$\langle Au, u-v\rangle \le \Phi(u,v)-\Phi(u,u)+I_\Delta(\psi^0(\gamma_1 u,\gamma_2 u;\gamma_2 v-\gamma_2 u)) - \langle f, v-u\rangle.$$

From the defining inequality (4.5) for the numerical solution u,

$$\langle Au^h, u^h-v^h\rangle \le \Phi(u^h,v^h)-\Phi(u^h,u^h)+I_\Delta(\psi^0(\gamma_1 u^h,\gamma_2 u^h;\gamma_2 v^h-\gamma_2 u^h)) - \langle f, v^h-u^h\rangle.$$

Using these upper bounds in (4.7), after some rearrangement of the terms, we have

$$m_A\|u-u^h\|_V^2 \le \langle Au-Au^h, u-v^h\rangle + R_u(v^h,u) + R_u(v,u^h) + I_\Phi(u^h,v^h) + I_\psi(v,v^h), \tag{4.8}$$

where

$$\begin{aligned} R_u(v,w) :={}& \langle Au, v-w\rangle + \Phi(u,v) - \Phi(u,w) \\ &+ I_\Delta(\psi^0(\gamma_1 u,\gamma_2 u;\gamma_2 v-\gamma_2 w)) - \langle f, v-w\rangle, \end{aligned} \tag{4.9}$$

$$I_\Phi(u^h,v^h) := \Phi(u,u^h)+\Phi(u^h,v^h)-\Phi(u,v^h)-\Phi(u^h,u^h),$$

$$\begin{aligned} I_\psi(v,v^h) :={}& I_\Delta(\psi^0(\gamma_1 u,\gamma_2 u;\gamma_2 v-\gamma_2 u)+\psi^0(\gamma_1 u^h,\gamma_2 u^h;\gamma_2 v^h-\gamma_2 u^h) \\ &- \psi^0(\gamma_1 u,\gamma_2 u;\gamma_2 v^h-\gamma_2 u) - \psi^0(\gamma_1 u,\gamma_2 u;\gamma_2 v-\gamma_2 u^h)). \end{aligned}$$

Let us bound the first and the last two terms on the right hand side of (4.8). First, by (2.5),

$$\langle Au-Au^h, u-v^h\rangle \le L_A\|u-u^h\|_V\|u-v^h\|_V.$$

Apply (2.10) to see, for an arbitrary $\varepsilon>0$ and a constant c depending on ε,

$$\langle Au-Au^h, u-v^h\rangle \le \varepsilon\,\|u-u^h\|_V^2 + c\,\|u-v^h\|_V^2. \tag{4.10}$$

By (2.1), we have

$$\begin{aligned} I_\Phi(u^h,v^h) &\le \alpha_\Phi\|u-u^h\|_V\|u^h-v^h\|_V \\ &\le \alpha_\Phi\left(\|u-u^h\|_V^2 + \|u-u^h\|_V\|u-v^h\|_V\right). \end{aligned}$$

Then, apply the inequality (2.10) on the term $\alpha_\Phi\|u-u^h\|_V\|u-v^h\|_V$ to obtain

$$I_\Phi(u^h,v^h) \le (\alpha_\Phi+\varepsilon)\,\|u-u^h\|_V^2 + c\,\|u-v^h\|_V^2 \tag{4.11}$$

for another constant c depending on $\varepsilon > 0$. By the subadditivity of the generalized directional derivative, we have

$$\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v - \gamma_2 u) \le \psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v - \gamma_2 u^h) + \psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 u^h - \gamma_2 u),$$
$$\psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 v^h - \gamma_2 u^h) \le \psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 v^h - \gamma_2 u) + \psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 u - \gamma_2 u^h).$$

Thus

$$I_\psi(v, v^h) \le I_\Delta(\psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 v^h - \gamma_2 u)) - I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v^h - \gamma_2 u))$$
$$+ I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 u^h - \gamma_2 u) + \psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 u - \gamma_2 u^h)).$$

