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SOLUTIONS FOR A NONSTRICTLY HYPERBOLIC AND GENUINELY NONLINEAR SYSTEM

XIANTING WANG[1], YUN-GUANG LU[2], QINGYOU SUN[2], CHANGFENG XUE[3,*]

[1] *Department of Mathematics, Wuxi Institute of Technology, Wuxi, China*
[2] *School of Mathematics, Hangzhou Normal University, Hangzhou, China*
[3] *School of Mathematics and Physics, Yancheng Institute of Technology, Yancheng, China*

Abstract. In this paper, we study the existence of global entropy solutions for the Cauchy problem of an isentropic gas dynamics system with the special pressure $P(\rho) = \frac{1}{1-\rho}$. After the gas density ρ is fixed in the region $\rho \in (0,1)$, by the method of the artificial viscosity and the maximum principle, this system is nonstrictly hyperbolic and genuinely nonlinear, and its global entropy solutions are obtained by the famous compactness framework introduced by DiPerna in the paper "Convergence of approximate solutions to conservation laws " (Arch. Rat. Mech. Anal., (82) (1983), 27-70).

Keywords. Compensated compactness; Gas dynamics system; Global weak solution.

1. INTRODUCTION

Systems of hyperbolic conservation laws are very important mathematical models for a variety of physical phenomena that appear in traffic flow, theory of elasticity, gas dynamics, fluid dynamics, and so on [1, 2]. In general, the classical solution of the Cauchy problem for nonlinear hyperbolic conservation laws exists only locally in time even if the initial data are small and smooth. This means that shock waves always appear in the solution for a suitable large time. Since the solution is discontinuous and does not satisfy the given partial differential equations in the classical sense, we have to study the generalized solutions, or functions which satisfy the equations in the sense of distributions.

An important aspect of the theory of nonlinear system of conservation laws is the question of existence of solutions to these equations. It helps to answer the question if the modelling of the natural phenomena at hand has been done correctly, and if the problem is well posed.

It is well-known that there are three most important arguments to study the global existence of weak solutions for a given nonlinear hyperbolic conservation systems. They are: (1) the Glimm's scheme method [3]; (2) the compensated compactness method [4, 5]; and (3) BV estimates coupled with the vanishing viscosity method [6]. One could use the Glimm's scheme to construct a subsequence converging to a weak solution of a hyperbolic system of arbitrary number equations, but the global existence result is valid for small BV initial data and for

*Corresponding author.
E-mail addresses: wangxt@wxit.edu.cn (X. Wang), yglu2010@hznu.edu.cn (Y.G. Lu), qysun@hznu.edu.cn (Q. Sun), cfxue@ycit.edu.cn (C. Xue).

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strictly hyperbolic system. If, instead of a BV bound, only an uniform bound on the L^∞ norm of solutions is available, one can use the compensated compactness argument to construct a subsequence converging pointwisely to a weak solution of a nonstrictly hyperbolic system.

In the last four decades, there has been a growing interest in the application of compensated compactness method to system of conservation laws. We mention here that Tartar [4] first used this technique to scalar conservation law. Later, many people considered using this technique to prove the existence of solution to system of two conservation laws (cf. [7, 8, 9, 10, 11, 12, 13] and the references cited therein).

In [7], DiPerna started the study of the Cauchy problem for general hyperbolic systems of two equations

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0, \tag{1.1}$$

with bounded measurable initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \tag{1.2}$$

where u and v are in R. We let $U = (u,v)$ and $F(U) = (f,g)$ so that the equations in (1.1) can be written as

$$U_t + dF(U)U_x = 0, \tag{1.3}$$

where $dF(U)$ is the Jacobian matrix of F. The following definitions can be found from Smoller's book [2].

Definition 1.1. We say that system (1.1) is hyperbolic if dF has two real eigenvalues $\lambda_1$and λ_2. System (1.1) is called strictly hyperbolic if λ_1 and λ_2 are distinct, i.e., $\lambda_1 < \lambda_2$. If λ_1 and λ_2 coincide at some points or domains, system (1.1) is called nonstrictly hyperbolic or hyperbolically degenerate.

Let $l_{\lambda_1}, l_{\lambda_2}, r_{\lambda_1}$, and r_{λ_2} denote the corresponding left and right eigenvectors.

