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OPTIMAL PAYOFFS FOR DIRECTIONALLY CLOSED ACCEPTANCE SETS

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Abstract. Acceptance sets are used for modeling regulatory preconditions for financial institutes. In this paper, we consider directionally closed acceptance sets in the linear space of capital positions. Assuming finitely many eligible assets, the decision maker of a financial institution has to decide how to invest into these assets to secure acceptability for the financial position, meaning that the resulting capital position belongs to the acceptance set. We study the risk measure that describes the costs for reaching acceptability in the linear space of capital positions. The aim of our paper is to derive a general characterization of the solution set of the optimal payoff map based on our previously published results concerning the properties of the risk measure. Furthermore, we also give some more details about (weakly) efficient points of the acceptance set.

Keywords. Financial positions; Investments; Ordering cones; Risk measures; Translation invariance; Weakly efficient points.

1. Introduction

In financial mathematics, risk measures with respect to acceptance sets have been studied for many years. Financial institutions face many different risks like credit risk (or default risk), market risk, or liquidity risk, which are issue of many research; see, e.g., Ghabri et al. [1] for a study about the influence of Bitcoin on liquidity risk or Redeker and Wunderlich [2] for a study of credit risk with asymmetric information and non-constant default threshold. Consequently, there are many different ways of modeling or measuring risk; see [3]. For example, the term "risk" could be identified with the extend of the deviation of an outcome from an expected value, but it could also be understood as the valuation of the potential of a possible probability-based loss. Given a space of financial positions $\mathscr{X}$ like $\mathscr{L}^\infty$, (monetary) risk measures $\rho\colon \mathscr{X} \to \mathbb{R}$ are common for modeling risk (see Artzner et al. [4]), for example the Value-at-Risk (see, e.g., Pritsker [5]) and the Conditional-Value-at-Risk, sometimes known as Expected Shortfall (see, e.g., Rockafellar and Uryasev [6], [7], and Acerbi and Tasche [8, 9]). A famous and practical class of risk measures are coherent risk measures (see Artzner et al. [4]), and, more generally, convex risk measures (see Föllmer and Schied [10]). Since the first models of modern portfolio theory (especially, the mean-variance-optimization model introduced by Markowitz [11]), many extended models have been invented; see, e.g., Ehrgott et al. [12]. With the increased interest in

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general risk measures, these measures play an important role in practical portfolio optimization problems nowadays, either as a part of the objective function (see, e.g., Gaivoronski and Pflug [13]), or in the sense of a risk constraint (see, e.g., Gabih et al. [14], or Akume et al. [15]).

In recent years, regulatory preconditions have much more influence on the way of measuring and managing risk than ever before. One origin can be found in the massive failures of banks and misbehavior of financial institutions in dealing with risk which have been revealed during the financial crisis 2007 and afterwards, see, e.g., Hull [16] for details about the crisis itself, and Vazquez and Federico [17] for an empirical study about some origins of failures. Consequently, by expanded regulatories like Basel III (see [18] and [19]), there are much more restrictions that institutions have to fulfill to secure their survivability, but also to avoid institutional and systemic effects.

Artzner et al. proved in [4] an one-to-one-correspondence between coherent risk measures and convex, closed acceptance sets which was generalized by Föllmer and Schied in [20] to monetary risk measures and general acceptance sets. Thus, there is a direct connection between risk measures and acceptance sets. Obviously, it is from practical interest to find effective algorithms to solve the resulting portfolio problems (see, e.g., Feng et al. in [21]). To do so, it is important to study the mathematical properties of the solution set, first. Following Baes et al. in [22], this paper focuses on the following optimization problem:

$$\pi(Z) \to \min_{X+Z\in\mathscr{A},\, Z\in\mathscr{M}}, \qquad (P_\pi(X))$$

where $\mathscr{X}$ is a real linear space, $\mathscr{A} \subseteq \mathscr{X}$ is an acceptance set, $\pi\colon \mathscr{M} \to \mathbb{R}$ is a pricing functional and $\mathscr{M} \subseteq \mathscr{X}$ is a subspace of $\mathscr{X}$. Our aim is to generalize the results from Baes et al. [22], Marohn and Tammer [23] to real linear spaces $\mathscr{X}$ by weaker assumptions on the acceptance set $\mathscr{A}$. In mathematical finance, especially in arbitrage theory (see the class of financial market models in [24, Section 2] and [25]), it is of interest to consider general real linear spaces of financial positions $\mathscr{X}$ instead of topological linear spaces $\mathscr{X}$ to improve the applicability for practical purposes. In the literature concerning monetary measures of risk, the class of capital positions is supposed to be a linear space $\mathscr{X}$ of bounded functions containing the constants (see Föllmer and Schied [20]). Sometimes, it is not convenient to consider $\mathscr{X}$ endowed with the supremum norm (inducing the topology) such that it is preferable to consider the linear space $\mathscr{X}$. Especially, an industrial user or economic researcher who is working with financial data (e.g. samples) might not know which topology to choose that is suitable for the corresponding situation. Hence, if the topology does not matter for deriving suitable results for practical problems, the user can directly apply these outcomes more easily which does not pose the danger of generating an lack of interest by unnecessary mathematical and non-economical assumptions.

The aim of this paper is to use our results from [26] for the risk measure $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ given by (see Baes et al. [22] and Farkas et al. [27])

$$\rho_{\mathscr{A},\mathscr{M},\pi}(X) := \inf\{\pi(Z) \mid X+Z \in \mathscr{A}, Z \in \mathscr{M}\}$$

to derive a characterization of the solution set for $X \in \mathscr{X}$

$$\mathscr{E}(X) := \left\{Z \in \mathscr{M} \mid X+Z \in \mathscr{A}, \pi(Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X)\right\} \qquad (1.1)$$

when the acceptance set $\mathscr{A} \subseteq \mathscr{X}$ is a subset of the linear space of capital positions $\mathscr{X}$ and directionally closed. We call the set-valued map $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ optimal (eligible) payoff map and each $Z \in \mathscr{E}(X)$ is known as optimal (eligible) payoff for X. The optimal payoff map was introduced and studied in [22] for closed acceptance sets under stronger assumptions. After these studies about the solution set $\mathscr{E}(X)$, we will give a short outlook how these results can be used to extend our observations for efficient and weakly efficient points in [23] to the setting in this paper by use of a more common definition of the efficient set.

The paper is organized as follows. We give a short overview about the needed mathematical background, especially with respect to directional properties of sets in Section 2. In Section 3, we describe the used financial model and recall useful properties of the risk measure proved in [26]. For directional closed acceptance sets $\mathscr{A}$, we derive a more general characterization of the solution set $\mathscr{E}(X)$ given by (1.1) in Section 4. In Section 5, we give an outlook on properties of (weakly) efficient points of directionally closed acceptance sets $\mathscr{A}$ by using cones that coincide with the geometric view on finding solutions of the optimization problem and the economic background. In contrast to our paper [23], we use a more common definition of efficient points and weaker assumptions. Section 6 ends this paper.

2. Preliminaries

In this article, certain standard notions and terminology are used. Let $\mathscr{X}$ be a *real linear space* with origin $\mathbf{0}$. We only consider real linear spaces and write "linear spaces" for simplicity. The extended set of real numbers is $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, the set of non-negative real numbers $\mathbb{R}_+$, the set of positive real numbers $\mathbb{R}_>$, and the set of non-positive real numbers $\mathbb{R}_-$. Take $\mathscr{A}, \mathscr{B} \subseteq \mathscr{X}$ arbitrary. Then, the *Minkowksi sum* of $\mathscr{A}$ and $\mathscr{B}$ is

$$\mathscr{A} + \mathscr{B} := \{X + Y \mid X \in \mathscr{A}, Y \in \mathscr{B}\}.$$

For simplicity, we often write $X + \mathscr{A}$ for $\{X\} + \mathscr{A}$ with $X \in \mathscr{X}$. We set

$$\lambda \mathscr{A} := \{\lambda X \mid X \in \mathscr{A}\} \text{ with } \lambda \in \mathbb{R},$$

which implies $\mathscr{A} - \mathscr{B} := \mathscr{A} + (-\mathscr{B}) = \{X - Y \mid X \in \mathscr{A}, Y \in \mathscr{B}\}$. The cardinality of $\mathscr{A}$ is denoted by $|\mathscr{A}|$. $\mathscr{A}$ is *convex* if $\forall X, Y \in \mathscr{A}$ and $\forall \lambda \in [0,1]$, $\lambda X + (1-\lambda)Y \in \mathscr{A}$ holds. A nonempty set $\mathscr{A}$ is called a *cone* if $\forall\, X \in \mathscr{A}$ and $\forall \lambda \geq 0$, $\lambda X \in \mathscr{A}$ is fulfilled. Let $\mathscr{C} \subseteq \mathscr{X}$ be a cone. We call $\mathscr{C}$ *pointed* if $\mathscr{C} \cap (-\mathscr{C}) = \{\mathbf{0}\}$ holds. The *recession cone* of $\mathscr{A}$ is

$$\operatorname{rec} \mathscr{A} := \{X \in \mathscr{X} \mid Y + \lambda X \in \mathscr{A} \;\text{ for all } Y \in \mathscr{A}, \lambda \in \mathbb{R}_+\}.$$

Take $K \in \mathscr{X} \setminus \{\mathbf{0}\}$ arbitrary. Following Gutiérrez et al. [28] (see also Martínez-Legaz et al. [29], and Qiu and He [30]), the *K-directional closure* of a nonempty subset $\mathscr{A} \subseteq \mathscr{X}$ is

$$\operatorname{cl}_K(\mathscr{A}) := \{X \in \mathscr{X} \mid \forall \lambda \in \mathbb{R}_> \;\exists t \in \mathbb{R}_+ \text{ with } t < \lambda \text{ and } X - tK \in \mathscr{A}\}$$

and we say that $\mathscr{A}$ is *K-directionally closed* if $\mathscr{A} = \operatorname{cl}_K(\mathscr{A})$ holds. It holds that (see Tammer and Weidner [31, Lemma 2.3.24])

$$\operatorname{cl}_K(\mathscr{A}) = \{X \in \mathscr{X} \mid \exists (t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+ : t_n \downarrow 0 \;\forall n \in \mathbb{N} : X - t_n K \in \mathscr{A}\}.$$

Furthermore, the *K-directional interior of* $\mathscr{A}$ is

$$\operatorname{int}_K(\mathscr{A}) := \{X \in \mathscr{X} \mid \exists \lambda \in \mathbb{R}_> \;\forall t \in [0,\lambda] : X + tK \in \mathscr{A}\}$$

and the *K-directional boundary of* $\mathscr{A}$ is $\operatorname{bd}_K(\mathscr{A}) := \operatorname{cl}_K(\mathscr{A}) \setminus \operatorname{int}_K(\mathscr{A})$.