By (4.1),

$$\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 u^h - \gamma_2 u) + \psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 u - \gamma_2 u^h) \le \alpha'_{\psi,1} |\gamma_1 (u - u^h)|_{\mathbb{R}^{m_1}} |\gamma_2 (u - u^h)|_{\mathbb{R}^{m_2}}$$
$$+ \alpha'_{\psi,2} |\gamma_2 (u - u^h)|^2_{\mathbb{R}^{m_2}}.$$

Then,

$$I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 u^h - \gamma_2 u) + \psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 u - \gamma_2 u^h))$$
$$\le \alpha'_{\psi,1} \|\gamma_1 (u - u^h)\|_{L^2(\Delta;\mathbb{R}^{m_1})} \|\gamma_2 (u - u^h)\|_{L^2(\Delta;\mathbb{R}^{m_2})} + \alpha'_{\psi,2} \|\gamma_2 (u - u^h)\|^2_{L^2(\Delta;\mathbb{R}^{m_2})}.$$

By (4.2),

$$\left|\psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 v^h - \gamma_2 u)\right| \le c \left(1 + |\gamma_1 u^h|_{\mathbb{R}^{m_1}} + |\gamma_2 u^h|_{\mathbb{R}^{m_2}}\right) |\gamma_2 (v^h - u)|_{\mathbb{R}^{m_2}},$$
$$\left|\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v^h - \gamma_2 u)\right| \le c\,(1 + |\gamma_1 u|_{\mathbb{R}^{m_1}} + |\gamma_2 u|_{\mathbb{R}^{m_2}})\, |\gamma_2 (v^h - u)|_{\mathbb{R}^{m_2}}.$$

Then,

$$\left|I_\Delta(\psi^0(\gamma_1 u^h, \gamma_2 u^h; \gamma_2 v^h - \gamma_2 u))\right|$$
$$\le c \left(1 + \|\gamma_1 u^h\|_{L^2(\Delta;\mathbb{R}^{m_1})} + \|\gamma_2 u^h\|_{L^2(\Delta;\mathbb{R}^{m_2})}\right) \|\gamma_2 (v^h - u)\|_{L^2(\Delta;\mathbb{R}^{m_2})}$$
$$\le c \left(1 + \|u^h\|_V\right) \|\gamma_2 (v^h - u)\|_{L^2(\Delta;\mathbb{R}^{m_2})}.$$

Similarly,

$$\left|I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v^h - \gamma_2 u))\right| \le c\,(1 + \|u\|_V)\, \|\gamma_2 (v^h - u)\|_{L^2(\Delta;\mathbb{R}^{m_2})}.$$

Combining the above four inequalities and using the fact that $\|u^h\|_V$ is bounded, we find that

$$I_\psi(v, v^h) \le \left(\alpha'_{\psi,1} \|\gamma_1\|\, \|\gamma_2\| + \alpha'_{\psi,2} \|\gamma_2\|^2\right) \|u - u^h\|^2_V + c\, \|\gamma_2 (u - v^h)\|_{L^2(\Delta;\mathbb{R}^{m_2})} \tag{4.12}$$

for some constant $c > 0$ independent of h. Using (4.10), (4.11), and (4.12) in (4.8), we have

$$\left(m_A - \alpha_\Phi - \alpha'_{\psi,1} \|\gamma_1\|\, \|\gamma_2\| - \alpha'_{\psi,2} \|\gamma_2\|^2 - 2\,\varepsilon\right) \|u - u^h\|^2_V$$
$$\le c\, \|u - v^h\|^2_V + c\, \|\gamma_2 (u - v^h)\|_{L^2(\Delta;\mathbb{R}^{m_2})} + R_u(v^h, u) + R_u(v, u^h).$$

Recall the smallness condition (4.3), we can take

$$\varepsilon = \left(m_A - \alpha_\Phi - \alpha'_{\psi,1} \|\gamma_1\|\, \|\gamma_2\| - \alpha'_{\psi,2} \|\gamma_2\|^2\right) / 4 > 0$$

and obtain the inequality

$$\|u-u^h\|_V^2 \le c \inf_{v^h\in K^h}\Big[\|u-v^h\|_V^2+\|\gamma_2(u-v^h)\|_{L^2(\Delta;\mathbb{R}^{m_2})}+R_u(v^h,u)\Big]+c\inf_{v\in K}R_u(v,u^h). \quad (4.13)$$

We summarize the result in the form of a theorem.