Definition 1.2. We say that (1.1) is genuinely nonlinear in the λ_1 characteristic field if $\nabla\lambda_1 \cdot r_{\lambda_1} \neq 0$ and genuinely nonlinear in the λ_2 characteristic field if $\nabla\lambda_2 \cdot r_{\lambda_2} \neq 0$. If $\nabla\lambda_1 \cdot r_{\lambda_1} = 0$ or $\nabla\lambda_2 \cdot r_{\lambda_2} = 0$ at some domain D, then system (1.3) is called linearly degenerate in D in the λ_1 characteristic field or in the λ_2 characteristic field.

Furthermore, we add a small parabolic perturbation term to the right-hand side of (1.1) and consider the Cauchy problem of the following parabolic system

$$u_t + f(u,v)_x = \varepsilon u_{xx}, \quad v_t + g(u,v)_x = \varepsilon v_{xx}, \tag{1.4}$$

with initial data (1.2), where $\varepsilon > 0$ is a constant. Then the famous compactness framework of DiPerna is as follows:

Theorem 1.1. (R.J. DiPerna 1983) *Suppose that the viscosity solutions $(u^\varepsilon, v^\varepsilon)$ of the Cauchy problem (1.4) and (1.2) are in a uniformly bounded domain $D \in R^2$, and system (1.1) is strictly hyperbolic and genuinely nonlinear in D. Then there exists a subsequence (still labelled) $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ such that $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \to (u(x,t), v(x,t))$ a.e. on Ω, where $\Omega \subset R \times R^+$ is any bounded open set, and $(u(x,t), v(x,t))$ is a weak solution of Cauchy problem (1.1) and (1.2).*

The above DiPerna's theorem is for general hyperbolic systems of two conservation laws. However, to the best of our knowledge, it seems not easy to find a conservation system, with physical background, whose two eigenvalues are distinct, two characteristic fields are genuinely nonlinear in D, and especially, whose viscosity solutions are also in the same bounded domain D.

The aim of this paper is to look for hyperbolic systems satisfying all the conditions in DiPerna's theorem.

2. Main Results

In this section, we introduce our main results in this paper and their proofs.

We consider the following isentropic gas dynamics system of two conservation laws in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \tag{2.1}$$

with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad 0 \le \rho_0(x) \le \bar{\rho} < 1, \tag{2.2}$$

where ρ is the density of gas, $\bar{\rho}$ is a constant, u is the velocity, and $P = P(\rho)$ is the pressure satisfying $P'(\rho) \ge 0$. For the polytropic gas, P takes the special form $P(\rho) = c\rho^\gamma$, where $\gamma > 1$ corresponds to the adiabatic exponent and c is a positive constant.

The ideas of compensated compactness developed in [4, 5] were used in [8] to establish a global existence theorem for the Cauchy problem (2.1) with large initial data for $\gamma = 1 + \frac{2}{N}$, where $N \ge 5$ is odd, with the use of the viscosity method. The convergence of the Lax-Friedrichs scheme and the existence of a global solution in L^∞ for large initial data with adiabatic exponent $\gamma \in (1, \frac{5}{3}]$ were proved in [9, 10, 11]. In [12], the global existence of a weak solution was proved for $\gamma \ge 3$ with the use of the kinetic setting in combination with the compensated compactness method. The method in [12] was finally improved in [13] to fill the gap $\gamma \in (\frac{5}{3}, 3)$, and a new proof of the existence of a global solution for all $\gamma > 1$ was given there.

The main contribution of this paper is to verify that system (2.1) satisfies all the conditions in DiPerna's theorem if the pressure function $P(\rho) = \frac{1}{1-\rho}$ when $\rho \in (0,1)$.

By simple calculations, two eigenvalues of system (2.1) are

$$\lambda_1 = \frac{m}{\rho} - \sqrt{P'(\rho)} = \frac{m}{\rho} - \frac{1}{1-\rho}, \quad \lambda_2 = \frac{m}{\rho} + \sqrt{P'(\rho)} = \frac{m}{\rho} + \frac{1}{1-\rho}, \tag{2.3}$$

where $m = \rho u$ denotes the momentum, with corresponding right eigenvectors

$$r_1 = (1, \lambda_1)^T, \quad r_2 = (1, \lambda_2)^T$$

and

$$\begin{aligned} \nabla\lambda_1 \cdot r_1 &= \left(-\frac{m}{\rho^2} - \frac{P''(\rho)}{2\sqrt{P'(\rho)}}, \frac{1}{\rho}\right)(1, \lambda_1)^T \\ &= -\frac{\rho P''(\rho) + 2P'(\rho)}{2\rho\sqrt{P'(\rho)}} = -\frac{1}{\rho(1-\rho)^2}, \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
\nabla\lambda_2\cdot r_2 &= (-\frac{m}{\rho^2}+\frac{P''(\rho)}{2\sqrt{P'(\rho)}},\frac{1}{\rho})(1,\lambda_2)^T \\
&= \frac{\rho P''(\rho)+2P'(\rho)}{2\rho\sqrt{P'(\rho)}}=\frac{1}{\rho(1-\rho)^2}.
\end{aligned}
\tag{2.5}
$$

Therefore, system (2.1) is strictly hyperbolic from (2.3), and genuinely nonlinear from (2.4)-(2.5) when $\rho\in(0,1)$.

