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ON THE STRONG CONVERGENCE OF A PERTURBED ALGORITHM TO THE UNIQUE SOLUTION OF A VARIATIONAL INEQUALITY PROBLEM

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Abstract. We introduce a general iterative algorithm that generates strongly convergent to the unique solution of a general type of variational inequality over the set of the fixed points of a nonexpansive mapping. The strong convergence of the algorithm is established under general and simple conditions on the parameters. Moreover, we study a perturbed version of the algorithm and provide some numerical experiments that highlight the effects of the parameters on the stability of the algorithm and its rate of convergence.

Keywords. Fixed points; Monotone operators; Nonexpansive mappings; Variational inequalities.

1. INTRODUCTION

Throughout this paper, $\mathscr{H}$ is assumed to be a real Hilbert space with inner product $\langle .,. \rangle$ and associated norm $\|.\|$, Q is assumed to be a closed, convex, and nonempty subset of $\mathscr{H}$, $P_Q : \mathscr{H} \to Q$ is assumed to be the metric projection onto Q, $T : Q \to Q$ is assumed to be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in Q$) such that $C := F_{ix}(T) = \{x \in Q : Tx = x\}$ is nonempty, $f : Q \to \mathscr{H}$ is assumed to be a Lipschitzian mapping with coefficient $\alpha \geq 0$ (i.e. $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in Q$), and $F : Q \to \mathscr{H}$ is assumed to be a Lipschitizian mapping with coefficient $\kappa > 0$. In addition, F is assumed to be strongly monotone with coefficient $\eta > 0$, which means that

$$\langle F(x) - F(y), x - y \rangle \geq \eta \|x - y\|^2 \text{ for all } x, y \in Q.$$

We also assume that $\alpha < \eta$. Then it is easily seen that the operator $g := F - f$ is strongly monotone with coefficient $\eta - \alpha$. Hence, the variational inequality problem

$$\text{Find } q \in C \text{ such that } \langle F(q) - f(q), x - q \rangle \geq 0 \text{ for all } x \in C, \tag{VIP}$$

has a unique solution which we denote by q^* (see Lemma 2.1 below).

In the present work, we are concerned with the construction of a general iterative algorithm that generates sequences converging strongly to q^*. Let us first recall some previous results

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related to this subject. In the particular case that $Q=\mathscr{H}$, $f\equiv u$, a constant, and $F=I$, the identity mapping from $\mathscr{H}$ into itself, Halpern [1] introduced the iterative process

$$x_0\in\mathscr{H},\ x_{n+1}=\alpha_n u+(1-\alpha_n)Tx_n, \tag{1.1}$$

where $\{\alpha_n\}\in[0,1]$. He established that if $\alpha_n=\frac{1}{n^\theta}$ for all $n\geq 0$ with $\theta\in]0,1[$, then the generated sequence $\{x_n\}$ converges strongly to q^*, which is in this case equal to $P_C(u)$, where $P_C:\mathscr{H}\to C$ is the metric projection from $\mathscr{H}$ onto the closed and convex subset $C=F_{ix}(T)$. He also pointed out that the conditions (C1) $\lim_{n\to+\infty}\alpha_n=0$ (C2) $\sum_{n=0}^{+\infty}\alpha_n=+\infty$, are necessary for the strong convergence of algorithm (1.1). In 1977, Lions [2] extended the Halpern's result. In fact, he proved the strong convergence of the sequences $\{x_n\}$ generated by process (1.1) to q^* provided that $\{\alpha_n\}$ satisfies the necessary conditions (C1)-(C2) and the supplementary condition (C3) $\lim_{n\to+\infty}\frac{\alpha_{n+1}-\alpha_n}{\alpha_n^2}=0$. In 2000, Moudafi [3] considered the case when $Q=\mathscr{H}$, $f:\mathscr{H}\to\mathscr{H}$ is a contraction with coefficient $\alpha\in[0,1[$, and $F=I$ the identity mapping from $\mathscr{H}$ into itself. He introduced the algorithm

$$x_0\in\mathscr{H},\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Tx_n, \tag{1.2}$$

where $\{\alpha_n\}\in]0,1]$. He established, under conditions (C1), (C2), and (C4) $\lim_{n\to+\infty}\frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}\alpha_n}=0$, the strong convergence of any sequence generated by this algorithm to q^* which in this case is equal to the unique fixed point of the contraction mapping $P_C\circ f$. In 2004, Xu [4] improved Moudafi's result. In fact, he followed a new approach to prove the strong convergence of algorithm (1.2) provided that sequence $\{\alpha_n\}$ satisfies conditions (C1), (C2) and (C5) $\lim_{n\to+\infty}\frac{\alpha_{n+1}-\alpha_n}{\alpha_n}=0$ or $\sum_{n=0}^{+\infty}|\alpha_{n+1}-\alpha_n|<+\infty$. Xu [5] has also considered the case that $Q=\mathscr{H}$, $f=u$ a constant, and $F=A$, an $\eta-$ strongly positive self adjoint bounded linear operator from $\mathscr{H}$ to $\mathscr{H}$. He established the strong convergence of the algorithm

$$x_0\in\mathscr{H},\ x_{n+1}=\alpha_n u+(I-\alpha_n A)Tx_n$$

to the unique solution q^* of (VIP) provided that real sequence $\{\alpha_n\}$ satisfies conditions (C1), (C2), and (C5). Let us note here that, in this case, q^* is the unique minimizer of the strongly quadratic convex function $\frac{1}{2}\langle Ax,x\rangle-\langle u,x\rangle$ over the closed and convex subset $C=F_{ix}(T)$.