Theorem 4.3. *Assume $H(K)$, $H(A)$, $H(\Phi)_2$, $H(\psi)_2'$, $H(f)$, and (4.3). Then for the solution u of Problem 4.1 and the solution u^h of Problem 4.2, we have the Céa-type inequality (4.13).*

To proceed further, we need to bound the residual term (4.9) and this depends on the problem to be solved. We illustrate this point in Sections 6 in the context of a contact problem.

5. A Contact Problem

The contact problem we consider in this section describes the deformation of an elastic body that is fixed on a part of its boundary, is acted upon by body forces and surface tractions and is or may arrive in contact with a foundation.

Denote by $\Omega\subset\mathbb{R}^d$ ($d=2,3$) the reference configuration of the body. We assume that Ω is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ that is split as follows:

$$\partial\Omega=\overline{\Gamma_D}\cup\overline{\Gamma_N}\cup\overline{\Gamma_C},$$

where Γ_D, Γ_N, and Γ_C are mutually disjoint relatively open subsets. We assume meas$(\Gamma_D)>0$ and meas$(\Gamma_C)>0$, yet allow Γ_N to be empty. We use ν for the unit outward normal vector on $\partial\Omega$. The displacement variable is an $\mathbb{R}^d$-valued function $u\colon\Omega\to\mathbb{R}^d$ with the components u_i, $1\le i\le d$. We adopt the summation convention over a repeated index. Over $\mathbb{R}^d$, we use the canonical inner product

$$u\cdot v=u_iv_i,\quad u,v\in\mathbb{R}^d.$$

The stress tensor σ is a $\mathbb{S}^d$-valued function in Ω, where $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner product over $\mathbb{S}^d$ is

$$\sigma:\tau=\sigma_{ij}\tau_{ij},\quad \sigma=(\sigma_{ij}),\tau=(\tau_{ij})\in\mathbb{S}^d.$$

The linearized strain tensor associated to a differentiable (in the classical sense or the weak sense) displacement field u is the $\mathbb{S}^d$-valued function given by $\varepsilon(u)=(\nabla u+(\nabla u)^T)/2$. Here and below, to simplify the notation, we usually do not indicate explicitly the dependence of various functions on the spatial variable $x\in\Omega\cup\partial\Omega$. For a vector field v, its normal and tangential components on the boundary $\partial\Omega$ are defined as $v_\nu=v\cdot\nu$ and $v_\tau=v-v_\nu\nu$. For a tensor field σ, its normal and tangential components on $\partial\Omega$ are defined by $\sigma_\nu=(\sigma\nu)\cdot\nu$ and $\sigma_\tau=\sigma\nu-\sigma_\nu\nu$.

The pointwise formulation of the contact problem we consider is the following

Problem 5.1. *Find a displacement field $u:\Omega\to\mathbb{R}^d$ and a stress field $\sigma:\Omega\to\mathbb{S}^d$ such that*

$$\sigma=\mathscr{F}\varepsilon(u)\quad\text{in}\quad\Omega, \quad (5.1)$$

$$\operatorname{Div}\sigma+f_0=0\quad\text{in}\quad\Omega, \quad (5.2)$$

$$u=0\quad\text{on}\quad\Gamma_D, \quad (5.3)$$

$$\sigma\nu=f_N\quad\text{on}\quad\Gamma_N, \quad (5.4)$$

$$-\sigma_\nu=p_\nu(u_\nu-g_0)\quad\text{on}\quad\Gamma_C, \quad (5.5)$$

$$-\sigma_\tau\in p_\tau(u_\nu-g_0)\partial\psi_\tau(u_\tau)\quad\text{on}\quad\Gamma_C. \quad (5.6)$$