Consider a map $f \colon \mathscr{X} \to \overline{\mathbb{R}}$. Then, we call f *linear* if the following holds:

$$\forall X, Y \in \mathscr{X}, \forall \lambda, \mu \in \mathbb{R} : \quad f(\lambda X + \mu Y) = \lambda f(X) + \mu f(Y).$$

If f is linear, then $\ker f := \{X \in \mathscr{X} \mid f(X) = 0\}$ is a subspace of $\mathscr{X}$ called *kernel* or *null space* of f. For any map $f\colon \mathscr{X} \to \overline{\mathbb{R}}$, the sets

$$\begin{aligned} \operatorname{lev}_{f,=}(\alpha) &:= \{X \in \mathscr{X} \mid f(X) = \alpha\}, \\ \operatorname{lev}_{f,\le}(\alpha) &:= \{X \in \mathscr{X} \mid f(X) \le \alpha\}, \\ \operatorname{lev}_{f,<}(\alpha) &:= \{X \in \mathscr{X} \mid f(X) < \alpha\} \end{aligned}$$

are called *level line*, *sublevel set*, and *strict sublevel set* of f to the level $\alpha \in \mathbb{R}$, respectively.

For our applications, it is necessary to compare elements $X, Y \in \mathscr{X}$; see [32] for the standard terminology. If $\mathscr{X}$ is partially ordered by $\le$, the natural ordering cone in $\mathscr{X}$ is given by the *positive cone* $\mathscr{X}_+ = \{X \in \mathscr{X} \mid \mathbf{0} \le X\}$, whose elements are called *positive*. The corresponding partial order that we consider here is also simplified denoted $\le$, i.e.,

$$\forall X, Y \in \mathscr{X}: \quad X \le Y \quad :\Longleftrightarrow \quad Y - X \in \mathscr{X}_+.$$

3. The Financial Market

First, we shortly present the framework we are working in. For details and more information to the economical background, we refer to, e.g., Marohn and Tammer [26] or Marohn [33]. The model is motivated by the works of Baes et al. [22] and Farkas et al. [27].

Let $\mathscr{X}$ be the space of capital positions which is assumed to be a linear space partially ordered by the positive cone $\mathscr{X}_+$. In general, $\mathscr{X}$ is a space of random variables. Thus, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, where Ω is the set of future states, $\mathscr{F}$ is a $\sigma-$Algebra on Ω, and $\mathbb{P}\colon \mathscr{F} \to [0,1]$ is a probability measure. We identify $X = Y$ for each $X, Y \in \mathscr{X}$ with $\mathbb{P}(X = Y) = 1$ and write as shortcut $X = c$ for a constant random variable $X = 1_\Omega$ with $c \in \mathbb{R}$ arbitrary and $1_\Omega \in \mathscr{X}$ being the random variable that equals 1 in every possible state $\omega \in \Omega$. The market is regulated by conditions on the capital positions that are summarized in so called acceptance sets (see Artzner et al. [4] and Baes et al. [22]).

Definition 3.1. Let $\mathscr{X}$ be a linear space. We call $\mathscr{A} \subseteq \mathscr{X}$ an *acceptance set* if $\mathscr{A}$ is a proper subset, i.e., $\mathscr{A} \neq \mathscr{X}$, fulfilling $\mathbf{0} \in \mathscr{A}$ and the following monotonicity property:

$$\mathscr{A} + \mathscr{X}_+ \subseteq \mathscr{A}, \tag{3.1}$$

i.e., $\mathscr{X}_+ \subseteq \operatorname{rec}(\mathscr{A})$ holds.

For remarks about the terminology of acceptance sets, compare, e.g., [26, Remark 4.2]. Note that (3.1) in Definition 3.1 directly implies

$$\mathscr{A} + \mathscr{X}_+ = \mathscr{A} \tag{3.2}$$

for an acceptance set $\mathscr{A}$ by $\mathbf{0} \in \mathscr{X}_+$. Property (3.2) is also known as *free disposal assumption* with respect to the ordering cone $\mathscr{X}_+$ which is a basic assumption in production theory; see, e.g., [34] for references related to the condition (3.2). Hence, an acceptance set $\mathscr{A}$ is also called *free disposal set*. Free disposal sets are widely used in optimization and mathematical economics, see, e.g., [35]. For $\mathscr{X}_+ = \mathbb{R}^k_+$ being the set of k-dimensional vectors with non-negative real components, sets fulfilling (3.2) are investigated under the terminology *downward set*, see, e.g., [29].

If the current capital position $X \in \mathscr{X}$ is not acceptable, i.e., $X \notin \mathscr{A}$, for a given acceptance set $\mathscr{A} \subseteq \mathscr{X}$, the decision maker or investor has to invest the capital in predefined *eligible assets* given by

$$S^i := (S_0^i, S_1^i)^T \in \mathbb{R} \times \mathscr{X} \quad \text{for } i = 0, 1, \ldots, n \tag{3.3}$$

with $n \in \mathbb{N}$ such that the resulting capital position is acceptable. For convenience, we set

$$S_j := (S_j^0, S_j^1, \ldots, S_j^n)^T, \qquad j \in \{0, 1\}. \tag{3.4}$$

for the vector collecting all prices or payoffs, respectively. We define the eligible asset $i = 0$ as a secure investment opportunity paying an interest rate $r \in \mathbb{R}_+$, namely

$$S^0 := (1, (1+r)1_\Omega)^T = (1, 1+r)^T. \tag{3.5}$$

Assuming a secure investment opportunity is justified by economical reasons and follows the most common models in modern financial economics, for example the Capital Asset Pricing Model (see Sharpe [36], Lintner [37], and Mossin [38]). For convenience, we set $r = 0$ in this paper, which, of course, is a major simplification as well as that there is only one secure investment opportunity independently from time horizon.

The allowed operations for changing the current capital position of an institute or investor can be summarized by the subspace

$$\mathscr{M} := \operatorname{span}\{S_1^i \in \mathscr{X} \mid i = 0, 1, \ldots, n\} \tag{3.6}$$

called *space of eligible payoffs*. Note that we have $\mathbb{R} = \operatorname{span} S_1^0 \subseteq \mathscr{M}$, since $m = mS_1^0$ holds for all $m \in \mathbb{R}$ by our assumption $r = 0$ for the secure investment opportunity S^0. Moreover, if $\mathscr{A} \subseteq \mathscr{X}$ is an acceptance set, it holds that $m = m \cdot 1_\Omega \in \mathscr{A}$ for all $m \in \mathbb{R}_+$ by Definition 3.1.

We consider an one period model, i.e., the investor buys (or sells) shares of the assets S^i ($i = 0, 1, \ldots, n$) with price $S_0^i \in \mathbb{R}$ for each share in $t = 0$ (today) and obtains some, in general random, payoff $S_1^i \in \mathscr{X}$ for each share holding in $t = 1$. The resulting capital position is $X + Z \in \mathscr{X}$ for the corresponding $Z \in \mathscr{M}$.

The following assumption on the financial market is typical:

Assumption 3.1. Let $\mathscr{M} \subseteq \mathscr{X}$ be the subspace given by (3.6). It holds the *Law of One Price*, i.e., for $Z \in \mathscr{M}$ arbitrary, there is some $c \in \mathbb{R}$ fulfilling $S_0^T x = c$ for all $x \in \mathbb{R}^{n+1}$ with $Z = S_1^T x$, where S_j with $j \in \{0, 1\}$ is defined by (3.4). Moreover, the *no-arbitrage principle* holds, i.e., there is no arbitrage opportunity. An arbitrage opportunity (see Remark 3.1) is defined as $x \in \mathbb{R}^{n+1}$ with

$$S_0^T x \leq 0 \wedge \mathbb{P}(S_1^T x \geq \mathbf{0}) = 1 \quad \text{and it holds that} \quad S_0^T x < 0 \vee \mathbb{P}(S_1^T x > \mathbf{0}) > 0.$$

By the Law of One Price in Assumption 3.1, we can measure the price for the change $Z \in \mathscr{M}$ of the current capital position by the linear functional $\pi \colon \mathscr{M} \to \mathbb{R}$ defined as

$$\pi(Z) := \sum_{i=0}^n x_i S_0^i \quad \text{with } x \in \mathbb{R}^{n+1} \text{ fulfilling } Z = \sum_{i=0}^n x_i S_1^i. \tag{3.7}$$

Especially, $\pi(S_1^i) = S_0^i$ holds for $i = 0, 1, \ldots, n$ and $\pi(m) = \pi(mS_1^0) = m$ is fulfilled for each $m \in \mathbb{R} \subseteq \mathscr{M}$ by linearity of π. The set

$$\pi_m := \{Z \in \mathscr{M} \mid \pi(Z) = m\} \tag{3.8}$$

for $m \in \mathbb{R}$ collects all eligible assets with price m. We suppose that the payoff $S_1^0 = 1$ of the secure investment opportunity is linear independent from the payoffs S_1^i of the finitely many risky assets $i = 1, \dots, n$. Hence,

$$1 < \dim \mathscr{M} < +\infty \tag{3.9}$$

holds. Moreover, if $\mathscr{X}$ is a topological linear space, the linear pricing functional $\pi\colon \mathscr{M} \to \mathbb{R}$ defined by (3.7) is continuous.