In 2006, Marino and Xu [6] established that the previous strong convergence result remains true in the more general case that $f:\mathscr{H}\to\mathscr{H}$ is a Lipschitzian mapping with constant α strictly less than η. On the other hand, Yamada [7] studied the particular case that $Q=\mathscr{H}$ and $f\equiv 0$. He proved that if $\{\alpha_n\}$ satisfies conditions (C1), (C2), and (C3), then, for every starting point $x_0\in\mathscr{H}$, the sequence $\{x_n\}$ generated by the iterative process $x_{n+1}=(I-\alpha_n F)Tx_n$ converges strongly to q^*.

In 2010, Tian [8], by combining the iterative method of Yamada and the method of Mariano and Xu, introduced the following general algorithm

$$x_0\in\mathscr{H},\ x_{n+1}=\alpha_n f(x_n)+(I-\alpha_n F)Tx_n.$$

He established the strong convergence of this algorithm to q^* provided that $\{\alpha_n\}$ satisfies conditions (C1), (C2), and (C5).

In 2011, Ceng, Ansari and Yao [9] extended Tian's result to the case that Q is not necessary equal to the whole space $\mathscr{H}$. Precisely, they proved that if sequence $\{\alpha_n\}$ satisfies conditions (C1), (C2) and (C5), then, for any starting point x_0 in Q, the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = P_Q\left(\alpha_n f(x_n) + (I - \alpha_n F)Tx_n\right)$$

converges strongly to q^*.

In this paper, in order to generalize and unify the previous results and to take account of the possible computational errors, we introduce the following relaxed and perturbed algorithm:

$$x_0 \in Q,\ x_{n+1} = \beta_n x_n + (1-\beta_n)P_Q\left(\alpha_n f(x_n) + (I - \alpha_n F)Tx_n + e_n\right), \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ and $\{e_n\}$ is a sequence in $\mathscr{H}$ representing the perturbation. We will prove that any sequence $\{x_n\}$ generated by the algorithm converges strongly to q^* provided that the sequence $\{\alpha_n\}$ in (1.3) satisfies only the necessary conditions (C1) and (C2), the sequence $\{\beta_n\}$ in (1.3) is not too close to 0 or 1, and the perturbation $\{e_n\}$ is relatively small with respect to $\{\alpha_n\}$.

The paper is organized as follows. In Section 2, we recall some essential lemmas that are used frequently in the proof of the results of the paper. Section 3 is devoted to the study of the convergence of an implicit version of algorithm (1.3). The strong convergence of the iterative algorithm (1.3) is investigated in Section 4. Section 5 is devoted to the study of the limit case that the strong monotonicity coefficient of F is equal to the Lipschitzian coefficient of f. In the last section, Section 6, we investigate, through some numerical experiments, the effect of the sequences $\{\beta_n\}$ and $\{\alpha_n\}$ on the stability and the rate of convergence of algorithm (1.3).

2. Preliminaries

In this section, we recall some classical results that are useful in the proof of the main theorems of the paper. The first result is on the existence and the uniqueness of solutions of the variational problem (VIP).

Lemma 2.1. *Let $\lambda \in]0, \frac{\mu-\alpha}{(\kappa+\alpha)^2}[$. Then the mapping $\Psi_\lambda : Q \to Q$ defined by $\Psi_\lambda(x) = P_C(x - \lambda(F(x) - f(x)))$, where $C = F_{ix}(T)$ is a contraction with coefficient $\rho = \sqrt{1-\lambda(\mu-\alpha)}$. Moreover, the unique solution q^* of the problem (VIP) is also the unique fixed point of Ψ_λ.*

Proof. Define a mapping $G : Q \to H$ by $G(x) = F(x) - f(x)$. Let $x, y \in Q$. Using the facts that P_C is a nonexpancive and G is $(\mu - \alpha)$ strongly monotone and Lipschitz continuous with coefficient $\kappa + \alpha$, we have

$$\begin{aligned}
\|\Psi_\lambda(x) - \Psi_\lambda(y)\|^2 &\le \|(x-y) - \lambda(G(x) - G(y))\|^2 \\
&\le \left(1 - 2\lambda(\mu-\alpha) + \lambda^2(\kappa+\alpha)^2\right)\|x-y\|^2 \\
&\le \rho^2\|x-y\|^2,
\end{aligned}$$

where $\rho = \sqrt{1-\lambda(\mu-\alpha)} \in]0,1[$. This ends the proof of the first part of the lemma.

The second part of the lemma is a simple consequence of the variational characterization of the metric projection P_C. In fact, $q = \Psi_\lambda(q)$ is equivalent to $q \in C$ and $\langle q - (q - \lambda G(q), x - q\rangle \ge 0$, $\forall x \in C$, which is clearly equivalent to q, a solution to the problem (VIP). □

The next result is a powerful lemma proved by Xu in [10]. This lemma is a generalization of a result due to Bertsekas (see [11, Lemma 1.5.1]).

Lemma 2.2. *Let $\{a_n\}$ be a nonnegative real sequence with $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n r_n + \delta_n$, $n \geq 0$, where $\{\gamma_n\} \in [0,1]$, and $\{r_n\}$ and $\{\delta_n\}$ are two real sequences such that $\sum_{n=0}^{+\infty} \gamma_n = +\infty$; $\sum_{n=0}^{+\infty} |\delta_n| < +\infty$; $\limsup_{n\to+\infty} r_n \leq 0$. Then sequence $\{a_n\}$ converges to 0.*

The third result is the following lemma due to Suzuki [12].

Lemma 2.3. *Let $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $z_{n+1} = \beta_n z_n + (1-\beta_n)w_n$, $n \geq 0$ and $\limsup_{n\to\infty} (\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - w_n\| = 0$.*

The last result of this section is a particular case of the well-known demiclosedness principle (see [13, Corollary 4.18]).