Let us briefly comment on the equations and boundary conditions in Problem 5.1. Equation (5.1) represents the constitutive law of an elastic material, in which $\mathscr{F}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$ is the elasticity operator, assumed to satisfy the condition

$$\left\{\begin{array}{l}\text{(a) there exists a constant } L_{\mathscr{F}}>0 \text{ such that}\\ \qquad \|\mathscr{F}(x,\varepsilon_1)-\mathscr{F}(x,\varepsilon_2)\|\le L_{\mathscr{F}}\|\varepsilon_1-\varepsilon_2\|\\ \qquad\quad \text{for all } \ \varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega;\\ \text{(b) there exists a constant } m_{\mathscr{F}}>0 \text{ such that}\\ \qquad (\mathscr{F}(x,\varepsilon_1)-\mathscr{F}(x,\varepsilon_2)):(\varepsilon_1-\varepsilon_2)\ge m_{\mathscr{F}}\,\|\varepsilon_1-\varepsilon_2\|^2\\ \qquad\quad \text{for all } \ \varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega;\\ \text{(c) } \mathscr{F}(\cdot,\varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon\in\mathbb{S}^d;\\ \text{(d) } \mathscr{F}(x,0)=0 \ \text{ for a.e. } x\in\Omega.\end{array}\right. \tag{5.7}$$

Equation (5.2) is the equation of equilibrium in which f_0 represents a given density of the external body force, while conditions (5.3) and (5.4) represent the displacement and the traction boundary conditions, respectively. Condition (5.3) reflects the fact that the body is clamped on Γ_D, and condition (5.4) describes the force boundary condition on Γ_N, f_N being a given density of the surface traction.

On the contact boundary Γ_C, along the normal direction, the contact condition is (5.5), in which p_ν is a given normal compliance function, and g_0 denotes the initial gap between the body and a foundation. Such contact conditions were introduced in [29] and then used in a large number of papers, including [30, 31]. Along the tangential direction, the friction law is (5.6), in which p_τ and ψ_τ are given functions and $\partial\psi_\tau$ represents the generalized gradient (or the Clarke subdifferential) of the function ψ_τ. Conditions (5.5) and (5.6) are of general forms, and they include many particular contact conditions and friction laws as special cases, as explained in [4, Section 6.3]. Note that the functions p_ν and p_τ in these conditions are supposed to vanish for a negative argument. This restriction is imposed from physical reasons since it reflects the fact that when there is separation between the body and the foundation then the reaction of the foundation vanishes.

For the normal compliance function $p_\nu\colon \Gamma_C\times\mathbb{R}\to\mathbb{R}$, we assume that

$$\left\{\begin{array}{l}\text{(a) there exists a constant } L_{p_\nu}\ge 0 \text{ such that}\\ \qquad |p_\nu(x,r_1)-p_\nu(x,r_2)|\le L_{p_\nu}|r_1-r_2|\\ \qquad\quad \text{for all } r_1,\,r_2\in\mathbb{R}, \text{ a.e. } x\in\Gamma_C;\\ \text{(b) } p_\nu(\cdot,r) \text{ is measurable on } \Gamma_C \text{ for all } r\in\mathbb{R};\\ \text{(c) } p_\nu(x,r)=0 \ \text{ for a.e. } x\in\Gamma_C, \text{ all } r\le 0.\end{array}\right. \tag{5.8}$$

Moreover, for the functions $\psi_\tau\colon \Gamma_C\times\mathbb{R}^d\to\mathbb{R}$ and $p_\tau\colon \Gamma_C\times\mathbb{R}\to\mathbb{R}$, we assume that

$$\left\{\begin{array}{l}\text{(a) } \psi_\tau(\cdot,\xi) \text{ is measurable on } \Gamma_C \text{ for all } \xi\in\mathbb{R}^d \text{ and there}\\ \qquad \text{exists } \bar{e}_\tau\in L^2(\Gamma_C)^d \text{ such that } \psi_\tau(\cdot,\bar{e}_\tau(\cdot))\in L^1(\Gamma_C);\\ \text{(b) } \psi_\tau(x,\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } x\in\Gamma_C;\\ \text{(c) } |\partial\psi_\tau(x,\xi)|\le\bar{c}_{0\tau} \text{ for a.e. } x\in\Gamma_C, \text{ for } \xi\in\mathbb{R}^d \text{ with } \bar{c}_{0\tau}\ge 0;\\ \text{(d) } \psi_\tau^0(x,\xi_1;\xi_2-\xi_1)+\psi_\tau^0(x,\xi_2;\xi_1-\xi_2)\le\alpha_{\psi_\tau}\|\xi_1-\xi_2\|^2\\ \qquad \text{for a.e. } x\in\Gamma_C, \text{ all } \xi_1,\,\xi_2\in\mathbb{R}^d \text{ with } \alpha_{\psi_\tau}\ge 0;\end{array}\right. \tag{5.9}$$

