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EXISTENCE OF RADIAL SIGN-CHANGING SOLUTIONS FOR FRACTIONAL KIRCHHOFF-TYPE PROBLEMS IN $\mathbb{R}^3$

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Abstract. In this paper, the following fractional Kirchhoff-type problem

$$\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+V(x)u=f(x,u),\ \ x\in\mathbb{R}^3,$$

where $a,b>0$ are constants, $s\in(\frac{3}{4},1)$, $2_s^*=\frac{6}{3-2s}$, $V:\mathbb{R}^3\to\mathbb{R}$ is a continuous function, and $f:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ is a continuous function, is considered. It is demonstrated that the fractional Kirchhoff-type equation has a radial sign-changing solution u_b and a radial solution $\bar{u}_b$ when f does not satisfy the subcritical growth condition and the usual Nehari-type monotonicity condition. The main tools are the constraint variational method and some analysis techniques.

Keywords. Fractional Kirchhoff type problems; Sign-changing solution; Variational method.

1. Introduction and Main Results

This paper is concerned with the existence of radial sign-changing solutions for the following fractional Kirchhoff-type problem

$$\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+V(x)u=f(x,u),\ \ x\in\mathbb{R}^3, \tag{1.1}$$

where a and b are positive parameters, $s\in(\frac{3}{4},1)$, and $2_s^*=\frac{6}{3-2s}$ is the Sobolev embedding exponent. The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u=C_{3,s}P.V.\int_{\mathbb{R}^3}\frac{u(x)-u(y)}{|x-y|^{3+2s}}dy=-\frac{C_{3,s}}{2}\int_{\mathbb{R}^3}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}dy,\ u\in\mathscr{S}(\mathbb{R}^3),$$

where $C_{3,s}$ is a normalization constant depending on 3 and s, $P.V.$ stands for the Cauchy principal value of the integration, and $\mathscr{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying functions.

For the potential $V(x)$, we impose the following conditions:

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(V_1) $V \in \mathscr{C}(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$, where V_0 is a positive constant; $V(x) = V(|x|)$, and the operator $(-\Delta)^s + V(x) : H^s(\mathbb{R}^3) \to H^{-s}(\mathbb{R}^3)$ satisfies

$$\inf_{u\in H^s(\mathbb{R}^3),\|u\|_2=1} \int_{\mathbb{R}^3} \left(a|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2\right) \mathrm{d}x > 0;$$

(V_2) there exists a sequence $\{t_n\} \subset (0,\infty)$ such that $t_n \to \infty$ and $\sup_{x\in\mathbb{R}^3,n\in\mathbb{N}} \frac{V(t_n x)}{t_n^{5-4s}V(x)} < \infty$.

For the nonlinearity f, we assume that:

(F_1) $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^3$;

(F_2) $f \in \mathscr{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and $\lim_{|t|\to\infty} \frac{f(x,t)}{t^{2_s^*-1}} = 0$ uniformly in $x \in \mathbb{R}^3$;

(F_3) $\lim_{|t|\to\infty} \frac{|t|^{4s-3} f(x,t)}{t^3} = +\infty$ uniformly in $x \in \mathbb{R}^3$;

(F_4) $\frac{f(x,t)-V(x)t}{|t|^3}$ is nondecreasing in t on both $(-\infty,0)$ and $(0,\infty)$ for every $x \in \mathbb{R}^3$.

In (1.1), if we set $s = 1$, $V(x) = 0$, and replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^N$, respectively, we gave the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u) & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega. \end{cases} \tag{1.2}$$

In recent years, the following fractional Kirchhoff type equation:

$$-\left(a + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)(-\Delta)^s u = f(u) \ \ ,x \in \mathbb{R}^N,$$

was studied extensively by using various nonlinear analytical methods. We refer to [1] when f is subcritical growth, and to [2] for the critical nonlinearity f. For more existence results of fractional Kirchhoff type problems, we refer to [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein.

When $a = 1$ and $b = 0$, then problem (1.1) reduces to the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u) \ \ \text{in}\,\mathbb{R}^N. \tag{1.3}$$

This was proposed by Laskin [13] in fractional quantum mechanics as a result of the extension of Feynman integrals from the Brownian like to the Lèvy like quantum mechanical paths. For the existence, the multiplicity, and the behavior of solutions to (1.3), we refer the reader to [14, 15, 16, 17, 18] and the references therein.