Lemma 2.4. *Let $\{x_n\}$ be a sequence in Q. If $\{x_n\}$ converges weakly to some x and $\{x_n - Tx_n\}$ converges strongly to 0, then $x \in F_{ix}(T)$.*

3. The Convergence of An Implicit Version of the Algorithm (HPA)

In this section, we prove the strong convergence of the perturbed and implicit algorithm $x_t = P_Q(tf(x_t) + (I - tF)Tx_t + e(t))$. $\{x_t\}$, as $t \to 0^+$, converges to the unique solution q^* of the variational inequality problem (VIP) provided that the perturbation $e(t)$ is sufficiently small. More precisely, we prove the following theorem.

Theorem 3.1. *Set $\delta_0 := \min\{1, \frac{1}{2\eta}, \frac{\eta-\alpha}{\kappa^2}\}$. Let $e :]0,\delta_0] \to \mathscr{H}$ such that $\lim_{t\to0^+} \frac{\|e(t)\|}{t} = 0$. Then, for every $t \in]0,\delta_0]$, there exists a unique $x_t \in Q$ such that $x_t = P_Q(tf(x_t) + (I-tF)Tx_t + e(t))$. Moreover, x_t converges strongly in $\mathscr{H}$ as $t \to 0^+$ toward q^*, the unique solution of the variational inequality problem (VIP).*

The proof essentially relies on the following lemma, which will also be used in the next section, devoted to the study of the strong convergence of the algorithm (HPA).

Lemma 3.1. *For every $t \in]0,\delta_0]$, the mapping $S_t : Q \to \mathscr{H}$ defined by $S_t(x) = tf(x) + (I-tF)Tx$ is Lipschitzian with coefficient $\mu_t = 1 - t\sigma_0 \in]0,1[$, where $\sigma_0 = \frac{\eta-\alpha}{2}$.*

Proof. Let $t \in]0,\delta_0]$ and $x, y \in Q$. We have

$$\begin{aligned} &\|(I-tF)Tx - (I-tF)Ty\|^2 \\ &= \|Tx - Ty\|^2 - 2t\langle F(Tx) - F(Ty), Tx - Ty\rangle + t^2 \|F(Tx) - F(Ty)\|^2 \\ &\leq \left(1 - 2t(\eta - \frac{t\kappa^2}{2})\right) \|x - y\|^2 . \end{aligned}$$

Since $0 < t \leq \delta_0 \leq \min\{\frac{2\eta}{\kappa^2}, \frac{1}{2\eta}\}$, we have $0 \leq 2t(\eta - \frac{t\kappa^2}{2}) \leq 1$. Then, by using the elementary inequality $\sqrt{1-x} \leq 1 - \frac{x}{2}$, for all $x \in [0,1]$, we deduce that

$$\|(I-tF)Tx - (I-tF)Ty\| \leq \left(1 - t(\eta - \frac{t\kappa^2}{2})\right) \|x - y\| .$$

Therefore,

$$\begin{aligned}\|S_t(x)-S_t(y)\| &\leq t\,\|f(x)-f(y)\|+\|(I-tF)Tx-(I-tF)Ty\| \\ &\leq \left(t\alpha+1-t(\eta-\frac{t\kappa^2}{2})\right)\|x-y\| \\ &\leq \left(1-t(\eta-\alpha-\frac{\kappa^2\delta_0}{2})\right)\|x-y\| \\ &\leq (1-\frac{\eta-\alpha}{2}t)\,\|x-y\|\,(\text{since } \delta_0\leq\frac{\eta-\alpha}{\kappa^2}) \\ &= (1-\sigma_0 t)\,\|x-y\|,\end{aligned}$$

Finally, the facts $\eta>\alpha$ and $\delta_0<\frac{1}{2\eta}$ ensure that $\sigma_0>0$ and that, for every $t\in]0,\delta_0]$, $\mu_t = 1-\sigma_0 t\in]0,1[$. This completes the proof. □

Now, we are in position to prove Theorem 3.1.

Proof. Let $t\in]0,\delta_0]$. Since P_Q is nonexpansive, it follows from the previous lemma that the two mapping φ_t and ϕ_t, defined from Q to Q by $\varphi_t(x)=P_Q(S_t(x))$ and $\phi_t(x)=P_Q(S_t(x)+e(t))$ are contractions with the same coefficient $1-t\sigma_0\in[0,1[$. Hence the classical Banach fixed point theorem ensures the existence and the uniqueness of x_t and y_t in Q such that $x_t=P_Q(S_t(x_t)+e(t))$ and $y_t=P_Q(S_t(y_t))$. Reusing Lemma 3.1 and the fact that P_Q is nonexpansive, we obtain $\|x_t-y_t\|\leq(1-t\sigma_0)\|x_t-y_t\|+\|e(t)\|$, which implies that $\|x_t-y_t\|\leq\frac{\|e(t)\|}{t\sigma_0}$. Hence, from the assumption on $e(t)$, we have $\|x_t-y_t\|\to 0$ as $t\to 0^+$. Therefore, in order to prove that $x_t\to q^*$ as $t\to 0^+$, it suffices to prove that $y_t\to q^*$ as $t\to 0^+$. To do this, let us first prove that the family $(y_t)_{0<t\leq\delta_0}$ is bounded in $\mathscr{H}$. Pick $q\in F_{ix}(T)$. By using the fact that P_Q is nonexpansive and Lemma 3.1, we easily deduce that, for every $t\in]0,\delta_0]$,

$$\begin{aligned}\|y_t-q\| &\leq \|P_Q(S_t(y_t))-P_Q(S_t(q))\|+\|P_Q(S_t(q))-P_Q(q)\| \\ &\leq \|S_t(y_t)-S_t(q)\|+\|S_t(q)-q\| \\ &\leq (1-t\sigma_0)\,\|y_t-q\|+t\,\|f(q)-F(q)\|.\end{aligned}$$