Now, we consider the following parabolic system

$$
\begin{cases}
\rho_t+m_x=\varepsilon\rho_{xx} \\
m_t+(\frac{m^2}{\rho}+P(\rho))_x=\varepsilon m_{xx},
\end{cases}
\tag{2.6}
$$

with the initial data

$$
(\rho^\varepsilon(x,0),m^\varepsilon(x,0))=(\rho_0^\varepsilon(x),m_0^\varepsilon(x)),
\tag{2.7}
$$

where

$$
(\rho_0^\varepsilon(x),m_0^\varepsilon(x))=(\rho_0(x)+\varepsilon,\rho_0(x)u_0(x))*G^\varepsilon
$$

and G^ε is a mollifier. Then

$$
(\rho_0^\varepsilon(x),m_0^\varepsilon(x))\in C^\infty\times C^\infty,
$$
$$
(\rho_0^\varepsilon(x),m_0^\varepsilon(x))\to(\rho_0(x),m_0(x)) \text{ a.e., as } \varepsilon\to 0,
$$

and

$$
\varepsilon\le\rho_0^\varepsilon(x)\le M_1<1,\quad |u_0^\varepsilon(x)|=|\frac{m_0^\varepsilon(x)}{\rho_0^\varepsilon(x)}|\le M_2
$$

for a suitable large constant M_2, which depends only on the L^∞ bound of $(\rho_0(x),u_0(x))$, but is independent of ε.

We have the main result in this paper as follows.

Theorem 2.1. *Let the initial data $(\rho_0(x),u_0(x))$ be bounded measurable and $0\le\rho_0(x)\le\bar{\rho}<1, P(\rho)=\frac{1}{1-\rho}$. Then, for fixed $\varepsilon>0$, the smooth viscosity solution $(\rho^\varepsilon(x,t),m^\varepsilon(x,t))$ of the Cauchy problem (2.6), (2.7) exists and satisfies*

$$
0<c(\varepsilon,t)\le\rho^\varepsilon(x,t)<1,\quad |u^\varepsilon(x,t)|=|\frac{m^\varepsilon(x,t)}{\rho^\varepsilon(x,t)}|\le M_3,
\tag{2.8}
$$

where M_3 is a positive constant, being independent of ε; and $c(\varepsilon,t)$ is a positive function, which could tend to zero as ε tends to zero or t tends to infinity. Moreover, there exists a subsequence (still labelled) $(\rho^\varepsilon(x,t),\rho^\varepsilon(x,t)u^\varepsilon(x,t))$ which converges almost everywhere on any bounded and open set $\Omega\subset R\times R^+$:

$$
(\rho^\varepsilon(x,t),\rho^\varepsilon(x,t)u^\varepsilon(x,t))\to(\rho(x,t),\rho(x,t)u(x,t)),\ \textit{as}\ \varepsilon\downarrow 0^+,
\tag{2.9}
$$

where the limit pair of functions $(\rho(x,t),\rho(x,t)u(x,t))$ is a weak solution to Cauchy problem (2.1), (2.2).

Proof of Theorem 2.1. The Riemann invariants of (2.1) are functions $w(\rho,m)$ and $z(\rho,m)$ satisfying the equations

$$
(w_\rho,w_m)\cdot dF=\lambda_2(w_\rho,w_m) \text{ and } \quad (z_\rho,z_m)\cdot dF=\lambda_1(z_\rho,z_m).
\tag{2.10}
$$

One solution of (2.10) is

$$w(\rho,m)=\frac{m}{\rho}+\ln\frac{\rho}{1-\rho},\quad z(\rho,m)=\frac{m}{\rho}-\ln\frac{\rho}{1-\rho}.$$

To prove the existence of smooth viscosity solutions $(\rho^\varepsilon(x,t),m^\varepsilon(x,t))$ for the Cauchy problem (2.6), (2.7), by Theorem 2.1, we only need to prove the a priori estimates given in (2.8).