In recent years, Cheng and Gao [19] studied the existence and asymptotic behavior of sign-changing solutions for (1.1), where f satisfies (F_1) and the following assumptions:

$(F_{2'})$ $f \in \mathscr{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $C_0 > 0$ and $2 < p < 2_s^*$ such that $|f(x,t)| \leq C_0(1 + |t|^{p-1})$, $\forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$;

$(F_{3'})$ $\lim_{|t|\to\infty} \frac{f(x,t)}{t^3} = +\infty$ uniformly in $x \in \mathbb{R}^3$;

$(F_{4'})$ $\frac{f(x,t)}{|t|^3}$ is nondecreasing in t on both $(-\infty,0)$ and $(0,\infty)$ for every $x \in \mathbb{R}^3$.

Recently, Chen, Tang and Liao [20] proved the existence of radial sign-changing solutions of (1.1) when f satisfies (F_1), $(F_{2'})$, (F_3), and (F_4). The inspiration of this paper mainly comes from [20]. It is worthwhile pointing out that, under our assumptions, condition (F_2) is weaker than condition $(F_{2'})$. The main purpose of this paper is to study the existence of radial sign-changing solutions of problem (1.1) when f does not satisfy the subcritical growth condition and the usual Nehari-type monotonicity condition. Based on the constraint variational method and some analysis techniques, we prove the same result under more generic conditions, which generalizes the results presented in [20]. From the technical points of view, the difficulty in finding sign-changing solutions of (1.1) results from two nonlocal terms: $(-\Delta)^s u$ and $\|(-\Delta)^{\frac{s}{2}}u\|_2^2(-\Delta)^s u$. In this sense, (1.1) is different from the classical case $s=1$ and the methods of finding sign-changing solutions for (1.3) with $s\in(0,1]$, and (1.2) cannot be directly applied to (1.1). This gives rise to some mathematical difficulties that make the study of the sign-changing solutions for (1.1) particularly interesting. In this paper, by combining the constraint variational method with some new inequalities, we prove that (1.1) with $b\geq 0$ has a radial sign-changing solution u_b and a radial solution $\overline{u}_b$.

Before stating our main result, let us consider the fractional Laplacian in the weak sense. As a rule, for any $s\in(0,1)$, we have

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v = C_{3,s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y$$

and

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = C_{3,s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{[u(x)-u(y)]^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y,$$

and define the fractional Sobolev space $H^s(\mathbb{R}^3)$ as follows $H^s(\mathbb{R}^3)=\left\{u\in L^2(\mathbb{R}^3):(-\Delta)^{\frac{s}{2}}u\in L^2(\mathbb{R}^3)\right\}$, equipped with the scalar product $(u,v)_{H^s(\mathbb{R}^3)}=\int_{\mathbb{R}^3}\left[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v+uv\right]\mathrm{d}x$, and the corresponding norm

$$\|u\|_{H^s(\mathbb{R}^3)}=\left(\int_{\mathbb{R}^3}\left[|(-\Delta)^{\frac{s}{2}}u|^2+u^2\right]\mathrm{d}x\right)^{\frac{1}{2}}.$$

Throughout this paper, we define $H_r^s(\mathbb{R}^3)=\left\{u\in H^s(\mathbb{R}^3):u(x)=u(|x|)\right\}$, and denote the fractional Sobolev space for (1.1) by $H=\left\{u\in H_r^s(\mathbb{R}^3):\int_{\mathbb{R}^3}V(x)u^2\mathrm{d}x<\infty\right\}$, where the scalar product is given by $(u,v)=\int_{\mathbb{R}^3}\left[a(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v+V(x)uv\right]\mathrm{d}x$, and the associated norm is

$$\|u\|=\left(\int_{\mathbb{R}^3}\left[a|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2\right]\mathrm{d}x\right)^{\frac{1}{2}}.$$

Under the condition (V_1) and $a>0$, the embedding $H\hookrightarrow H_r^s(\mathbb{R}^3)$ is continuous. We know from [21] that the embedding $H\hookrightarrow L^q(\mathbb{R}^3)$ is compact for $2<q<2_s^*$ when $s\in(0,1)$.