Hence, $\sup_{0<t\leq\delta_0}\|y_t-q\|\leq\frac{\|f(q)-F(q)\|}{\sigma_0}$, which implies that $(y_t)_{0<t\leq\delta_0}$ is bounded in $\mathscr{H}$, and so is $(f(y_t)-F(Ty_t))_{0<t\leq\delta_0}$ since the mapping $f-F\circ T$ is Lipschitz continuous (with Lipschitz constant $\alpha+\kappa$). Therefore,

$$y_t-Ty_t\to 0 \text{ in } \mathscr{H} \text{ as } t\to 0^+. \tag{3.1}$$

For every $t\in]0,\delta_0]$, we have

$$\|y_t-Ty_t\|=\|P_Q(S(y_t))-P_Q(Ty_t)\|\leq\|S_t(y_t)-Ty_t\|=t\,\|f(y_t)-F(Ty_t)\|.$$

On the other hand, since $(y_t)_{0<t\leq\delta_0}$ is bounded in $\mathscr{H}$, there exists a sequence $(t_n)_n\in]0,\delta_0]$ which converges to 0 such that the sequence $\{y_{t_n}\}$ converges weakly in $\mathscr{H}$ to some y and

$$\limsup_{t\to 0^+}\langle y_t-q^*,f(q^*)-F(q^*)\rangle=\lim_{n\to+\infty}\langle y_{t_n}-q^*,f(q^*)-F(q^*)\rangle=\langle y-q^*,f(q^*)-F(q^*)\rangle.$$

Thanks to Lemma 2.4, we deduce from (3.1) that $y \in F_{ix}(T)$. Hence, from the definition of q^*, we infer that $\limsup_{t\to 0^+}\langle y_t - q^*, f(q^*) - F(q^*)\rangle \leq 0$, which together with (3.1) obtains

$$\limsup_{t\to 0^+}\langle Ty_t - q^*, f(q^*) - F(q^*)\rangle \leq 0. \tag{3.2}$$

Now, for every $t \in]0, \delta_0]$, we have

$$\begin{aligned}
\|y_t - q^*\|^2 &\leq \|S_t(y_t) - q^*\|^2 \\
&= \|S_t(y_t) - S_t(q^*)\|^2 + 2\langle S_t(y_t) - q^*, S_t(q^*) - q^*\rangle - \|S_t(q^*) - q^*\|^2 \\
&\leq \|S_t(y_t) - S_t(q^*)\|^2 + 2\langle S_t(y_t) - q^*, S_t(q^*) - q^*\rangle \\
&\leq (1 - t\sigma_0)^2\,\|y_t - q^*\|^2 + 2t\langle Ty_t - q^*, f(q^*) - F(q^*)\rangle \\
&\quad + 2t^2\langle f(y_t) - F(Ty_t), f(q^*) - F(q^*)\rangle,
\end{aligned}$$

where we used the facts that $S_t(y_t) = Ty_t + t(f(y_t) - F(Ty_t)$ and $S_t(q^*) - q^* = t(f(q^*) - F(q^*)$. Recalling that $(y_t)_{0<t\leq\delta_0}$ and $(f(y_t) - F(Ty_t))_{0<t\leq\delta_0}$ are bounded in $\mathscr{H}$, we deduce that there exists a positive constant C such that, for every $t \in]0, \delta_0]$,

$$\|y_t - q^*\|^2 \leq (1 - 2\sigma_0 t)\,\|y_t - q^*\|^2 + 2t\,\langle Ty_t - q^*, f(q^*) - F(q^*)\rangle + Ct^2.$$

Therefore, for every $t \in]0, \delta_0]$,

$$\|y_t - q^*\|^2 \leq \frac{1}{\sigma_0}\left(\langle Ty_t - q^*, f(q^*) - F(q^*)\rangle + \frac{C}{2}t\right)$$

Hence, by using (3.2), we conclude that $y_t \to q^*$ in $\mathscr{H}$ as $t \to 0^+$. This completes the proof. $\square$

4. The Convergence of the Algorithm (HPA)

In this section, we study the strong convergence of the averaged and perturbed algorithm (HPA)

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)P_Q\left(\alpha_n f(x_n) + (I - \alpha_n F)Tx_n + e_n.\right) \tag{HPA}$$

We prove the following result.

Theorem 4.1. *Let $\{e_n\}$ be a sequence in $\mathscr{H}$, and let $\{\alpha_n\} \in]0, 1]$ and $\{\beta_n\} \in [0, 1]$ be two real sequences such that:*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{+\infty}\alpha_n = +\infty$

(ii) One of the two following two conditions is satisfied:

(h1) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

(h2) $\limsup_{n\to\infty}\beta_n < 1$, *either* $\frac{\beta_{n+1}-\beta_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty}|\beta_{n+1} - \beta_n| < \infty$ *and either* $\frac{\alpha_{n+1}-\alpha_n}{\alpha_n} \to 1$ *or* $\sum_{n=0}^{+\infty}|\alpha_{n+1} - \alpha_n| < \infty$.

(iii) $\sum_{n=0}^{+\infty}\|e_n\| < \infty$ *or* $\frac{\|e_n\|}{\alpha_n} \to 0$.

Then, for every $x_0 \in Q$, the sequence $\{x_n\}$ generated by the algorithm (HPA) converges strongly in $\mathscr{H}$ to q^* the unique solution of the variational inequality problem (VIP).