Remark 3.1. In the no-arbitrage principle in Assumption 3.1, the arbitrage terminology is taken from Irle [39, Def. 1.10]. Especially, for $Z \in \mathscr{M}$ arbitrary, it holds under Assumption 3.1 that

$$(\pi(Z) \le 0 \wedge \mathbb{P}(Z \ge \mathbf{0}) = 1) \quad \Longrightarrow \quad \pi(Z) = 0 = \mathbb{P}(Z > \mathbf{0}),$$

i.e., $\mathscr{X}_+ \cap \ker \pi = \{\mathbf{0}\}$, since we identify random variables $X, Y \in \mathscr{X}$ with $\mathbb{P}(X = Y) = 1$. Note that $\{\mathbf{0}\} \subseteq \mathscr{X}_+ \cap \ker \pi \subseteq \mathscr{M}$ by $\mathbf{0} \in \ker \pi \subseteq \mathscr{M}$. In [39, Anmerkung 1.11], Irle notices that an one-period-model is arbitrage-free if there is no arbitrage opportunity $x \in \mathbb{R}^{n+1}$ with $S_0^T x = 0$. Note that in [20, Def. 1.2], Föllmer and Schied define arbitrage without the case $S_0^T x < 0$.

If Assumption 3.1 is fulfilled, the pricing functional $\pi\colon \mathscr{M} \to \mathbb{R}$ given by (3.7) is monotonically increasing on $\mathscr{M}$ (see [26, Lemma 3.1]), i.e.,

$$\forall Z^1, Z^2 \in \mathscr{M}: \quad Z^2 - Z^1 \in \mathscr{X}_+ \Longrightarrow \pi(Z^1) \le \pi(Z^2) \tag{3.10}$$

holds. Indeed, π is *strictly* monotonically increasing on $\mathscr{M}$, see [33, Rem. 1.3.22]. Because of $\mathscr{X}_+ \cap \ker \pi = \{\mathbf{0}\}$ by Assumption 3.1 (see Remark 3.1), we obtain

$$\forall Z \in (\mathscr{M} \cap \mathscr{X}_+) \setminus \{\mathbf{0}\}: \quad \pi(Z) > 0. \tag{3.11}$$

We assume that there is some positive eligible payoff, which price can be set as 1 by linearity of the pricing functional:

Assumption 3.2. Let $\mathscr{M} \subseteq \mathscr{X}$ be the subspace given by (3.6) and $\pi\colon \mathscr{M} \to \mathbb{R}$ the pricing functional in (3.7). There is some eligible payoff $U \in \mathscr{M} \cap \mathscr{X}_+$ fulfilling $\pi(U) = 1$.

Although (3.11) holds, Assumption 3.2 is not automatically fulfilled because $(\mathscr{M} \cap \mathscr{X}_+) \setminus \{\mathbf{0}\} \neq \varnothing$ does not have to hold, e.g., if we consider $\mathscr{X} = \mathbb{R}^3$ and $\mathscr{M} = \{Z \in \mathbb{R}^3 \mid Z_3 = -Z_2\}$. Note that we can choose $U = S_1^0 = 1$ and, thus, Assumption 3.2 is fulfilled, but we explicitly note where Assumption 3.2 is necessary and work with some arbitrary payoff $U \in \mathscr{M} \cap \mathscr{X}_+$ for easier application to other models where, e.g., no secure investment opportunity exists. Note that the Law of One Price in Assumption 3.1 is automatically fulfilled in Assumption 3.2 by requiring the existence of the pricing functional $\pi\colon \mathscr{M} \to \mathbb{R}$ in (3.7). Obviously, if $\mathscr{A} \subseteq \mathscr{X}$ is an acceptance set, it holds that

$$\forall U \in \mathscr{M} \cap \mathscr{X}_+: \quad \mathscr{A} + \mathbb{R}_+ U = \mathscr{A} \tag{3.12}$$

by (3.2) and $\mathbf{0} \in \mathbb{R}_+ U$. Obviously, (3.12) is equivalent to $U \in \operatorname{rec}(\mathscr{A})$ for all $U \in \mathscr{M} \cap \mathscr{X}_+$. Moreover, the role of $U \in \mathscr{M} \cap \mathscr{X}_+$ in Assumption 3.2 is highlighted by the relationship (see [26, Lemma 3.2] and [27])

$$\forall m \in \mathbb{R}: \quad \pi_m = mU + \ker \pi \tag{3.13}$$

with $\pi_m \subseteq \mathscr{M}$ being the set defined by (3.8), which implies $\mathscr{M} = \mathbb{R}U + \ker \pi$.

We summarize our financial model as follows:

(FM) : $\mathscr{X}$ is the linear space of capital positions, partially ordered by $\mathscr{X}_+$,
$(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space,
$S^i \in \mathbb{R} \times \mathscr{X}$ $(i = 0, 1, \ldots, n)$ is the i-th eligible asset in (3.3)
with S^0 being the secure investment opportunity in (3.5),
$\mathscr{M} \subseteq \mathscr{X}$ is the linear subspace of eligible payoffs given by (3.6)
with $1 < \dim \mathscr{M} < +\infty$,
$\pi\colon \mathscr{M} \to \mathbb{R}$ is the pricing functional given by (3.7),
$\mathscr{A} \subseteq \mathscr{X}$ is an acceptance set according to Definition 3.1
and Assumption 3.1 is fulfilled.

4. Optimal Eligible Payoffs

In this section, given an origin capital position $X \in \mathscr{X}$, we focus on the set of eligible payoffs $Z \in \mathscr{M}$ with $\mathscr{M}$ being the subspace defined by (3.6) such that the resulting capital position is acceptable, i.e., $X + Z \in \mathscr{A}$ for a given acceptance set $\mathscr{A} \subseteq \mathscr{X}$. As before, we consider (FM). Let $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ be the risk measure given by

$$\rho_{\mathscr{A},\mathscr{M},\pi}(X) := \inf\{\pi(Z) \mid X + Z \in \mathscr{A}, Z \in \mathscr{M}\}; \tag{4.1}$$

see Farkas et al. [27] and Baes et al. [22]. The functional (4.1) was introduced in [40] and further studied in [41] with $\mathscr{M}$ being an arbitrary subset of $\mathscr{X}$ and π being an arbitrary functional. Farkas et al. [27] observed for topological linear spaces (which we proved for linear spaces in [26]) that $\rho_{\mathscr{A},\mathscr{M},\pi}$ is a risk measure. Indeed, find the following lemma.

Lemma 4.1 (see Farkas et al. [27, Lemma 2], and Marohn, Tammer [23, Lemma 3.14])**.** *Consider* (FM)*. Let $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ be the functional given by* (4.1) *and Assumption 3.2 be fulfilled. Then, $\rho_{\mathscr{A},\mathscr{M},\pi}$ fulfills the following properties:*

(i) *$\rho_{\mathscr{A},\mathscr{M},\pi}(X) \geq \rho_{\mathscr{A},\mathscr{M},\pi}(Y)$ for all $X, Y \in \mathscr{X}$ with $Y \in X + \mathscr{X}_+$,*
(ii) *$\rho_{\mathscr{A},\mathscr{M},\pi}(X + Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X) - \pi(Z)$ for all $X \in \mathscr{X}, Z \in \mathscr{M}$.*

Note that the terminology "risk measure" is used differently in the literature, compare [26, Remark 4.3] and [33, Remark 3.3.7]. Here, especially, we do not assume $\rho_{\mathscr{A},\mathscr{M},\pi}(\mathbf{0}) \in \mathbb{R}$ for the functional $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ in (4.1). It is important to mention that $\rho_{\mathscr{A},\mathscr{M},\pi}$ being monotonically decreasing (in the sense of Lemma 4.1(i)) implies that adding a positive amount $c \in \mathbb{R}_>$ of capital to the current position $X \in \mathscr{X}$ decreases the risk measured by $\rho_{\mathscr{A},\mathscr{M},\pi}$, while the costs measured by the pricing functional $\pi\colon \mathscr{M} \to \mathbb{R}$ increase by (3.10), since $\pi(c) = c > 0$ with $c \in \mathscr{M}$ by Definition of $\mathscr{M}$ as in (3.6) and $\operatorname{span}(S_1^0) = \operatorname{span}(1) = \mathbb{R} \subseteq \mathscr{M}$ hold.

The following lemma, which is crucial for our studies, states that the risk measure $\rho_{\mathscr{A},\mathscr{M},\pi}$ can be reduced to a type of nonlinear functional as it is used by Gerstewitz in [42] as scalarization functional of vector optimization problems or for deriving separation theorems for not necessarily convex sets. It is well known that scalarization and separation of sets are important topics in a wide field of research, including financial mathematics and risk theory. Thus, they can be applied in different problem settings in mathematics and mathematical economics.