We say that $u\in H$ is a weak solution to (1.1) if

$$\begin{aligned}0&=\langle I_b'(u),\varphi\rangle\\&=\int_{\mathbb{R}^3}\left[a(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi+V(x)u(x)\varphi(x)\right]\mathrm{d}x\\&\quad+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)\varphi(x)\mathrm{d}x\end{aligned}\tag{1.4}$$

for any $\varphi \in H$. We will omit weak throughout this paper for convenience. Define the corresponding energy functional $I_b : H \to \mathbb{R}$ to problem (1.1) as below:

$$I_b(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left[a|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2\right]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}F(x,u)\mathrm{d}x. \quad (1.5)$$

Analogously to [22, 23, 24, 25], also in the case that the nonlinear term does not satisfy the subcritical growth condition, by (F_1) and (F_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|F(x,t)| \le \varepsilon t^2 + C_\varepsilon |t|^{2_s^*}$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. It is easy to see that I_b belongs to $\mathscr{C}^1(H,\mathbb{R})$ and the critical points of I_b are the solutions to (1.1). Furthermore, if $u \in H$ is a solution to (1.1) and $u^{\pm} \neq 0$, we say that u is a radial sign-changing solution of (1.1), where $u^+(x) = \max\{u(x),0\}$ and $u^-(x) = \min\{u(x),0\}$.

Our goal in this paper is to seek the sign-changing solutions of (1.1). So, we borrow some ideals from [19, 20, 26, 27, 28]. We first try to seek a minimizer of the energy functional I_b over the following constraints:

$$\mathscr{M}_b := \{u \in H : u^{\pm} \neq 0, \langle I_b'(u), u^+\rangle = \langle I_b'(u), u^-\rangle = 0\},$$
$$m_b := \inf_{u\in\mathscr{M}_b} I_b(u),\ \ \forall b \ge 0,$$
$$\mathscr{N}_b := \{u \in H : u \neq 0, \langle I_b'(u), u\rangle = 0\},$$
$$c_b := \inf_{u\in\mathscr{N}_b} I_b(u),\ \ \forall b \ge 0,$$

and then prove that the minimizers of m_b is radial sign-changing solutions of (1.1) and the minimizers of c_b are ground state solutions to (1.1).

When $s = 1, b = 0$, and $a = 1$, (1.1) turns out to be the (1.3) mentioned above. There are several ways in the literature to obtain sign-changing solution for (1.3); see [29, 30, 31, 32] and the references therein. However, there only exist few results on the sign-changing solutions of (1.1). Indeed, in the case $s \in (0,1)$, we have the following decomposition:

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}u^+ + (-\Delta)^{\frac{s}{2}}u^-\|_2^2 &= \|(-\Delta)^{\frac{s}{2}}u^+\|_2^2 + \|(-\Delta)^{\frac{s}{2}}u^-\|_2^2 \\ &\quad -4C_{3,s}\int_{\mathbb{R}^6}\frac{u^+(x)u^-(y)}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y,\ \ \forall u \in H^s(\mathbb{R}^3). \end{aligned} \quad (1.6)$$

Since $\langle u^+, u^-\rangle_{H^s(\mathbb{R}^3)} > 0$ when $u^{\pm} \neq 0$, a straightforward computation yields that

$$\begin{aligned} I_b(u) &= I_b(u^+) + I_b(u^-) + 2aP(u^+,u^-) + \frac{b}{2}\|(-\Delta)^{\frac{s}{2}}u^+\|_2^2\|(-\Delta)^{\frac{s}{2}}u^-\|_2^2 \\ &\quad +2bP(u^+,u^-)\left[\|(-\Delta)^{\frac{s}{2}}u^+\|_2^2 + \|(-\Delta)^{\frac{s}{2}}u^-\|_2^2 + 2P(u^+,u^-)\right] \\ &\quad > I_b(u^+) + I_b(u^-), \\ &\langle I_b'(u), u^+\rangle > \langle I_b'(u^+), u^+\rangle \ \text{ and } \ \langle I_b'(u), u^-\rangle > \langle I_b'(u^-), u^-\rangle, \\ &\forall u \in H,\ u^+, u^- \neq 0, \end{aligned} \quad (1.7)$$

where

$$P(u^+,u^-) := -C_{3,s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^+(x)u^-(y)}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y > 0, \qquad \forall u \in H,\ u^+, u^- \neq 0,$$

which implies that $u^{\pm} \notin \mathscr{N}_b$ for $u \in \mathscr{M}_b$.

The main result can be stated as follows.