Proof. Since we are only interested in the study of the asymptotic behavior of the sequence $\{x_n\}$ and $\alpha_n \to 0$ as $n \to \infty$, we can assume without loss of generality that, for all $n \in \mathbb{N}$, $\alpha_n \in]0, \delta_0]$, where δ_0 is the real defined in Theorem 3.1. Let $\{y_n\}$ be the sequence defined as follows

$$y_0 = x_0,\ y_{n+1} = \beta_n y_n + (1-\beta_n) P_Q(\alpha_n f(y_n) + (I - \alpha_n F)Ty_n),\ n \geq 0.$$

Using the fact P_Q is nonexpansive and Lemma 3.1, we obtain

$$\begin{aligned}\|y_{n+1} - x_{n+1}\| &\leq \beta_n \|y_n - x_n\| + (1-\beta_n)\|P_Q(S_{\alpha_n}(y_n)) - P_Q(S_{\alpha_n}(x_n) + e_n)\| \\ &\leq [\beta_n + (1-\beta_n)(1-\sigma_0\alpha_n)]\|y_n - x_n\| + (1-\beta_n)\|e_n\| \\ &\leq (1-\gamma_n)\|y_n - x_n\| + \|e_n\|,\end{aligned}$$

where $\gamma_n = \sigma_0(1-\beta_n)\alpha_n$. Since $\limsup_{n\to\infty}\beta_n < 1$, there exist $a > 0$ and $n_0 \in \mathbb{N}$ such that $a\alpha_n \leq \gamma_n \leq 1$ for all $n \geq n_0$. Hence, by applying Lemma 2.2, we deduce that

$$y_n - x_n \to 0. \tag{4.1}$$

Therefore, it suffices to prove that $\{y_n\}$ converges strongly to q^* to conclude that $\{x_n\}$ also converges strongly to q^*.

Let us first prove that $\{y_n\}$ is bounded in $\mathscr{H}$. Let $q \in F_{ix}(T)$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|y_{n+1} - q\| &\leq \beta_n\|y_n - q\| + (1-\beta_n)[\|P_Q(S_{\alpha_n}(y_n)) - P_Q(S_{\alpha_n}(q))\| + \|P_Q(S_{\alpha_n}(q)) - P_Q(q)\|] \\ &\leq \beta_n\|y_n - q\| + (1-\beta_n)[\|S_{\alpha_n}(y_n) - S_{\alpha_n}(q)\| + \|S_{\alpha_n}(q) - q\|] \\ &\leq \beta_n\|y_n - q\| + (1-\beta_n)[(1-\sigma_0\alpha_n)\|y_n - q\| + \alpha_n\|f(q) - F(q)\|].\end{aligned}$$

The last inequality immediately implies that the sequence $v_n := \max\{\|y_n - q\|, \frac{\|f(q)-F(q)\|}{\sigma_0}\}$ is decreasing. Therefore, $\{y_n\}$ is bounded in $\mathscr{H}$, so are $\{f(y_n)\}$ and $\{F(Ty_n)\}$ since the mappings f and $F \circ T$ are Lipschitz continuous.

Now we prove that

$$y_n - Ty_n \to 0. \tag{4.2}$$

Let us first assume that condition (h1) is satisfied. For every $n \in \mathbb{N}$, setting

$$z_n = P_Q(\alpha_n f(y_n) + (I - \alpha_n F)Ty_n),$$

we have, the sequences $\{y_n\}$ and $\{z_n\}$ are bounded in $\mathscr{H}$, $y_{n+1} = \beta_n y_n + (1-\beta_n)z_n$ for every n, and

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq (\alpha_n + \alpha_{n+1})\sup_{m\geq 0}\|f(y_m) - F(Ty_m)\| + \|Ty_{n+1} - Ty_n\| \\ &\leq (\alpha_n + \alpha_{n+1})\sup_{m\geq 0}\|f(y_m) - F(Ty_m)\| + \|y_{n+1} - y_n\|\end{aligned}$$

which implies $\limsup_{n\to\infty}\|z_{n+1} - z_n\| - \|y_{n+1} - y_n\| \leq 0$. Therefore, from Lemma 2.3, we deduce that $z_n - y_n \to 0$, which together with the fact that

$$\begin{aligned}\|z_n - Ty_n\| &= \|P_Q(\alpha_n f(y_n) + (I - \alpha_n F)Ty_n) - P_Q(Ty_n)\| \\ &\leq \alpha_n \sup_{m\geq 0}\|f(y_m) - F(Ty_m)\| \to 0 \text{ as } n \to +\infty,\end{aligned}$$

implies the required result (4.2).

Let us now establish (4.2) under the assumption (h2). A simple computation with the fact that the sequence $\{y_n\}$ and $\{P_Q(\alpha_n f(y_n) + (I-\alpha_n F)Ty_n)\}$ are bounded in $\mathscr{H}$ ensures the existence of two real constants $M_1, M_2 > 0$ such that, for every $n \in \mathbb{N}$,

$$\begin{aligned}\|y_{n+1}-y_n\| &\leq \beta_n \|y_n - y_{n-1}\| + (1-\beta_n)\left\|P_Q(S_{\alpha_n}(y_n)) - P_Q(S_{\alpha_{n-1}}(y_{n-1}))\right\| + M_1|\beta_n - \beta_{n-1}| \\ &\leq \beta_n \|y_n - y_{n-1}\| + (1-\beta_n)\left\|S_{\alpha_n}(y_n) - S_{\alpha_{n-1}}(y_{n-1})\right\| + M_1|\beta_n - \beta_{n-1}| \\ &\leq \beta_n \|y_n - y_{n-1}\| + (1-\beta_n)\|S_{\alpha_n}(y_n) - S_{\alpha_n}(y_{n-1})\| \\ &\quad + (1-\beta_n)\left\|S_{\alpha_n}(y_{n-1}) - S_{\alpha_{n-1}}(y_{n-1})\right\| + M_1|\beta_n - \beta_{n-1}| \\ &\leq (1-\sigma_0(1-\beta_n)\alpha_n)\|y_n - y_{n-1}\| + M_1|\beta_n - \beta_{n-1}| + M_2|\alpha_n - \alpha_{n-1}|.\end{aligned}$$