$$\begin{cases} \text{(a) there exists a constant } c_{p_\tau} \geq 0 \text{ such that} \\ \qquad |p_\tau(x,r_1) - p_\tau(x,r_2)| \leq c_{p_\tau} |r_1 - r_2| \\ \qquad\quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C; \\ \text{(b) } p_\tau(\cdot, r) \text{ is measurable on } \Gamma_C \text{ for all } r \in \mathbb{R}; \\ \text{(c) } 0 \leq p_\tau(x,r) \leq \overline{p}_\tau \text{ for all } r \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C \text{ with } \overline{p}_\tau \geq 0; \\ \text{(d) } p_\tau(x,r) = 0 \text{ for a.e. } x \in \Gamma_C, \text{ all } r \leq 0. \end{cases} \tag{5.10}$$

We note that (5.9) (b) and (c) are equivalent to the property that $\psi_\tau(x,\cdot)$ is Lipschitz continuous on $\mathbb{R}^d$ for a.e. $x \in \Gamma_C$ with a Lipschitz constant $\bar{c}_{0\tau}$.

Finally, we assume that the rest of the data are such that

$$f_0 \in L^2(\Omega;\mathbb{R}^d), \quad f_N \in L^2(\Gamma_N;\mathbb{R}^d), \tag{5.11}$$

$$g_0 \in L^2(\Gamma_C), \quad g_0(x) \geq 0 \quad \text{a.e. } x \in \Gamma_C. \tag{5.12}$$

In the variational analysis of Problem 5.1, we use the function space

$$V = \left\{ v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \right\},$$

which is a Hilbert space with the inner product

$$(u,v)_V = \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx.$$

Assume that (u,σ) are sufficiently regular functions which satisfy (5.1)–(5.6), and let $v \in V$. Then, using standard arguments based on integration by parts and the definition of the generalized gradient, it follows that $u \in V$ and

$$\int_\Omega \mathscr{F}(\varepsilon(u)) \cdot \varepsilon(v)\, dx + \int_{\Gamma_C} \left[p_\nu(u_\nu - g_0)\, v_\nu + p_\tau(u_\nu - g_0)\, \psi_\tau^0(u_\tau; v_\tau) \right] da$$
$$\geq \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_N} f_N \cdot v\, da.$$

Define the operator $A\colon V \to V^*$, the function $\Phi : V \times V \to \mathbb{R}$ and the element $f \in V^*$ as follows:

$$\langle Au, v\rangle = \int_\Omega \mathscr{F}\varepsilon(u) \cdot \varepsilon(v)\, dx, \quad \forall u, v \in V, \tag{5.13}$$

$$\Phi(u,v) = \int_{\Gamma_C} p_\nu(u_\nu - g_0) v_\nu\, da, \quad \forall u, v \in V, \tag{5.14}$$

$$\langle f, v\rangle = \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_N} f_N \cdot v\, da, \quad \forall v \in V. \tag{5.15}$$

Then, using notation (5.13)–(5.15), we deduce the following variational formulation of the contact problem (5.1)–(5.6).

Problem 5.2. *Find a displacement field $u \in V$ such that*

$$\langle Au, v\rangle + \Phi(u,v) + \int_{\Gamma_C} p_\tau(u_\nu - g_0)\, \psi_\tau^0(u_\tau; v_\tau)\, da \geq \langle f, v\rangle, \quad \forall v \in V.$$

For the part of error estimation in numerical analysis later, we let $m_1 = 1$, $m_2 = d$, and let $\gamma_1 \colon V \to L^2(\Gamma_C)$ and $\gamma_2 \colon V \to L^2(\Gamma_C;\mathbb{R}^d)$ be the normal component and tangential component trace operators on V, that is,

$$\gamma_1 v = v_\nu,\ \gamma_2 v = v_\tau, \quad \forall v \in V.$$

We have $\|\gamma_1\| = \lambda_\nu^{-1/2}$ and $\|\gamma_2\| = \lambda_\tau^{-1/2}$, where $\lambda_\nu > 0$ is the smallest eigenvalue of the eigenvalue problem

$$u \in V, \quad \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx = \lambda \int_{\Gamma_C} u_\nu v_\nu\, da, \quad \forall v \in V,$$

and $\lambda_\tau > 0$ is the smallest eigenvalue of the eigenvalue problem

$$u \in V, \quad \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx = \lambda \int_{\Gamma_C} u_\tau \cdot v_\tau\, da, \quad \forall v \in V.$$