Theorem 1.1. *Suppose that* $(V_1)-(V_2)$ *and* $(F_1)-(F_4)$ *hold. Then problem* (1.1) *has a radial sign-changing solution* $u_b \in \mathscr{M}_b$ *such that* $I_b(u_b) = \inf_{\mathscr{M}_b} I_b > 0$ *and has a radial solution* $\overline{u}_b \in \mathscr{N}_b$ *such that* $I_b(\overline{u}_b) = \inf_{\mathscr{N}_b} I_b > 0$.

Remark 1.1. We know that (F_2) is obviously weaker than $(F_{2'})$. There exist functions satisfying the generalized subcritical condition (F_2) and not satisfying the subcritical growth condition $(F_{2'})$. For example, for the sake of simplicity, drop the x-dependence. Let $F(t) = \frac{t^{2_s^*}}{\ln(e+t^2)}$. Then

$$f(t) = \frac{2_s^* t^{2_s^*-1}(e+t^2)\ln(e+t^2) - 2t^{2_s^*+1}}{(e+t^2)(\ln(e+t^2))^2}.$$

Moreover, when $\lim_{x\in\mathbb{R}^3} V(x) \geq 1$, there exist functions satisfying (F_3) and (F_4), but do not satisfy $(F_{3'})$ or $(F_{4'})$. For example, $f(x,t) = K(x)t^3 - |t|^{\frac{3}{2}}t + |t|t$, where $K \in \mathscr{C}(\mathbb{R}^3, [m,n])$ with $m,n > 0$. Then,

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t^3} = \lim_{|t|\to\infty} \frac{K(x)t^3 - |t|^{\frac{3}{2}}t + |t|t}{t^3} = \lim_{|t|\to\infty} K(x) - \frac{1}{|t|^{\frac{1}{2}}} + \frac{1}{|t|} = K(x) \in [m,n] \neq +\infty,$$

and

$$\begin{aligned}\lim_{|t|\to\infty} \frac{|t|^{4s-3} f(x,t)}{t^3} &= \lim_{|t|\to\infty} \frac{K(x)t^3|t|^{4s-3} - |t|^{4s-\frac{3}{2}}t + |t|^{4s-2}t}{t^3} \\ &= \lim_{|t|\to\infty} K(x)|t|^{4s-3} - |t|^{4s-\frac{7}{2}} + |t|^{4s-4} = +\infty,\end{aligned}$$

where $s \in (\frac{3}{4}, 1)$. Therefore, $f(x,t)$ satisfies (F_3) but not $(F_{3'})$. Similarly, when $\inf_{x\in\mathbb{R}^3} V(x) \geq 1$, we can prove by some simple computation that $f(x,t)$ satisfies (F_4) but not $(F_{4'})$.

This paper is organized as follows. In Section 2, we give the proof to Theorem 1.1 by combining the constraint variational method with some new inequalities. Throughout this paper, we use the following notations: $\|u\|_p$ denotes the L^p-norm of the space $L^p(\mathbb{R}^3)$ for $p \geq 2$; $B_r(x) = \{y \in \mathbb{R}^3 : |y-x| < r\}$; and $C_i (i = 1,2,\ldots)$ are some positive constant could change from line to line.

2. Proof of Theorem 1.1

Proof of theorem 1.1. The proof is split into three steps.

We first prove that, for $b \geq 0$, the following sets

$$\begin{aligned}\mathscr{E}_b \quad := \quad &\Big\{u \in H : b\|(-\Delta)^{\frac{s}{2}}u\|_2^2 \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}u^{\pm}\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \left[V(x)(u^{\pm})^2 - f(x,u^{\pm})u^{\pm}\right]\mathrm{d}x < 0\Big\}\end{aligned}$$

and

$$\overline{\mathscr{E}}_b \quad := \quad \left\{u \in H : b\|(-\Delta)^{\frac{s}{2}}u\|_2^4 + \int_{\mathbb{R}^3} \left[V(x)u^2 - f(x,u)u\right]\mathrm{d}x < 0\right\}$$

are not empty by scaling technique (see [20, Lemma 2.5]).

STEP 1. for each $u \in \mathscr{E}_b$, there is a unique pair $(\alpha_u, \beta_u) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ such that $\alpha_u u^+ + \beta_u u^- \in \mathscr{M}_b$; and for each $u \in \overline{\mathscr{E}}_b$, there is a unique $\overline{\beta}_u > 0$ such that $\overline{\beta}_u u \in \mathscr{N}_b$.