Hence, by proceeding as in the proof of (4.1), we infer that

$$\|y_{n+1}-y_n\| \to 0. \tag{4.3}$$

On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|y_{n+1}-Ty_n\| &\leq \beta_n\|y_n - Ty_n\| + (1-\beta_n)\|P_Q(S_{\alpha_n}(y_n)) - P_Q(Ty_n)\| \\ &\leq \beta_n\|y_{n+1}-Ty_n\| + \beta_n\|y_{n+1}-y_n\| + \|S_{\alpha_n}(y_n) - Ty_n\| \\ &\leq \beta_n\|y_{n+1}-Ty_n\| + \|y_{n+1}-y_n\| + \alpha_n\|f(y_n) - F(Ty_n)\|.\end{aligned}$$

Hence, we obtain the inequality

$$\|y_{n+1}-Ty_n\| \leq \frac{1}{1-\beta_n}\left(\|y_{n+1}-y_n\| + \alpha_n \sup_{m\geq 0}\|f(y_m) - F(Ty_m)\|\right),$$

which together with (4.3) and the fact that $\limsup_{n\to+\infty}\beta_n < 1$ implies that $\|y_{n+1}-Ty_n\| \to 0$. Hence, from (4.3), we obtain (4.2).

Next, we use the fundamental lemma, Lemma 2.2, to conclude that $\{y_n\}$ converges strongly to q^*. By proceeding as in the proof of Theorem 3.1 and by using (4.2) and the fact that $\{y_n\}$ is bounded, we deduce that $\limsup_{n\to\infty}\langle y_n - q^*, f(q^*) - F(q^*)\rangle \leq 0$, which together with (4.2) yields

$$\limsup_{n\to\infty}\langle Ty_n - q^*, f(q^*) - F(q^*)\rangle \leq 0. \tag{4.4}$$

Finally, for every $n \in \mathbb{N}$,

$$\begin{aligned}\|y_{n+1}-q^*\|^2 &\leq \beta_n\|y_n - q^*\|^2 + (1-\beta_n)\|P_Q(S_{\alpha_n}(y_n)) - P_Q(q^*)\|^2 \\ &\leq \beta_n\|y_n - q^*\|^2 + (1-\beta_n)\|S_{\alpha_n}(y_n)) - q^*\|^2 \\ &= \beta_n\|y_n - q^*\|^2 + (1-\beta_n)[\|S_{\alpha_n}(y_n)) - S_{\alpha_n}(q^*)\|^2 \\ &\quad + 2\langle S_{\alpha_n}(y_n)) - q^*, S_{\alpha_n}(q^*) - q^*\rangle - \|S_{\alpha_n}(q^*) - q^*\|^2] \\ &\leq \beta_n\|y_n - q^*\|^2 + (1-\beta_n)[(1-\alpha_n\sigma_0)^2\|y_n - q^*\|^2 \\ &\quad + 2\alpha_n^2\langle f(y_n) - F(Ty_n), f(q^*) - F(q^*)\rangle + 2\alpha_n\langle Ty_n - q^*, f(q^*) - F(q^*)\rangle] \\ &\leq (1-\gamma_n)\|y_n - q^*\|^2 + \gamma_n r_n\end{aligned}$$

where $C > 0$ is a constant independent of n, $\gamma_n = 2\sigma_0(1-\beta_n)\alpha_n$, and

$$r_n = \frac{1}{\sigma_0}\left(\langle Ty_n - q^*, f(q^*) - F(q^*)\rangle + C\alpha_n\right).$$

Using estimate (4.4), we obtain $\limsup r_n \leq 0$. Hence, by applying Lemma 2.2, we conclude as previously that the sequence $\{y_n\}$ converges strongly to q^*. This completes the proof of Theorem 4.1. □

5. The Study of the Limit Case $\mu = \alpha$

Throughout this section, we assume that $\mu = \alpha$. In this limit case, the operator $F - f$ is monotone but not necessary strongly monotone; so the uniqueness of the solution of the variational inequality problem (VIP) is no long assured. Moreover, We assume that (VIP) has at least one solution. We denote by S_{VIP} the set of the solutions of (VIP). The following theorem provides a method to approximate a particular element of the set S_{VIP}.

The following lemma is essential to the theorem in this section.

Lemma 5.1. *(Bruck [14]) Let $A : D(A) \subset \mathscr{H} \to 2^{\mathscr{H}}$ be a maximal monotone operator with $A^{-1}(0) \neq \varnothing$. Then for any $u \in \mathscr{H}$, $(I+tA)^{-1}u \to P_{A^{-1}(0)}(u)$ as $t \to +\infty$.*

Theorem 5.1. *Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{e_n\}$ satisfy the same assumptions as in Theorem 4.1. Then, for every $\varepsilon > 0$ and $x_0 \in Q$, the sequence $\{x_n^\varepsilon\}$ defined by the recursive formula*

$$x_{n+1}^\varepsilon = \beta_n x_n^\varepsilon + (1-\beta_n)P_Q\left(\alpha_n f(x_n^\varepsilon) + ((1-\alpha_n\varepsilon)I - \alpha_n F)Tx_n^\varepsilon + e_n\right),\ n \geq 0,$$

converges strongly to q^ε, the unique solution to the variational inequality problem

$$\text{Find } q \in C \text{ such that } \langle F(q) + \varepsilon q - f(q), x - q\rangle \geq 0, \quad \forall x \in C. \qquad (\mathrm{VIP}_\varepsilon)$$

Moreover, set S_{VIP} is closed and convex and q^ε converges strongly as $\varepsilon \to 0$ to the nearest element of S_{VIP} to the origin.