Lemma 4.2 (Reduction Lemma, see Farkas et al. [27, Lemma 3])**.** *Consider* (FM). *Let* $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ *be the functional defined in* (4.1) *and Assumption 3.2 be fulfilled by the eligible payoff* $U \in \mathscr{M} \cap \mathscr{X}_+$. *Then, the following holds:*

$$\forall X \in \mathscr{X}: \quad \rho_{\mathscr{A},\mathscr{M},\pi}(X) = \inf\{m \in \mathbb{R} \mid X + mU \in \mathscr{A} + \ker \pi\}.$$

Farkas et al. formulated the Reduction Lemma 4.2 for topological linear spaces, but they did not use any topological properties in the proof. Hence, we reformulate it here for linear spaces. Note that $\mathscr{A} + \ker \pi \subseteq \mathscr{X}$ is an acceptance set itself if it is proper (see [26, Remark 5.4]). Nevertheless, it holds for every acceptance set $\mathscr{A}$ and $U \in \mathscr{M} \cap \mathscr{X}_+$ that (see [26, Cor. 5.1])

$$\mathscr{A} + \ker \pi + \mathbb{R}_+ U = \mathscr{A} + \ker \pi. \tag{4.2}$$

In our paper [26], we studied the properties of the functional (4.1), in more detail, especially the level lines, sublevel sets and strict sublevel sets for geometrical interpretation and quantification of risk corresponding to the acceptance set. We want to recall some of these properties here for use in later proofs concerning optimal payoffs:

Theorem 4.1 (see Marohn and Tammer [26, Theorem 3.1])**.** *Consider* (FM). *Let* $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ *be the functional defined in* (4.1) *and Assumption 3.2 be fulfilled by the eligible payoff* $U \in \mathscr{M} \cap \mathscr{X}_+$. *Then, for* $m \in \mathbb{R}$ *be arbitrary, the following conditions hold:*

$$\begin{aligned}
\text{(i) } \operatorname{lev}_{\rho_{\mathscr{A},\mathscr{M},\pi},<}(m) &= \operatorname{int}_{-U}(\mathscr{A} + \ker \pi) - mU = \mathscr{A} + \ker \pi + \mathbb{R}_> U - mU \\
&\subseteq \mathscr{A} + \ker \pi - mU, \\
\text{(ii) } \operatorname{lev}_{\rho_{\mathscr{A},\mathscr{M},\pi},\leq}(m) &= \operatorname{cl}_{-U}(\mathscr{A} + \ker \pi) - mU \\
&= \operatorname{cl}_{-U}(\mathscr{A} + \ker \pi) + \mathbb{R}_+ U - mU, \\
\text{(iii) } \operatorname{lev}_{\rho_{\mathscr{A},\mathscr{M},\pi},=}(m) &= \operatorname{bd}_{-U}(\mathscr{A} + \ker \pi) - mU.
\end{aligned}$$

As mentioned at the beginning, we will focus on the set-valued mapping $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ which is introduced in the following definition and that we study in a more general setting as in [22] and [23], where a closed acceptance set $\mathscr{A}$ and a locally convex Hausdorff space $\mathscr{X}$ was considered:

Definition 4.1 (see Baes et al. [22])**.** Consider (FM). The set-valued mapping $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ defined by

$$\mathscr{E}(X) := \{Z \in \mathscr{M} \mid X + Z \in \mathscr{A}, \pi(Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X)\} \tag{4.3}$$

is called *optimal payoff map* and $Z \in \mathscr{E}(X)$ is called *optimal payoff for* $X \in \mathscr{X}$. Furthermore, we call every $X^0 := X + Z \in \mathscr{A}$ with $Z \in \mathscr{E}(X)$ *(cost-)optimal acceptable capital position* for a given capital position $X \in \mathscr{X}$.

Obviously, $\mathscr{E}(X)$ is the solution set of the optimization problem

$$\pi(Z) \to \min_{X+Z \in \mathscr{A},\, Z \in \mathscr{M}}. \tag{$P_\pi(X)$}$$

We will generalize the results by Baes et al. [22] and Marohn and Tammer [23] to linear spaces and by weaker assumptions on the acceptance set.

Assumption 4.1. Consider (FM) and let Assumption 3.2 be fulfilled by the eligible payoff $U \in \mathscr{M} \cap \mathscr{X}_+$. The acceptance set $\mathscr{A}$ is $(-U)$-directionally closed.

As noticed, e.g., in [26, Remark 4.2], there are many different definitions of acceptance sets in the literature. Directionally closed acceptance sets were considered, for example, in [43] and [44]. Considering directionally closed sets makes sense in different situations. Artzner et al. studied in [4] the risk measure $\rho_{\mathscr{A},r_0}\colon \mathscr{X} \to \overline{\mathbb{R}}$ given by

$$\rho_{\mathscr{A},r_0}(X) := \inf\{m \in \mathbb{R} \mid X + mr_0 1_\Omega \in \mathscr{A}\},$$

where $r_0 \in \mathbb{R}$ is the return of a risk-free reference instrument (e.g. the return $S_1^0 = 1 + r$ of our secure investment opportunity S^0, where we assume $r = 0$ for convenience), $\mathscr{A} \subseteq \mathscr{X}$ is an acceptance set and $1_\Omega \in \mathscr{X}$ is the random variable that equals 1 for each $\omega \in \Omega$. If $X \in \mathscr{X}$ fulfills $\rho_{\mathscr{A},r_0}(X) \in \mathbb{R}$, then

$$X + \rho_{\mathscr{A},r_0}(X) r_0 \cdot 1_\Omega \in \operatorname{bd}_{-r_0 1_\Omega}(\mathscr{A})$$

holds. Consequently, for securing that every $X \in \mathscr{X}$ with $\rho_{\mathscr{A},r_0}(X) \in \mathbb{R}$ results in an optimal capital position belonging to $\mathscr{A}$, we have to require $\operatorname{bd}_{-r_0 1_\Omega}(\mathscr{A}) \subseteq \mathscr{A}$. Since $\operatorname{bd}_{-r_0 1_\Omega}(\mathscr{A}) \subseteq \operatorname{cl}_{-r_0 1_\Omega}(\mathscr{A})$, this requirement is fulfilled if $\mathscr{A}$ is an acceptance set according to Assumption 4.1 with $U = r_0 1_\Omega$ because $\operatorname{cl}_{-r_0 1_\Omega}(\mathscr{A}) = \mathscr{A}$ holds, then.

Another situation where directionally closed acceptance sets might occur is when scenarios for the development of prices or financial markets are considered; see [33, Example 4.1.2].

Nevertheless, Assumption 4.1 can not secure the existence of an optimal acceptable capital position in a more general setting. The reason is, as it is formulated and proved in Theorem 4.2 below, that optimal acceptable capital positions $X^0 \in \mathscr{X}$ must be elements of $\operatorname{bd}_{-U}(\mathscr{A})$, but also they have to be an element of $\operatorname{bd}_{-U}(\mathscr{A} + \ker\pi)$. Thus, this intersection might be empty. Note that in the case of one eligible asset as in [4] (which we excluded here), we would have $\ker\pi = \{\mathbf{0}\}$, and, thus the second condition is obsolet, but in our setting $\ker\pi \neq \{\mathbf{0}\}$ always holds by (3.9).

The Definition 3.1 of an acceptance set was motivated by Baes et al. in [22], where $\mathscr{A}$ was additionally assumed to be a closed set in a locally convex Hausdorff space over $\mathbb{R}$. Under Assumption 4.1, we automatically have a closed acceptance set $\mathscr{A} \subseteq \mathscr{X}$ when considering (FM) if $\mathscr{A}$ fulfills the following (see [33, Lemma 4.1.3]):

$$\operatorname{cl}(\mathscr{A}) + \mathbb{R}_> U \subseteq \mathscr{A}. \tag{4.4}$$

An easy example for a $(-U)$-directionally closed acceptance set $\mathscr{A}$ with $U \in \mathscr{M} \cap \mathscr{X}_+$ such that $\mathscr{A}$ is not closed (and, thus, not fulfilling (4.4)) can be found in [33, Example 4.1.4].

Baes et al. [22, Th. 3.2] derived the following description for the optimal payoff map $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ given by (4.3) for a closed acceptance set $\mathscr{A}$ in a locally convex Hausdorff space over $\mathbb{R}$ under the assumption that $\rho_{\mathscr{A},\mathscr{M},\pi}$ is finite and continuous:

$$\forall X \in \mathscr{X}: \quad \mathscr{E}(X) = \{Z \in \mathscr{M} \mid X + Z \in \operatorname{bd}(\mathscr{A}) \cap \operatorname{bd}(\mathscr{A} + \ker\pi)\}. \tag{4.5}$$

Now, we present our main result. Since we consider linear spaces, we derive the following more general characterization of optimal payoffs than (4.5) without any assumption on the finiteness and continuity of $\rho_{\mathscr{A},\mathscr{M},\pi}$ or use of topological properties. This characterization will be used afterwards to study optimal acceptable capital positions for a given origin capital position.

Theorem 4.2. *Consider* (FM). *Let Assumption 3.2 be fulfilled by the eligible payoff* $U \in \mathscr{M} \cap \mathscr{X}_+$. *Furthermore, let* $\mathscr{E} \colon \mathscr{X} \rightrightarrows \mathscr{M}$ *be the set-valued optimal payoff map introduced in* (4.3). *Then, for* $X \in \mathscr{X}$ *arbitrary, it holds that*

$$\mathscr{E}(X) \subseteq \{Z \in \mathscr{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathscr{A}) \cap \operatorname{bd}_{-U}(\mathscr{A} + \ker \pi)\}. \tag{4.6}$$

Furthermore, it holds that

$$\mathscr{E}(X) = \{Z \in \mathscr{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathscr{A}) \cap \operatorname{bd}_{-U}(\mathscr{A} + \ker \pi)\} \tag{4.7}$$

if one of the following conditions is fulfilled:

(i) $\rho_{\mathscr{A},\mathscr{M},\pi}(X) \notin \mathbb{R}$,
(ii) $\mathscr{A}$ *is* $(-U)$*-directionally closed according to Assumption 4.1.*

Proof. For improved readability of the proof, we set

$$\mathscr{Z}(X) := \{Z \in \mathscr{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathscr{A}) \cap \operatorname{bd}_{-U}(\mathscr{A} + \ker \pi)\}.$$