To prove STEP 1, let us first prove $\mathscr{M}_b \neq \emptyset$. Let

$$\begin{aligned} g_1(\alpha,\beta) &= \langle I_b'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle \\ &= a\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}(\alpha u^+ + \beta u^-)(-\Delta)^{\frac{s}{2}}\alpha u^+ \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\left[V(x)(\alpha u^+)^2 - f(x,\alpha u^+)\alpha u^+\right]\mathrm{d}x \\ &\quad + b\|(-\Delta)^{\frac{s}{2}}(\alpha u^+ + \beta u^-)\|_2^2 \int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}(\alpha u^+ + \beta u^-)(-\Delta)^{\frac{s}{2}}(\alpha u^+)\mathrm{d}x \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} g_2(\alpha,\beta) &= \langle I_b'(\alpha u^+ + \beta u^-), \beta u^- \rangle \\ &= a\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}(\alpha u^+ + \beta u^-)(-\Delta)^{\frac{s}{2}}\beta u^- \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\left[V(x)(\beta u^-)^2 - f(x,\beta u^-)\beta u^-\right]\mathrm{d}x \\ &\quad + b\|(-\Delta)^{\frac{s}{2}}(\alpha u^+ + \beta u^-)\|_2^2 \int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}(\alpha u^+ + \beta u^-)(-\Delta)^{\frac{s}{2}}(\beta u^-)\mathrm{d}x. \end{aligned} \tag{2.2}$$

By (F_4), one has $f(x,\alpha\tau)\alpha\tau \geq f(x,\tau)\tau\alpha^4 - V(x)(\alpha^2-1)(\alpha\tau)^2$ for all $x \in \mathbb{R}^3, \alpha \geq 1, \tau \in \mathbb{R}$, which implies

$$\int_{\mathbb{R}^3}\left[V(x)(\alpha u^+)^2 - f(x,\alpha u^+)\alpha u^+\right]\mathrm{d}x \leq \alpha^4\int_{\mathbb{R}^3}\left[V(x)(u^+)^2 - f(x,u^+)u^+\right]\mathrm{d}x, \ \forall \alpha \geq 1. \tag{2.3}$$

From (2.1) and (2.3), we derive that

$$\begin{aligned} g_1(\alpha,\alpha) &= a\alpha^2\left[\|(-\Delta)^{\frac{s}{2}}u^+\|_2^2 + 2P(u^+,u^-)\right] + b\alpha^4\phi(u,u^+) \\ &\quad + \int_{\mathbb{R}^3}\left[V(x)(\alpha u^+)^2 - f(x,\alpha u^+)\alpha u^+\right]\mathrm{d}x \\ &\leq a\alpha^2\left[\|(-\Delta)^{\frac{s}{2}}u^+\|_2^2 + 2P(u^+,u^-)\right] + \alpha^4\Bigg\{b\phi(u,u^+) \\ &\quad + \int_{\mathbb{R}^3}\left[V(x)(u^+)^2 - f(x,u^+)u^+\right]\mathrm{d}x\Bigg\}, \ \forall \alpha \geq 1. \end{aligned} \tag{2.4}$$

where

$$\phi(u,u^+) := \|(-\Delta)^{\frac{s}{2}}u\|_2^2 \int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}u^+\mathrm{d}x, \quad \forall u \in H.$$

Using (2.4), it is easy to prove that $g_1(\alpha,\alpha) < 0$ for α large due to $u \in \mathscr{E}_b$. Similarly, we have $g_2(\beta,\beta) < 0$ for β large. Combining (2.1) with (2.2), we prove that there exists $r \in (0,R)$ such that

$$g_1(r,r) > 0, \ g_1(R,R) < 0; \quad g_2(r,r) > 0, \ g_2(R,R) < 0. \tag{2.5}$$

From (2.1) and (2.2), we have that $g_1(\alpha,\cdot)$ is increasing for any fixed $\alpha > 0$, and $g_2(\cdot,\beta)$ is increasing for any fixed $\beta > 0$. Hence, it follows from (2.1), (2.2), and (2.5) that

$$g_1(r,\beta) > g_1(r,r) > 0, \ g_1(R,\beta) < g_1(R,R) < 0, \quad \forall \beta \in [r,R],$$

and

$$g_2(\alpha,r)>g_2(r,r)>0,\ \ g_2(\alpha,R)<g_2(R,R)<0,\quad \forall\alpha\in[r,R].$$

Therefore, by applying Miranda's Theorem [33], there exists some point $(\alpha_u,\beta_u)\in[r,R]\times[r,R]$ such that $g_1(\alpha_u,\beta_u)=g_2(\alpha_u,\beta_u)=0$. So, $\alpha_u u^+ +\beta_u u^-\in\mathscr{M}_b$.