Proof. For every $\varepsilon > 0$, $F_\varepsilon := F + \varepsilon I$ is $\mu + \varepsilon$ strongly monotone and $\kappa + \varepsilon$ Lipschitizian. Since $\mu + \varepsilon > \alpha$, the first part of this theorem follows immediately from Theorem 4.1.

Let $\delta_C : C \to \mathscr{H}$ be the indicator function associated to the closed, convex, and nonempty subset C. Recall that δ_C is defined by

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

It is well known that δ_C is a proper, lower semi-continuous, and convex function. Hence, its sub-gradient $\partial\delta_C$ is a maximally monotone operator with the domain equal to C. We recall that, for every $x \in C$, $\partial\delta_C(x) = \{u \in \mathscr{H} : \langle u, y - x\rangle \leq 0\}$. It is then easily seen that set S_{VIP} is equal to $A^{-1}(0)$, the set of zeros of the operator $A := F - f + \delta_C$. From [15], operator A is maximally monotone. Therefore, S_{VIP} is a closed and convex subset of $\mathscr{H}$. On the other hand, the unique solution q^ε of (VIP_ε) satisfies the relation $-\left(F(q^\varepsilon) + \varepsilon q^\varepsilon - f(q^\varepsilon)\right) \in \delta_C(q^\varepsilon)$, which is equivalent to $0 \in q^\varepsilon + \frac{1}{\varepsilon}A(q^\varepsilon)$, since $\frac{1}{\varepsilon}\delta_C(q^\varepsilon) = \delta_C(q^\varepsilon)$. Therefore, $q^\varepsilon = J_{\frac{1}{\varepsilon}}(0)$, where for every $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1}$ is the resolvent of A (for more details, see the pioneer paper [16] of Minty). Hence, from Lemma 5.1, we deduce that q^ε converges strongly as $\varepsilon \to 0$ to $P_{A^{-1}(0)} = P_{S_{VIP}}(0)$ which is the element of S_{VIP} with minimal norm. □

Remark 5.1. From the previous theorem, we expect, but we do not yet have the justification, that, under some appropriate assumptions on the real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\varepsilon_n\}$, the sequences $\{x_n\}$ generated by the iterative process

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_Q\left(\alpha_n f(x_n) + ((1-\alpha_n\varepsilon_n)I - \alpha_n F)Tx_n\right),\ n \geq 0,$$

where x_0 is an arbitrary element of Q, converge strongly in $\mathscr{H}$ to $u^* = P_{S_{VIP}}(0)$. Observe that Reich and Xu in [17] raised a similar open question related to the constrained least squares problem

6. Numerical Experiments

In this section, we aim to study, through some simple numerical experiments, the effects of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ on the stability and the rate of convergence of the perturbed algorithm (HPA). We consider here the simple example:

(1) The Hilbert space $\mathscr{H}$ is $\mathbb{R}^2$ endowed with its natural inner product $\langle x,y\rangle = x_1y_1 + x_2y_2$ and associated Euclidean norm $\|x\| = \sqrt{\langle x,x\rangle}$.
(2) The closed and convex subset Q is given by: $Q = \{x = (x_1,x_2)^t \in \mathbb{R}^2 : x_1,x_2 \geq 0\}$.
(3) The mapping $f : Q \to Q$ is defined by $f(x) = (5+\cos(x_1+x_2), 6-\sin(x_1+x_2))^t$ for all $x = (x_1,x_2)^t \in Q$. One can easily verify that f is a Lipschitz continuous function with Lipschitz constant $\alpha = \sqrt{2} \simeq 1.41$.
(4) The mapping $F : Q \to \mathbb{R}^2$ is the given by: $F(x) = Ax$, for every $x = (x_1,x_2)^t \in Q$, where
$$A = \begin{pmatrix} 8 & 2 \\ 2 & 4 \end{pmatrix}.$$
Since the matrix A is symmetric and defined positive, F is strongly monotone with coefficient $\mu = \lambda_1(A) = 2(3-\sqrt{2}) \simeq 3.17$ (the smallest eigenvalue of A and Lipschitz continuous with coefficient $\kappa = \lambda_2(A) = 2(3+\sqrt{2}) \simeq$ (the largest eigenvalue of A).
(5) The nonexpansive mapping $T : Q \to Q$ is defined by: $T(x) = P_Q((x_2-4, x_1+4)^t)$ for every $x = (x_1,x_2)^t \in Q$. It is clear that $C = F_{ix}(T) = \{x = (x_1,x_2)^t \in Q : x_2 - x_1 = 4\}$. In this simple case, the projection P_C is explicitly defined by:
$$P_C((x_1,x_2)^t) = \begin{cases} (0,4)^t & \text{if } x_1+x_2 \leq 4, \\ (\frac{x_1+x_2}{2} - 2, \frac{x_1+x_2}{2} + 2)^t & \text{if } x_1+x_2 \geq 4. \end{cases}$$
Hence a simple routine on Matlab, using Lemma 2.1 and the iterative algorithm of Banach, gives a precise numerical approximation value of the solution q^* of the problem (VIP) $q^* = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.
(6) The perturbation (or error computational) term $\{e_n\}$ is given by $e_n = \frac{6}{n^2}X_n$, where $\{X_n\}$ is sequence of independent variables such that every X_n is uniform on the square $[-1,1]\times[-1,1]$.
(7) The sequence $\{\alpha_n\}$ takes the form $\alpha_n = \frac{1}{n^\theta}$ where $\theta \in]0,1]$.
(8) The sequence $\{\beta_n\}$ is equal to a constant $\beta \in]0,1[$.