6. Analysis and Approximation

In this section we apply the abstract results proved in Sections 3 and 4 in the study of inequality 5.2. We start with the following well-posedness result.

Theorem 6.1. *Assume* (5.7)–(5.12) *and*

$$L_{p_\nu}\lambda_\nu^{-1} + \overline{p}_\tau m_\tau \lambda_\tau^{-1} + c_{p_\tau} c_{\psi_\tau} (\lambda_\nu \lambda_\tau)^{-1/2} < m_{\mathscr{F}}. \tag{6.1}$$

Then, Problem 5.2 has a unique solution $u \in V$. Moreover, the operator $(f_0, f_N) \mapsto u = u(f_0, f_N)$ which maps any element $(f_0, f_N) \in L^2(\Omega, \mathbb{R}^d) \times L^2(\Gamma_N, \mathbb{R}^d)$ to the solution $u \in V$ of Problem 5.2 is Lipschitz continuous.

Proof. We apply Theorem 4.1. To this end, let $K = V$, $\Delta = \Gamma_C$, and define the function

$$\psi(y,z) = p_\tau(y - g_0)\, \psi_\tau(z), \quad y \in \mathbb{R}, z \in \mathbb{R}^d.$$

It is easy to see that $H(K)$, $H(A)$, and $H(f)$ hold, and the strong monotonicity constant of the operator A is $m_A = m_{\mathscr{F}}$. The rest of the conditions can be verified by using arguments similar to those used in [7]. However, for completeness, since the notation is different, we provide the details in proof.

Let us check the validity of the condition $H(\Phi)_2$. It is easy to see that $\Phi(u,\cdot) : V \to \mathbb{R}$ is a convex continuous function. For u_1, u_2, $v_1, v_2 \in V$, we have

$$p_\nu(u_{1,\nu} - g_0)(v_{2,\nu} - v_{1,\nu}) + p_\nu(u_{2,\nu} - g_0)(v_{1,\nu} - v_{2,\nu}) \le L_{p_\nu} |u_{1,\nu} - u_{2,\nu}||v_{1,\nu} - v_{2,\nu}|$$

a.e. on Γ_C. Thus

$$\int_{\Gamma_C} [p_\nu(u_{1,\nu} - g_0)(v_{2,\nu} - v_{1,\nu}) + p_\nu(v_{2,\nu} - g_0)(v_{1,\nu} - v_{2,\nu})]\, da$$
$$\le L_{p_\nu} \|u_{1,\nu} - u_{2,\nu}\|_{L^2(\Gamma_C)} \|v_{1,\nu} - v_{2,\nu}\|_{L^2(\Gamma_C)} \le L_{p_\nu} \lambda_\nu^{-1} \|u_1 - u_2\|_V^2.$$

Hence, the function Φ satisfies condition $H(\Phi)_2$ with

$$\alpha_\Phi = L_{p_\nu} \lambda_\nu^{-1}. \tag{6.2}$$

Then we turn to condition $H(\psi)_2'$. Obviously, ψ is locally Lipschitz continuous with respect to its second argument and

$$\psi^0(y, z_1; z_2) = p_\tau(y - g_0)\, \psi_\tau^0(z_1; z_2), \quad \forall y \in \mathbb{R},\ z_1, z_2 \in \mathbb{R}^d.$$

Regarding the condition (4.1), for $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$, we write

$$\begin{aligned} &p_\tau(y_1 - g_0)\, \psi_\tau^0(z_1; z_2 - z_1) + p_\tau(y_2 - g_0)\, \psi_\tau^0(z_2; z_1 - z_2) \\ &\quad = [p_\tau(y_2 - g_0) - p_\tau(y_1 - g_0)]\, \psi_\tau^0(z_2; z_1 - z_2) \\ &\qquad + p_\tau(y_1 - g_0) \left[\psi_\tau^0(z_1; z_2 - z_1) + \psi_\tau^0(z_2; z_1 - z_2)\right]. \end{aligned}$$