First, we show (4.6) for some arbitrary acceptance set $\mathscr{A} \subseteq \mathscr{X}$ and $X \in \mathscr{X}$. Since the case $\mathscr{E}(X) = \varnothing$ is trivial, we assume $\mathscr{E}(X) \neq \varnothing$. Thus $\rho_{\mathscr{A},\mathscr{M},\pi}(X) \in \mathbb{R}$. Let $Z \in \mathscr{E}(X)$, i.e.,

$$Z \in \mathscr{M} \quad \text{with} \quad X + Z \in \mathscr{A} \text{ and } \pi(Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X).$$

Furthermore, let $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_>$ with $t_n \downarrow 0$ for $n \to +\infty$. Then, $X + Z + t_nU \in \mathscr{A}$ for all $n \in \mathbb{N}$ by monotonicity of $\mathscr{A}$ (see Definition 3.1) since $t_nU \in \mathscr{X}_+$ for each $n \in \mathbb{N}$. We obtain by linearity of π

$$\pi(Z - t_nU) = \pi(Z) - t_n < \pi(Z) \tag{4.8}$$

because of $t_n \in \mathbb{R}_>$ for all $n \in \mathbb{N}$ and $\pi(U) = 1$ by Assumption 3.2. Hence, $X + Z - t_nU \notin \mathscr{A}$ holds for each $n \in \mathbb{N}$. Indeed, if $X + Z - t_nU \in \mathscr{A}$ is fulfilled for each $n \in \mathbb{N}$, (4.8) contradicts $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = \pi(Z)$. Consequently,

$$X + Z \in \operatorname{bd}_{-U}(\mathscr{A}). \tag{4.9}$$

Now, we show that $X + Z \in \operatorname{bd}_{-U}(\mathscr{A} + \ker \pi)$ holds, too. Set $m := \rho_{\mathscr{A},\mathscr{M},\pi}(X)$. By $Z \in \mathscr{E}(X)$, we have $X + Z \in \mathscr{A} \subseteq \mathscr{A} + \ker \pi$. Because of (3.13), there is some $Z^0 \in \ker \pi$ with $Z = mU + Z^0$. By (4.2), it holds that

$$\forall t \in \mathbb{R}_> : \quad X + (m+t)U + Z^0 \in \mathscr{A} + \ker \pi \tag{4.10}$$

since $X + mU + Z^0 \in \mathscr{A} + \ker \pi$ holds. Furthermore,

$$\forall t \in \mathbb{R}_> : \quad X + (m-t)U + Z^0 \notin \mathscr{A} + \ker \pi, \tag{4.11}$$

holds. Indeed, if there is some $\tilde{t} \in \mathbb{R}_>$ with $X + (m-\tilde{t})U + Z^0 \in \mathscr{A} + \ker \pi$, then the Reduction Lemma 4.2 implies

$$\rho_{\mathscr{A},\mathscr{M},\pi}(X) \leq \pi((m-\tilde{t})U + Z^0) = m - \tilde{t} < m = \pi(Z)$$

by $\pi(U) = 1$ according to Assumption 3.2 and $Z^0 \in \ker \pi$, in contradiction to $\pi(Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X)$. Thus (4.10) and (4.11) deliver $X + Z \in \operatorname{bd}_{-U}(\mathscr{A} + \ker \pi)$. Because of (4.9), we obtain $\mathscr{E}(X) \subseteq \mathscr{Z}(X)$, i.e., (4.6) is shown.

In order to show (4.7), we assume that (i) holds. We need to prove $\mathscr{Z}(X) \subseteq \mathscr{E}(X)$ by (4.6). Take $X \in \mathscr{X}$ arbitrarily with $\rho_{\mathscr{A},\mathscr{M},\pi}(X) \notin \mathbb{R}$. Then we obtain $\mathscr{E}(X) = \varnothing$ by (4.3) because of $\pi(Z) \in \mathbb{R}$ for all $Z \in \mathscr{M}$. For the proof of (4.7), we have to show $\mathscr{Z}(X) = \varnothing$. We assume

$\rho_{\mathscr{A},\mathscr{M},\pi}(X) = +\infty$ first. Then, the Reduction Lemma 4.2 implies $X + mU \notin \mathscr{A} + \ker\pi$ for all $m \in \mathbb{R}$. Consequently, for all $Z \in \mathscr{M}$, $X + Z \notin \mathscr{A} + \ker\pi$ holds, too, because of (3.13), i.e., $Z \in \pi(Z)U + \ker\pi$ for each $Z \in \mathscr{M}$. Especially, there is no $Z \in \mathscr{M}$ with $X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi)$. As a result, we obtain $\mathscr{Z}(X) = \varnothing = \mathscr{E}(X)$. Now, we assume $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = -\infty$. Suppose there is some $Z \in \mathscr{Z}(X)$. Then, Z fulfills $X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi)$, which implies $\rho_{\mathscr{A},\mathscr{M},\pi}(X + Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X) - \pi(Z) = 0$ by translation invariance of $\rho_{\mathscr{A},\mathscr{M},\pi}$ (see Lemma 4.1(ii)) and by Theorem 4.1, i.e., $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = \pi(Z)$ holds, which is a contradiction to $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = -\infty$. Consequently, we obtain $\mathscr{Z}(X) = \varnothing = \mathscr{E}(X)$, which completes the proof of (4.7) if (i) holds.

Now, we prove (4.7) for the case that condition (ii) holds, i.e., $\mathscr{A}$ is a $(-U)$-directionally closed acceptance set. Since (4.7) is shown under the condition (i), we take $X \in \mathscr{X}$ arbitrary with $\rho_{\mathscr{A},\mathscr{M},\pi}(X) \in \mathbb{R}$ and $\mathscr{E}(X) = \varnothing$. Then, by Definition 4.1, it holds that

$$\forall Z \in \mathscr{M} \text{ with } \pi(Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X): \quad X + Z \notin \mathscr{A}. \tag{4.12}$$

We have to show $\mathscr{Z}(X) = \varnothing$. By Theorem 4.1(iii), we have

$$X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi) \quad \Longleftrightarrow \quad \rho_{\mathscr{A},\mathscr{M},\pi}(X + Z) = 0$$

which is equivalent to $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = \pi(Z)$ by translation invariance of $\rho_{\mathscr{A},\mathscr{M},\pi}$ (see Lemma 4.1(ii)). Thus it holds that

$$X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi) \quad \Longleftrightarrow \quad \pi(Z) = \rho_{\mathscr{A},\mathscr{M},\pi}(X).$$

Consequently, (4.12) implies

$$\forall Z \in \mathscr{M}: \qquad X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi) \quad \Longrightarrow \quad X + Z \notin \mathscr{A}.$$

Since $\mathrm{bd}_{-U}(\mathscr{A}) \subseteq \mathrm{cl}_{-U}(\mathscr{A}) = \mathscr{A}$ holds by Assumption 4.1, we have

$$\forall Z \in \mathscr{M}: \qquad X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi) \quad \Longrightarrow \quad X + Z \notin \mathrm{bd}_{-U}(\mathscr{A}).$$

Thus $\mathscr{Z}(X) = \varnothing = \mathscr{E}(X)$.

Finally, we assume (ii) and take $X \in \mathscr{X}$ arbitrary with $\mathscr{E}(X) \neq \varnothing$. Thus, we have $\rho_{\mathscr{A},\mathscr{M},\pi}(X) \in \mathbb{R}$. Since (4.6) holds, it is sufficient to show $\mathscr{E}(X) \supseteq \mathscr{Z}(X)$. Take $Z \in \mathscr{Z}(X)$ arbitrary, i.e., $Z \in \mathscr{M}$ with

$$X + Z \in \mathrm{bd}_{-U}(\mathscr{A}) \cap \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi).$$

Then, $X + Z \in \mathscr{A}$ holds because of $\mathrm{bd}_{-U}(\mathscr{A}) \subseteq \mathrm{cl}_{-U}(\mathscr{A}) = \mathscr{A}$ since $\mathscr{A}$ is $(-U)$-directionally closed. Suppose $Z \notin \mathscr{E}(X)$ holds. Since we have $\mathscr{E}(X) \neq \varnothing$, there is some $\tilde{Z} \in \mathscr{M}$ with

$$X + \tilde{Z} \in \mathscr{A} \quad \text{and} \quad \pi(\tilde{Z}) < \pi(Z). \tag{4.13}$$

As $\mathscr{M}$ is a linear space and $Z \in \mathscr{Z}(X) \subseteq \mathscr{M}$, we have $\tilde{Z} - Z \in \mathscr{M}$ with $\tilde{m} := \pi(\tilde{Z} - Z) < 0$ by linearity of π and (4.13). Then, by (3.13), there is some $Z^0 \in \ker\pi$ with $\tilde{Z} - Z = \tilde{m}U + Z^0$, i.e., $Z + \tilde{m}U = \tilde{Z} - Z^0$. Since $X + \tilde{Z} \in \mathscr{A}$ by (4.13) and $-Z^0 \in \ker\pi$, we have

$$X + Z + \tilde{m}U = X + \tilde{Z} - Z^0 \in \mathscr{A} + \ker\pi.$$

By monotonicity of $\mathscr{A} + \ker\pi$ in (4.2), we obtain, for each $m \geq \tilde{m}$, $X + Z + mU \in \mathscr{A} + \ker\pi$ which contradicts $X + Z \in \mathrm{bd}_{-U}(\mathscr{A} + \ker\pi)$ by $\tilde{m} < 0$. Consequently, $Z \in \mathscr{E}(X)$ holds. That completes the proof of (4.7) if (ii) holds. □

Remark 4.1. Note that Theorem 4.2 (especially (4.7)) is really a generalization of the result (4.5) by Baes et al. in [22, Theorem 3.2] where $\mathscr{X}$ is a locally convex Hausdorff space over $\mathbb{R}$ and $\mathscr{A} \subseteq \mathscr{X}$ a closed acceptance set: As seen in the proof, the assumption (ii) of $\mathscr{A}$ being $(-U)$-directionally closed has only been used in showing

$$\mathscr{E}(X) \supseteq \{Z \in \mathscr{M} \mid X+Z \in \mathrm{bd}_{-U}(\mathscr{A}) \cap \mathrm{bd}_{-U}(\mathscr{A}+\ker\pi)\}$$

to conclude $X+Z \in \mathscr{A}$ for $X+Z \in \mathrm{bd}_{-U}(\mathscr{A})$. If $\mathscr{A}$ is a closed subset, then $\mathscr{A}$ is $(-U)$-directionally closed, too, and the conclusion $\mathrm{bd}_{-U}(\mathscr{A}) \subseteq \mathrm{bd}(\mathscr{A}) \subseteq \mathscr{A}$ holds. Nevertheless, $\mathrm{bd}(\mathscr{A}) = \mathrm{bd}_{-U}(\mathscr{A})$ does not have to hold in general, but

$$\mathrm{bd}_{-U}(\mathscr{A}) \cap \mathrm{bd}_{-U}(\mathscr{A}+\ker\pi) \subseteq \mathrm{bd}(\mathscr{A}) \cap \mathrm{bd}(\mathscr{A}+\ker\pi).$$

As a result, Theorem 4.2 demonstrates that, even in the case of $\mathscr{A}$ being closed, $\mathrm{bd}(\mathscr{A})\backslash\mathrm{bd}_{-U}(\mathscr{A})$ and $\mathrm{bd}(\mathscr{A}+\ker\pi)\backslash\mathrm{bd}_{-U}(\mathscr{A}+\ker\pi)$, respectively, are not important for determining $\mathscr{E}(X)$, see also the following Example 4.1.