We can summarize our numerical results in the following four points:

(A) The choice of the sequence $\{\beta_n\}$ close to 1 reduces the fluctuation of the algorithm due to the perturbation term $\{e_n\}$. This fact is illustrated by Figure 1 with $\theta = 0.8$ and β taking the values $0.1,\ 0.5$, and 0.9.

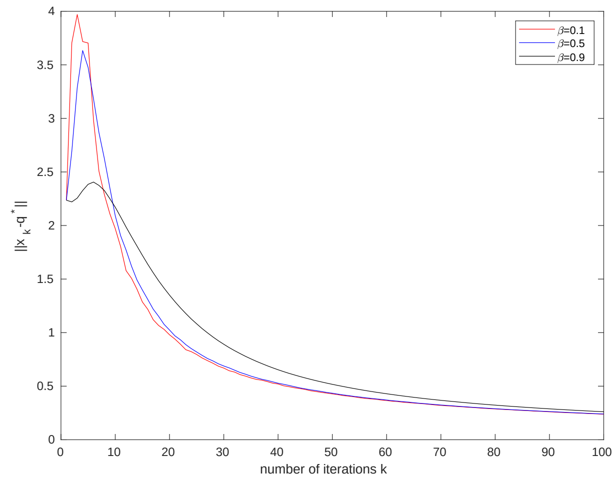

FIGURE 1. The effect of β on the fluctuation and stability the algorithm (HPA)

(B) The sequence $\{\beta_n\}$ has practically no effect on the convergence rate of the algorithm (HPA) as it is shown in Table 1 and Figure 2 with $\theta = 0.8$ and β taking the values $0.1, 0.5$, and 0.9.

ε	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$
0.1	291	292	300
0.05	687	687	695
0.01	5103	5104	5112

TABLE 1. $K(\varepsilon, \beta) := \min\{k \leq N_{\max} : \|x_k - q^*\| \leq \varepsilon\}$

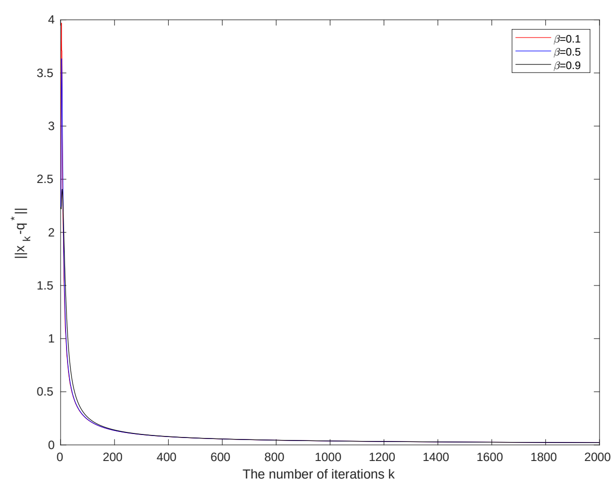

FIGURE 2. The effect of β on the convergence rate of the algorithm (HPA)

(C) If θ is small (close to 0), the convergence of $\{x_n\}$ to q^* is very slow and not clear (see Table 2).

θ	$\min_{k\leq N_{\max}} \|x_k - q^*\|$
0.1	2.2361
0.2	1.9077
0.3	0.7258
0.4	0.2916

TABLE 2. Slow convergence of the algorithm (HPA) for small values of θ

(D) The convergence rate of $\{x_n\}$ to q^* increases as θ approaches 1. This fact is illustrated in Table 3 and Figure 6 with $\beta = 0.5$ and θ taking the values $0.6, 0.8$, and 1. Observe here that, in Table 3, $N(\varepsilon, \theta) =$ ND (Not Defined) means that $\|x_k - q^*\| > \varepsilon\}$ for all the iterations $k \leq N_{max}$.

ε	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1.0$
0.50	141	43	22
0.10	1916	292	95
0.05	ND	687	188
0.01	ND	5104	928
0.005	ND	ND	1852
0.002	ND	ND	4625

TABLE 3. $N(\varepsilon, \theta) := \min\{k \leq N_{\max} : \|x_k - q^*\| \leq \varepsilon\}$

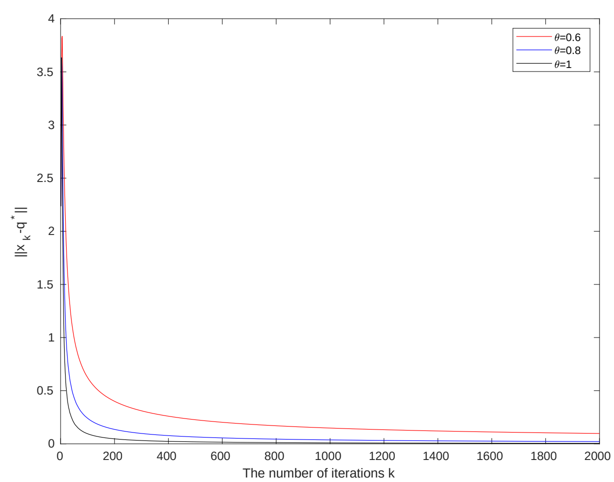

FIGURE 3. The convergence of the algorithm (HPA) for θ close to 1

Conclusion: The convergence of the algorithm (HPA) is in general very slow especially when the sequence $\{\alpha_n\}$ converges slowly towards zero. The sequence $\{\beta_n\}$ plays a role on the stability of the algorithm (HPA). In fact, it can reduce the fluctuation of the algorithm but $\{\beta_n\}$ has practically no effect on the speed of the convergence of the algorithm.

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