By assumption (5.10),

$$\begin{aligned} &0 \le p_\tau(x, r) \le \overline{p}_\tau, \\ &|p_\tau(y_1 - g_0) - p_\tau(y_2 - g_0)| \le c_{p_\tau} |y_1 - y_2|. \end{aligned}$$

By the Lipschitz continuity of ψ_τ,

$$\left|\psi_\tau^0(z_2; z_1 - z_2)\right| \le c_{\psi_\tau} |z_1 - z_2|.$$

Thus

$$\begin{aligned} &p_\tau(y_1 - g_0)\, \psi_\tau^0(z_1; z_2 - z_1) + p_\tau(y_2 - g_0)\, \psi_\tau^0(z_2; z_1 - z_2) \\ &\quad \le c_{p_\tau} c_{\psi_\tau} |y_1 - y_2|\, |z_1 - z_2| + \overline{p}_\tau m_\tau |z_1 - z_2|^2. \end{aligned}$$

Hence, (4.1) holds:

$$\psi^0(y_1, z_1; z_2 - z_1) + \psi^0(y_2, z_2; z_1 - z_2) \le \alpha'_{\psi,1} |y_1 - y_2|\, |z_1 - z_2|_{\mathbb{R}^d} + \alpha'_{\psi,2} |z_1 - z_2|^2_{\mathbb{R}^d}.$$

where

$$\alpha'_{\psi,1} = c_{p_\tau} c_{\psi_\tau}, \quad \alpha'_{\psi,2} = \overline{p}_\tau m_\tau. \tag{6.3}$$

We combine (6.2), (6.3), and (6.1) to see that condition (4.3) is satisfied. Theorem 6.1 is now a direct consequence of Theorem 4.1. □

We now consider the numerical solution of Problem 5.2 by the finite element method. For simplicity, we assume that Ω is a polygonal/tetrahedral domain. Let $\{\mathscr{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into Γ_D, Γ_N, and Γ_C, in the sense that if the intersection of one side/face of an element with one of the three sets has a positive surface measure, then the side/face lies entirely in that set. We now construct linear element spaces corresponding to $\mathscr{T}^h$:

$$V^h = \left\{ v^h \in C(\overline{\Omega})^d \mid v^h|_T \in \mathbb{P}_1(T)^d \text{ for } T \in \mathscr{T}^h,\ v^h = 0 \text{ on } \Gamma_D \right\}, \tag{6.4}$$

where $\mathbb{P}_1(T)$ stands for the space of polynomials of a degree less than or equal to 1 on T. Then, the finite element method for Problem 5.2 is as follows.

Problem 6.1. *Find a displacement field $u^h \in V^h$ such that*

$$\langle Au^h, v^h\rangle + \Phi(u^h, v^h) + \int_{\Gamma_C} p_\tau(u^h_\nu - g_0)\, \psi_\tau^0(u^h_\tau; v^h_\tau)\, da \ge \langle f, v^h\rangle, \quad \forall v^h \in V^h. \tag{6.5}$$

By a discrete analogue of Theorem 6.1, under the conditions stated in that theorem, Problem 6.1 has a unique solutions.

The properties (2.8)–(2.9) are obvious for the choice of V^h and $K^h = V^h$. By [6, Proposition 2.1.2] and the fact that a convergent sequence in V has a subsequence that converges pointwise on Γ_C, (4.6) is valid. Thus we can apply Theorem 4.2 to conclude the convergence of the numerical solutions

$$u^h \to u \text{ in } V \text{ as } h \to 0.$$

For error estimation, we note from (4.13) that, since $\inf_{v\in K} R_u(v,u^h)=0$,

$$\|u-u^h\|_V^2 \le c \inf_{v^h\in V^h}\left[\|u-v^h\|_V^2+\|u_\tau-v_\tau^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}+R_u(v^h,u)\right], \tag{6.6}$$

where

$$R_u(v^h,u)=\langle Au,v^h-u\rangle+\Phi(u,v^h-u)+\int_{\Gamma_C} p_\tau(u_\nu-g_0)\,\psi_\tau^0(u_\tau;v_\tau^h-u_\tau)\,da \\ -\langle f,v^h-u\rangle. \tag{6.7}$$