Example 4.1. Consider (FM). Let $\mathscr{X} = \mathbb{R}^3$, $\mathscr{M} = \{Z \in \mathbb{R}^3 \mid Z_3 = 0\}$, and $\pi(Z) := Z_1 + Z_2$. Thus $\ker\pi = \{Z \in \mathscr{M} \mid Z_2 = -Z_1\}$ holds. Take $U := (0,1,0)^T$ according to Assumption 3.2. Consider the (-U)-directionally closed acceptance set $\mathscr{A} = \mathscr{X}_+ = \mathbb{R}^3_+$, which is also closed with respect to the standard Euclidean topology. Obviously, it holds that $\mathrm{bd}_{-U}(\mathscr{A}) = \{X \in \mathbb{R}^3_+ \mid X_2 = 0\} \subsetneq \mathrm{bd}(\mathscr{A})$. Moreover, $\mathscr{A}+\ker\pi = \{X \in \mathbb{R}^3 \mid X_2 \geq -X_1, X_3 \geq 0\}$ holds. Hence, $\mathrm{bd}_{-U}(\mathscr{A}+\ker\pi) = \{X \in \mathbb{R}^3 \mid X_2 = -X_1, X_3 \geq 0\}$ and

$$\mathrm{bd}(\mathscr{A}+\ker\pi) = \mathrm{bd}_{-U}(\mathscr{A}+\ker\pi) \cup \{X \in \mathbb{R}^3 \mid X_2 \geq -X_1, X_3 = 0\},$$

i.e., $\mathrm{bd}_{-U}(\mathscr{A}+\ker\pi) \subsetneq \mathrm{bd}(\mathscr{A}+\ker\pi)$ holds. Consequently, $\mathrm{bd}_{-U}(\mathscr{A}) \cap \mathrm{bd}_{-U}(\mathscr{A}+\ker\pi) = \mathbb{R}_+(0,0,1)^T$ while we obtain

$$\mathrm{bd}(\mathscr{A}) \cap \mathrm{bd}(\mathscr{A}+\ker\pi) = \mathbb{R}_+(0,0,1)^T \cup \{X \in \mathbb{R}^3 \mid X_1, X_2 \geq 0, X_3 = 0\}.$$

Hence, Theorem 4.2 implies, for $X \in \mathscr{X}$ with $X_3 \geq 0$, the unique solution set,

$$\mathscr{E}(X) = \{(-X_1, -X_2, 0)^T\}.$$

Thus $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = -X_1 - X_2$ for $\rho_{\mathscr{A},\mathscr{M},\pi}\colon \mathscr{X} \to \overline{\mathbb{R}}$ given by (4.1), while $\rho_{\mathscr{A},\mathscr{M},\pi}(X) = +\infty$ for each $X \in \mathscr{X}$ with $X_3 < 0$ holds, i.e., $\mathscr{E}(X) = \varnothing$. Especially, $\rho_{\mathscr{A},\mathscr{M},\pi}$ is not continuous on $\mathscr{X}$.

Following [23], we study the *set of (cost-)optimal acceptable capital positions*

$$\mathscr{A}' := \mathrm{bd}_{-U}(\mathscr{A}) \cap \mathrm{bd}_{-U}(\mathscr{A}+\ker\pi). \tag{4.14}$$

We will give an overview about the results here and refer to [33] for details and most of the proofs. Note that we assumed $\mathscr{A}$ to be closed and $\rho_{\mathscr{A},\mathscr{M},\pi}$ to be continuous and finite in [23]. Furthermore, we considered the boundary of $\mathscr{A}$ and $\mathscr{A}+\ker\pi$, respectively, instead of the $(-U)$-directional boundary of these sets in [23]. Our aim is like in Theorem 4.2 to show that this was not necessary.

First, we note that the terminology for the set $\mathscr{A}'$ is justified, i.e., it really coincides with the set of all capital positions that can result through $(\mathrm{P}_\pi(X))$ for any $X \in \mathscr{X}$ arbitrary (see Definition 4.1).

Theorem 4.3. *Consider* (FM). *Let* $\mathscr{A}$ *be an acceptance set such that Assumption 4.1 is fulfilled with some eligible payoff* $U \in \mathscr{M} \cap \mathscr{X}_+$. *Furthermore, let* $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ *as in* (4.3) *and* $\mathscr{A}' \subseteq \mathscr{X}$ *given by* (4.14). *Then, it holds that* $\mathscr{A}' = \{X + Z \in \mathscr{X} \mid X \in \mathscr{X}, Z \in \mathscr{E}(X)\}$.

Theorem 4.3 can be seen as a reformulation of Theorem 4.2. For (FM) and $\mathscr{A} \subseteq \mathscr{X}$ being an acceptance set such that Assumption 4.1 is fulfilled with $U \in \mathscr{M} \cap \mathscr{X}_+$ being an eligible payoff according to Assumption 3.2, we have the following:

$$\forall X \in \mathscr{X}: \quad \mathscr{E}(X) = \mathscr{M} \cap \{X^0 - X \mid X^0 \in \mathscr{A}'\} = \mathscr{M} \cap (\mathscr{A}' - \{X\}). \tag{4.15}$$

Hence, for the case $\mathscr{M} = \mathscr{X}$ (implying $\dim \mathscr{X} < +\infty$ by (3.9)), it holds that $\mathscr{E}(X) \neq \varnothing$ for all $X \in \mathscr{X}$ if $\mathscr{A}' \neq \varnothing$ is fulfilled.

A direct consequence of Theorem 4.3 is

$$\left(\mathscr{A}' + \bigcup_{m \leq 0} \pi_m\right) \cap \mathscr{A} = (\mathscr{A}' + \ker \pi) \cap \mathscr{A} = \mathscr{A}' \tag{4.16}$$

with π_m defined as in (3.8). The proof can be found in [33]. The following lemma highlights the situation behind (4.16): It means, if we are able to move from a position in $\mathscr{A}'$ along $\ker \pi$ (movement with negative price do not have to be considered by (4.16)) while staying in the acceptance set $\mathscr{A}$, we do not leave $\mathscr{A}'$. Hence, if the difference of two arbitrary elements in $\mathscr{A}'$ belongs to $\mathscr{M}$, we obtain an element in $\ker \pi$. Note that $\mathscr{A}' - \mathscr{A}' \nsubseteq \mathscr{M}$ holds in general.

Lemma 4.3. *Consider* (FM). *Let* $\mathscr{A}$ *be an acceptance set such that Assumption 4.1 is fulfilled with some eligible payoff* $U \in \mathscr{M} \cap \mathscr{X}_+$ *and* $\mathscr{A}' \subseteq \mathscr{X}$ *given by* (4.14). *Take* $X^0, Y^0 \in \mathscr{A}'$ *with* $X^0 - Y^0 \in \mathscr{M}$ *arbitrary. Then, it holds that* $X^0 - Y^0 \in \ker \pi$.

The proof of Lemma 4.3 can be found in [33]. Lemma 4.3 implies

$$\mathscr{A}' - \mathscr{A}' \subseteq \mathscr{M} \quad \Longrightarrow \quad \mathscr{A}' - \mathscr{A}' \subseteq \ker \pi$$

which we can use to give a more precise description of $\mathscr{A}'$: if it is possible to switch between two arbitrary positions of $\mathscr{A}'$ via $\mathscr{M}$, the set $\mathscr{A}'$ is given by all optimal capital positions $X + Z$ that occur through solutions $Z \in \mathscr{E}(X)$ of $(\mathrm{P}_\pi(X))$ for *a single* arbitrary $X \in \mathscr{X}$ with $\mathscr{E}(X) \neq \varnothing$.