We multiply (2.6) by (w_ρ,w_m) and (z_ρ,z_m), respectively, to obtain

$$w_t+\lambda_2 w_x=\varepsilon w_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x w_x-\tfrac{1}{\rho^2(1-\rho)^2}\rho_x^2\le\varepsilon w_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x w_x, \tag{2.11}$$

and

$$z_t+\lambda_1 z_x=\varepsilon z_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x z_x+\tfrac{1}{\rho^2(1-\rho)^2}\rho_x^2\ge\varepsilon z_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x z_x. \tag{2.12}$$

Since the conditions on the initial data, we may choose a suitable large constant M such that $w(\rho_0^\varepsilon(x),m_0^\varepsilon(x))\le M, z(\rho_0^\varepsilon(x),m_0^\varepsilon(x))\ge -M$. By applying the maximum principle to (2.11) and (2.12), we obtain immediately $w(\rho^\varepsilon(x,t),m^\varepsilon(x,t))\le M, z(\rho^\varepsilon(x,t),m^\varepsilon(x,t))\ge -M$. This demonstrates that the region

$$\Sigma=\{(\rho,m):w(\rho,m)\le M,\quad z(\rho,m)\ge -M\}$$

is an invariant region. Thus we obtain the estimates $0\le\rho^\varepsilon\le\rho_0<1$ and $|u^\varepsilon(x,t)|=|\frac{m^\varepsilon(x,t)}{\rho^\varepsilon(x,t)}|\le M_3$ for a suitable constant M_3, where ρ_0 satisfies $\ln\frac{\rho_0}{1-\rho_0}=M$ or $\rho_0=1-\frac{1}{e^M+1}<1$.

After we obtain the upper bound of $u^\varepsilon(x,t)$, we have the positive lower bound estimate $\rho^\varepsilon(x,t)\ge c(\varepsilon,t)>0$ by using the method given in the book [14].

We rewrite the first equation in (2.6) as follows:

$$v_t+uv_x+u_x=\varepsilon(v_{xx}+v_x^2),$$

where $v=\log\rho$. Then

$$v_t=\varepsilon v_{xx}+\varepsilon(v_x-\frac{u}{2\varepsilon})^2-u_x-\frac{u^2}{4\varepsilon}. \tag{2.13}$$

We can represent the solution v of (2.13) with initial data $v_0(x)=\log(\rho_0(x))$ by a Green function $G(x-y,t)=\frac{1}{\sqrt{\pi\varepsilon t}}exp(-\frac{(x-y)^2}{4\varepsilon t})$:

$$\begin{aligned}v=&\int_{-\infty}^{\infty}G(x-y,t)v_0(y)dy\\&+\int_0^t\int_{-\infty}^{\infty}\left(\varepsilon(v_x-\frac{u}{2\varepsilon})^2-\frac{u^2}{4\varepsilon}-u_x\right)G(x-y,t-s)dyds.\end{aligned} \tag{2.14}$$

Since

$$\int_{-\infty}^{\infty}G(x-y,t)dy=1,\quad\int_{-\infty}^{\infty}|G_y(x-y,t)|dy\le\frac{M}{\sqrt{\varepsilon t}},$$

it follows from (2.14) that

$$\begin{aligned} v &\geq \int_{-\infty}^{\infty} G(x-y,t)v_0(y)dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left(-\frac{u^2}{4\varepsilon} - u_x\right) G(x-y,t-s)dyds \\ &= \int_{-\infty}^{\infty} G(x-y,t)v_0(y)dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left(g(u)G_y(x-y,t-s) - \frac{u^2}{4\varepsilon} G(x-y,t-s) \right) dyds \\ &\geq \log c_0 - \tfrac{Mt}{\varepsilon} - \tfrac{M_1 t^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}} \geq -C(t,c_0,\varepsilon) > -\infty. \end{aligned}$$

Thus ρ^ε has a positive lower bound $c(t,c_0,\varepsilon)$ for any fixed ε and $t < \infty$.

Therefore, the first part about the smooth viscosity solutions in Theorem 2.1 is proved. Since system (2.1) is strictly hyperbolic and genuinely nonlinear when $P(\rho) = \frac{1}{1-\rho}$ and $\rho \in (0,1)$, the DiPerna's compactness framework given in Theorem 1.1 deduces the convergence of $(\rho^\varepsilon(x,t), \rho^\varepsilon(x,t)u^\varepsilon(x,t))$. Theorem 2.1 is proved.

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