Assume the solution regularity conditions

$$u\in H^2(\Omega;\mathbb{R}^d),\quad \sigma=\mathscr{F}(\varepsilon(u))\in H^2(\Omega;\mathbb{S}^d). \tag{6.8}$$

We comment that in case $\mathscr{F}$ depends smoothly on x, the first condition in (6.8) implies the second condition in (6.8). Moreover, the second condition in (6.8) implies that

$$\sigma\nu\in L^2(\partial\Omega;\mathbb{R}^d).$$

Under the regularity conditions (6.8), arguments similar to those used in [22, Section 7] indicate that

$$\operatorname{Div}\sigma+f_0=0\quad \text{a.e. in }\Omega,$$
$$\sigma\nu=f_N\quad \text{a.e. on }\Gamma_N,$$

and then, for any $v^h\in V^h$,

$$\langle Au,v^h-u\rangle+\Phi(u,v^h-u)=\langle f,v^h-u\rangle+\int_{\Gamma_C}\sigma_\tau\cdot(v_\tau^h-u_\tau)\,da.$$

Hence, we have

$$R_u(v^h,u)=\int_{\Gamma_C} p_\tau(u_\nu-g_0)\,\psi_\tau^0(u_\tau;v_\tau^h-u_\tau)\,da+\int_{\Gamma_C}\sigma_\tau\cdot(v_\tau^h-u_\tau)\,da.$$

By the assumptions

$$|p_\tau(u_\nu-g_0)|\le \overline{p}_\tau,$$
$$\left|\psi_\tau^0(u_\tau;v_\tau^h-u_\tau)\right|\le \overline{c}_{0\tau}\left|v_\tau^h-u_\tau\right|,$$

since $\sigma_\tau\in L^2(\Gamma_C;\mathbb{R}^d)$, we have

$$\left|R_u(v^h,u)\right|\le c\,\|v_\tau^h-u_\tau\|_{L^2(\Gamma_C;\mathbb{R}^d)},$$

with a constant c depending on $\|\sigma_\tau\|_{L^2(\Gamma_C;\mathbb{R}^d)}$. Therefore, (6.6) implies that

$$\|u-u^h\|_V\le c\inf_{v^h\in V^h}\left[\|u-v^h\|_V+\|u_\tau-v_\tau^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}^{1/2}\right]. \tag{6.9}$$

We remark that in the literature on error analysis of numerical solutions of variational inequalities, it is standard that the Céa-type inequalities involve square root of approximation error of the solution in certain norms due to the inequality form of the problems. A reference in the field is [32], for instance.

Express the contact boundary $\overline{\Gamma_C}$ as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_C}=\cup_{i=1}^{i_C}\Gamma_{C,i}.$$

In addition to the solution regularity conditions (6.8), we assume further that

$$u|_{\Gamma_{C,i}} \in H^2(\Gamma_{C,i};\mathbb{R}^d),\ 1 \le i \le i_C. \tag{6.10}$$

Then, we can apply the standard finite element interpolation error estimates ([33, 34]) to derive from (6.9) the following optimal order error bound:

$$\|u - u^h\|_V \le c\,h. \tag{6.11}$$

We summarize the above derivation in the form of a theorem.

Theorem 6.2. *Assume* (5.7)–(5.12) *and* (6.1). *Let* V^h *be the linear finite element space defined by* (6.4) *corresponding to* $\mathscr{T}^h$ *from a regular family of partitions of* $\overline{\Omega}$ *into triangles/tetrahedrons that are compatible with the partition of the boundary* $\partial\Omega$ *into* Γ_D, Γ_N, *and* Γ_C. *Then, under the solution regularity conditions* (6.8) *and* (6.10), *for the finite element solution* $u^h \in V^h$ *of Problem 6.1, we have the optimal order error estimate* (6.11).

Acknowledgments

This work was partially supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH, and by Simons Foundation Collaboration Grants, No. 850737.

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