Theorem 4.4. *Consider* (FM). *Let* $\mathscr{A}$ *be an acceptance set such that Assumption 4.1 is fulfilled with some* $U \in \mathscr{M} \cap \mathscr{X}_+$. *Furthermore, let* $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ *as in* (4.3) *and* $\mathscr{A}' \subseteq \mathscr{X}$ *the set given by* (4.14). *Take an arbitrary fixed* $X \in \mathscr{X}$ *with* $\mathscr{E}(X) \neq \varnothing$. *Suppose that* $\mathscr{A}' - \mathscr{A}' \subseteq \mathscr{M}$ *holds, i.e.,*

$$\forall X^0, Y^0 \in \mathscr{A}': \quad X^0 - Y^0 \in \mathscr{M}. \tag{4.17}$$

Then,

$$\mathscr{A}' = \{X + Z \in \mathscr{X} \mid Z \in \mathscr{E}(X)\} \tag{4.18}$$

and, for all $Y \in \mathscr{X}$ *with* $\mathscr{E}(Y) \neq \varnothing$, *it holds that*

$$\mathscr{E}(Y) = \{Y^0 - Y \mid Y^0 \in \mathscr{A}'\} = \{X + Z - Y \mid Z \in \mathscr{E}(X)\}. \tag{4.19}$$

Proof. Take $X \in \mathscr{X}$ arbitrary and fixed. First, we show (4.18). Because of Theorem 4.3, we only need to show

$$\mathscr{A}' \subseteq \{X + Z \in \mathscr{X} \mid Z \in \mathscr{E}(X)\}. \tag{4.20}$$

By $\mathscr{E}(X) \neq \varnothing$, consider $Z^0 \in \mathscr{E}(X)$ arbitrary and let $X^0 := X + Z^0$. Thus, $X^0 \in \mathscr{A}'$ by Theorem 4.3. Now, let $Y^0 \in \mathscr{A}'$ arbitrary. Then,

$$Y^0 = X^0 + (Y^0 - X^0) = X + Z^0 + Y^0 - X^0. \tag{4.21}$$

Moreover, $Z^0 + Y^0 - X^0 \in \mathscr{M}$ holds because of $X^0, Y^0 \in \mathscr{A}'$ and (4.17), since $\mathscr{M}$ is a linear subspace of $\mathscr{X}$. Consequently, (4.21) implies $Z^0 + Y^0 - X^0 = Y^0 - X \in \mathscr{E}(X)$ by (4.15), i.e., $Y^0 \in \{X + Z \in \mathscr{X} \mid Z \in \mathscr{E}(X)\}$, showing (4.20). Hence, (4.18) is shown.

Next, we prove (4.19), but we only need to show the first equation by (4.18). By (4.15), it is sufficient to prove

$$\forall Y \in \mathscr{X} \text{ with } \mathscr{E}(Y) \neq \varnothing : \quad \{Y^0 - Y \mid Y^0 \in \mathscr{A}'\} \subseteq \mathscr{M}.$$

Let $Y \in \mathscr{X}$ with $\mathscr{E}(Y) \neq \varnothing$ and $Y^0 \in \mathscr{A}'$ arbitrary. By Theorem 4.3, there is some $Z \in \mathscr{E}(X)$ with $Y + Z \in \mathscr{A}'$. Taking into account (4.17), $Y^0 \in \mathscr{A}'$ and $Y + Z \in \mathscr{A}'$ imply

$$Y^0 - Y = \underbrace{Y^0 - (Y + Z)}_{\in \mathscr{M}} + Z \in \mathscr{M},$$

since $Z \in \mathscr{E}(Y) \subseteq \mathscr{M}$ and $\mathscr{M}$ is a linear subspace of $\mathscr{X}$. Thus the proof of (4.19) is completed. □

Remark 4.2. (4.18) and (4.19) show that it is sufficient to find one arbitrary $X \in \mathscr{X}$ with nonempty solution set $\mathscr{E}(X)$: Then, all optimal acceptable capital positions $X^0 \in \mathscr{A}'$ are determined that are also suitable for changing any other given capital position $Y \in \mathscr{X}$ into, provided there is a solution of $(P_\pi(Y))$. Hence, if an user has determined the solution set $\mathscr{E}(X)$ for an origin capital position X and detects that the capital position X was not determined correctly such that it has to be changed slightly to a capital position $\tilde{X}$ (e.g., because of data gathering errors), the user can easily derive solutions $Z \in \mathscr{E}(\tilde{X})$ by the already determined set $\mathscr{A}'$ if $\mathscr{E}(\tilde{X}) \neq \varnothing$ is fulfilled. An example that (4.17) is necessary in Theorem 4.4 is given by Example 4.1 and [23, Example 4.13].

We can simplify the assumptions of Theorem 4.4 in the following case:

Definition 4.2. Consider a financial market (FM). If $\mathscr{M} = \mathscr{X}$ holds, then the market is said to be *complete*. Otherwise, the market is called *incomplete*.

Because we have finitely many assets in our model (FM), it holds that $\dim \mathscr{M} < +\infty$ (see (3.6)). Thus the market is always incomplete in the case of infinite dimensional linear spaces $\mathscr{X}$. Note that $\mathscr{X}_+ \subseteq \mathscr{M}$ holds for complete markets. For a complete market (FM), the assumption (4.17) in Theorem 4.4 is obviously fulfilled. In that case, the following corollary shows that we may consider any arbitrary capital position in Theorem 4.4, nevertheless if $\mathscr{E}(X) \neq \varnothing$ holds or not (e.g., $X = \mathbf{0}$). The proof of Corollary 4.1 can be found in [33].

Corollary 4.1. *Consider* (FM). *Let $\mathscr{A}$ be an acceptance set such that Assumption 4.1 is fulfilled with some $U \in \mathscr{M} \cap \mathscr{X}_+$. Furthermore, let $\mathscr{E} \colon \mathscr{X} \rightrightarrows \mathscr{M}$ as in* (4.3)*, $\mathscr{A}' \subseteq \mathscr{X}$ the set given by* (4.14) *and suppose that the market is complete. Take any $X \in \mathscr{X}$ arbitrary fixed. Then, it holds that $\mathscr{A}' = \{X + Z \in \mathscr{X} \mid Z \in \mathscr{E}(X)\}$ and for all $Y \in \mathscr{X}$: $\mathscr{E}(Y) = \{Y^0 - Y \mid Y^0 \in \mathscr{A}'\}$.*

Remark 4.3. From a practical point of view, Corollary 4.1 leads to the following simplification: Normally, an investor wants to find solutions of $(P_\pi(X))$ for an origin capital position $X \in \mathscr{X}$

that has to be determined first and is a random variable. Thus, suitable probability models are required and errors might lead to non correct results. It is useful if we can decide upon $\mathscr{E}(X) \neq \varnothing$ without determining X (and, thus, possible errors by doing so) and, ideally, can conclude the solution set for (afterwards) determined X by the work so far. Corollary 4.1 shows that this is possible in complete markets because we can directly choose the capital position $X = \mathbf{0}$ to determine $\mathscr{A}' \subseteq \mathscr{X}$ given by (4.14). If $\mathscr{A}' = \varnothing$, we obtain $\mathscr{E}(X) = \varnothing$, too, and can conclude $\mathscr{E}(X)$ through $\mathscr{A}'$ by (4.15) after determining X.

Remark 4.4. Note that we consider multiple eligible assets by (FM), i.e., $\dim \mathscr{M} > 1$ and, thus, $\ker \pi \neq \{\mathbf{0}\}$ by the rank-nullity-theorem applied for $\pi\colon \mathscr{M} \to \mathbb{R}$ given by (3.7). The case $\dim \mathscr{M} = 1$ coincides with the case of one eligible asset that is well-studied, see, e.g., [4] and [45]. We do not consider only one eligible asset because of practical purposes where the multdimensional case is standard. Nevertheless, our results can also be proven for one eligible asset.

5. Some Remarks about Efficient and Weakly Efficient Points of Acceptance Sets

In the following, we give an overview about some interesting results for (FM) concerning (weakly) efficient points of $(-U)$-directionally closed acceptance sets $\mathscr{A}$ with $U \in \mathscr{M} \cap \mathscr{X}_+$ fulfilling Assumption 3.2. Proofs, more details, and additional results can be found in [33, Ch. 4]. Note that the results also work for closed acceptance sets $\mathscr{A}$ because these are $(-U)$-directionally closed, too, by $U \in \operatorname{rec}(\mathscr{A})$. In [23], we have analyzed (weakly) efficient points of closed acceptance sets for the introduced *kernel cone*

$$\mathscr{C}_{\ker} := \ker \pi + \mathscr{X}_+. \tag{5.1}$$

As mentioned at the beginning of [23, Sec. 5], optimal eligible payoffs (or, more precisely, the set $\mathscr{A}'$ given by (4.14)) can be found by a shift of $\ker \pi$ along $\mathscr{A}$ as far as possible in direction of $-U$, with $U \in \mathscr{M} \cap \mathscr{X}_+$ being the eligible payoff according to Assumption 3.2, and its intersection with $\operatorname{bd}_{-U}(\mathscr{A})$. That motivates the question under which circumstances there are no other acceptable capital positions that can be reached by some arbitrarily given $X^0 \in \mathscr{A}'$ with non-positive costs, i.e.,

$$\forall X^0 \in \mathscr{A}' \; \nexists X \in \mathscr{A} \backslash \{X^0\}: \quad X - X^0 \in -\bigcup_{m \geq 0} \pi_m, \tag{5.2}$$

where $\pi_m \subseteq \mathscr{M} \subseteq \mathscr{X}$ is the set of eligible payoffs with price equal to $m \in \mathbb{R}$, see (3.8). The condition (5.2) seems similar to determining some specific efficient points of $\mathscr{A}$ with respect to

$$\mathscr{C}_\pi := \bigcup_{m \in \mathbb{R}_+} \pi_m = \ker \pi + \mathbb{R}_+ U \tag{5.3}$$

with $U \in \mathscr{M} \cap \mathscr{X}_+$ according to Assumption 3.2. The second equation follows by (3.13). We call $\mathscr{C}_\pi$ the *price cone* (see [33, Def. 4.1.1]). For more details about the motivation for using $\mathscr{C}_\pi$ and $\mathscr{C}_{\ker}$, respectively, as well about their properties, we refer to [33, Sec. 4.1]. Note that (see [33, Lemma 4.1.5])

$$\mathscr{C}_\pi \subseteq \mathscr{C}_{\ker} \tag{5.4}$$

holds and $\mathscr{C}_\pi = \mathscr{C}_{\ker}$ is fulfilled if $\mathscr{X}_+ \subseteq \mathscr{M}$ holds (implying $\mathscr{C}_{\ker} \subseteq \mathscr{M}$).

It is important to mention that $\mathscr{C}_\pi$ and $\mathscr{C}_{\ker}$ are convex, non-pointed cones; see [33, Lemma 4.1.6]. Note that $\ker\pi = \{\mathbf{0}\}$ is excluded here; see Remark 4.4. Also note that

$$\mathbf{0} \in \ker\pi = \mathrm{bd}_{-U}(\mathscr{C}_\pi) \quad \text{and} \quad \mathbf{0} \in \ker\pi \subseteq \mathrm{bd}_{-U}(\mathscr{C}_{\ker})$$

hold, see [33, Lemma 4.3.3].

The set of *efficient points of $\mathscr{A}$ with respect to $\mathscr{C}_\pi$* is defined by

$$\mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) := \{X \in \mathscr{A} \mid \mathscr{A} \cap (\{X\} - \mathscr{C}_\pi) \subseteq \{X\}\}.$$

Analogously, we define $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker})$. Note that the sets of efficient points are subsets of $\mathrm{bd}(\mathscr{A})$ in topological linear spaces $\mathscr{X}$. By use of (5.4), we derive in [33, Theorem 4.2.4] that

$$\mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker}) \subseteq \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) \subseteq \mathrm{bd}_{-U}(\mathscr{A}).$$

holds. The following assertion is a main result which explains the conjectured relationship between $\mathscr{A}'$ given by (4.14) and the sets of efficient points of $\mathscr{A}$:

Theorem 5.1 (see [33, Theorem 4.2.6])**.** *Consider* (FM). *Let Assumption 4.1 be fulfilled by some $U \in \mathscr{M} \cap \mathscr{X}_+$ and an acceptance set $\mathscr{A} \subseteq \mathscr{X}$. Furthermore, let $\mathscr{C}_\pi \subseteq \mathscr{M}$ be the price cone given by* (5.3)*, $\mathscr{C}_{\ker} \subseteq \mathscr{X}$ be the kernel cone given by* (5.1) *and $\mathscr{A}' \subseteq \mathscr{X}$ the set given by* (4.14).

(i) *It holds that* $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker}) \subseteq \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) \subseteq \mathscr{A}'$.

(ii) *Suppose that $\mathscr{M}$ fulfills* (4.17)*, i.e., $\forall X^0, Y^0 \in \mathscr{A}' : X^0 - Y^0 \in \mathscr{M}$. If* $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker}) \neq \varnothing$, *then* $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker}) = \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) = \mathscr{A}'$. *Moreover, if* $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) \neq \varnothing$, *then* $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) = \mathscr{A}'$.

Under the assumptions of Theorem 5.1, $\mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker}) = \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) = \mathscr{A}'$ is automatically fulfilled if $\mathscr{A}' = \varnothing$ or $|\mathscr{A}'| = 1$ holds, see [33, Corollary 4.2.8]. Now, we consider weakly efficient points of $\mathscr{A}$ with respect to $\mathscr{C}_\pi$ and $\mathscr{C}_{\ker}$ in (FM) for a topological linear space $\mathscr{X}$, i.e., the set of *weakly efficient points of $\mathscr{A}$ with respect to $\mathscr{C}_\pi$*

$$\mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_\pi) := \mathrm{Eff}(\mathscr{A}, \mathrm{int}\,\mathscr{C}_\pi)$$

for $\mathscr{C}_\pi$ fulfilling $\mathrm{int}(\mathscr{C}_\pi) \neq \varnothing$ (analogously $\mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_{\ker})$ if $\mathrm{int}(\mathscr{C}_{\ker}) \neq \varnothing$ holds). For $X \in \mathscr{A}'$ arbitrary and (4.17) being fulfilled, it holds that

$$X \notin \mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker}) \quad \Longrightarrow \quad X \in \mathrm{Eff}(\mathscr{A}, \mathscr{C}_{\ker} \setminus \ker\pi)$$

and

$$X \notin \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi) \quad \Longrightarrow \quad X \in \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi \setminus \ker\pi)$$

by [33, Th. 4.3.6 and Rem. 4.3.7]. Moreover,

$$\mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_{\ker}) \subseteq \mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_\pi)$$

holds in general and, additionally, $\mathscr{A}' \subseteq \mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_{\ker})$ if (4.17) is fulfilled (see [33, Lemma 4.3.8 and Theorem 4.3.10]). Since $\mathscr{C}_\pi = \mathscr{C}_{\ker}$ holds for $\mathscr{X}_+ \subseteq \mathscr{M}$ (and, thus, complete markets), it can be shown for a topological linear space $\mathscr{X}$ and a complete market (FM) in the sense of Definition 4.2 $\ker\pi = \mathrm{bd}(\mathscr{C}_\pi) = \mathrm{bd}(\mathscr{C}_{\ker})$ (see [33, Cor. 4.3.4]). Thus

$$\mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_\pi) = \mathrm{Eff}_w(\mathscr{A}, \mathscr{C}_{\ker}) = \mathrm{Eff}(\mathscr{A}, \mathscr{C}_\pi \setminus \ker\pi).$$

This leads to the following result for complete markets (FM):

Theorem 5.2 (see [33, Theorem 4.3.10])**.** *Consider* (FM). *Let $\mathscr{X}$ be a topological linear space and Assumption 4.1 be fulfilled by some $U \in \mathscr{M} \cap \mathscr{X}_+$ and an acceptance set $\mathscr{A} \subseteq \mathscr{X}$. Furthermore, let $\mathscr{C}_\pi \subseteq \mathscr{M}$ be the price cone given by* (5.3) *and $\mathscr{C}_{\text{ker}} \subseteq \mathscr{X}$ be the kernel cone given by* (5.1)*. Suppose that $\operatorname{int}(\mathscr{C}_\pi) \neq \varnothing$ and $\operatorname{int}(\mathscr{C}_{\text{ker}}) \neq \varnothing$ hold. Let $\mathscr{A}' \subseteq \mathscr{X}$ be the set defined in* (4.14) *and $\mathscr{X}_+ \subseteq \mathscr{M}$. Then $\operatorname{Eff}_w(\mathscr{A},\mathscr{C}_{\text{ker}}) = \operatorname{Eff}_w(\mathscr{A},\mathscr{C}_\pi) = \mathscr{A}'$.*

Remark 5.1. As already mentioned in Remark 4.4, we consider multiple eligible assets in (FM), implying $\ker\pi \neq \{\mathbf{0}\}$ by $\dim\mathscr{M} > 1$. For the case of one eligible asset, i.e., $\dim\mathscr{M} = 1$, we obtain $\ker\pi = \{\mathbf{0}\}$ which leads to $\mathscr{C}_\pi$ and $\mathscr{C}_{\text{ker}}$ being pointed cones. Our results for (weakly) efficient points of $\mathscr{A}$ with respect to these cones can be transferred in that case.

6. Conclusion

In this paper, we studied the solution set $\mathscr{E}\colon \mathscr{X} \rightrightarrows \mathscr{M}$ of the optimization problem

$$\pi(Z) \to \min_{X+Z\in\mathscr{A},\, Z\in\mathscr{M}} \tag{$P_\pi(X)$}$$

in the setting (FM) introduced in Section 3 where, especially, $\mathscr{X}$ is a linear space, $\mathscr{A} \subseteq \mathscr{X}$ is an acceptance set and $\pi\colon \mathscr{M} \to \mathbb{R}$ is a pricing functional defined on a subspace of eligible payoffs $\mathscr{M} \subseteq \mathscr{X}$. A main result is a generalized characterization of the set of optimal eligible payoffs $Z \in \mathscr{E}(X)$ for $X \in \mathscr{X}$ and directionally closed acceptance sets $\mathscr{A}$, but we could also show that a subset-relation still holds for $\mathscr{E}(X)$ if $\mathscr{A}$ is not directionally closed. Moreover, we gave an overview about results concerning efficient and weakly efficient points of directionally closed acceptance sets that can be proved by use of the derived properties of $\mathscr{E}(X)$. Thereby, we used a more common definition of efficiency as in our paper [23] and, furthermore, considered an additional cone. An interesting observation is that there is a direct relationship between the set of efficient points with respect to the cones $\mathscr{C}_\pi$ and $\mathscr{C}_{\text{ker}}$, respectively, and the set $\mathscr{A}'$ of cost-optimal acceptable capital positions given by (4.14) which result by solutions of $\mathscr{E}(X)$. Especially, under additional assumption of $\mathscr{A}' - \mathscr{A}' \subseteq \mathscr{M}$, $\mathscr{A}'$ coincides with the set of efficient points with respect to $\mathscr{C}_\pi$ if $\operatorname{Eff}(\mathscr{A},\mathscr{C}_\pi) \neq \varnothing$ holds. A similar result holds for $\mathscr{C}_{\text{ker}}$.

Since we only considered weakly efficient points of $\mathscr{A}$ in complete markets (which require finite dimensional spaces in the setting here), research could focus on deriving a characterization for incomplete markets (and, thus, infinite dimensional spaces). Also, the model basis could be extended by, for example, transaction costs or multi-period financial markets. Furthermore, in several practical problems, it is from interest to replace the fixed acceptance set $\mathscr{A}$ by a variable domination structure given by a set-valued mapping $\mathscr{A}\colon \mathscr{X} \rightrightarrows \mathscr{X}$ or $\mathscr{A}\colon \mathscr{M} \rightrightarrows \mathscr{X}$ in the formulation of the risk measure $\rho_{\mathscr{A},\mathscr{M},\pi}$ given by (4.1) and, thus, the formulation of the optimal payoff map $\mathscr{E}$. By doing so, the acceptance sets may vary in dependence of the origin capital position of the financial institution. That could be motivated by the fact that some regulatory assumptions might depend, e.g., on the amount of capital and, thus, the size of the financial institute.

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