Now, we prove the uniqueness of the pair (α_u,β_u). Let (α_1,β_1) and (α_2,β_2) such that $\alpha_i u^+ + \beta_i u^-\in\mathscr{M}_b$, $i=1,2$. In view of [20, Lemma 2.2], one has

$$\begin{aligned}&I_b(\alpha_1u^++\beta_1u^-)\\&\geq I_b(\alpha_2u^++\beta_2u^-)+\frac{a(\alpha_1^2-\alpha_2^2)^2}{\alpha_1^2}\|(-\Delta)^{\frac{s}{2}}u^+\|_2^2+\frac{a(\beta_1^2-\beta_2^2)^2}{\beta_1^2}\|(-\Delta)^{\frac{s}{2}}u^-\|_2^2,\end{aligned}$$

and

$$\begin{aligned}&I_b(\alpha_2u^++\beta_2u^-)\\&\geq I_b(\alpha_1u^++\beta_1u^-)+\frac{a(\alpha_1^2-\alpha_2^2)^2}{\alpha_2^2}\|(-\Delta)^{\frac{s}{2}}u^+\|_2^2+\frac{a(\beta_1^2-\beta_2^2)^2}{\beta_2^2}\|(-\Delta)^{\frac{s}{2}}u^-\|_2^2.\end{aligned}$$

The above inequalities imply $(\alpha_1,\beta_1)=(\alpha_2,\beta_2)$.

Furthermore, we let $g(\beta)=\langle I_b'(\beta u),\beta u\rangle$ for $u\in\overline{\mathscr{E}}_b$. From (1.4) and (F_4), we derive that

$$g(\beta)\leq a\beta^2\|(-\Delta)^{\frac{s}{2}}u\|_2^2+\beta^4\left\{b\|(-\Delta)^{\frac{s}{2}}u\|_2^4+\int_{\mathbb{R}^3}\left[V(x)u^2-f(x,u)u\right]\mathrm{d}x\right\},\forall\beta\geq1,$$

which demonstrates that there exists $R_0>0$ sufficiently large such that $g(R_0)<0$. Choosing $r_0>0$ sufficiently small, we see that $g(r_0)>0$. Thus there exists $\overline{\beta}_u>0$ such that $g(\overline{\beta}_u)=0$ for $u\in\overline{\mathscr{E}}_b$. Similarly, we can deduce that $\overline{\beta}_u$ is unique. So we obtain that, for $u\in\overline{\mathscr{E}}_b$, there exists a unique $\overline{\beta}_u>0$ such that $\overline{\beta}_u u\in\mathscr{N}_b$. The proof of STEP 1 is complete.

STEP 2. $m_b=\inf_{u\in\mathscr{M}_b}I_b(u)>0$ and $c_b=\inf_{u\in\mathscr{N}_b}I_b(u)>0$ are achieved.

By (F_1) and (F_2), we see that, for every $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|f(x,t)t|\leq\varepsilon t^2+C_\varepsilon|t|^{2_s^*},\ \ |F(x,t)|\leq\varepsilon t^2+C_\varepsilon|t|^{2_s^*},\quad\forall(x,t)\in\mathbb{R}^3\times\mathbb{R}.\tag{2.6}$$

By (V_1), there exists $\gamma_0>0$ such that

$$\gamma_0\|u\|^2_{H^s(\mathbb{R}^3)}\leq\|u\|^2,\quad\forall u\in H.\tag{2.7}$$

First, we prove that $m_b>0$ and $c_b>0$. For $u\in\mathscr{M}_b$, it follows from (1.4), (2.6), (2.7), [20, Lemma 2.1], the expression for $P(u^+,u^-)$, and [19, Lemma 2.1] that

$$\begin{aligned}\gamma_0\|u^\pm\|^2_{H^s(\mathbb{R}^3)}\leq\|u^\pm\|^2\ &\leq\ a\|(-\Delta)^{\frac{s}{2}}(u^\pm)\|_2+2aP(u^+,u^-)+\int_{\mathbb{R}^3}V(x)(u^\pm)^2\mathrm{d}x\\&\qquad+b\|(-\Delta)^{\frac{s}{2}}u\|_2^2\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}u^\pm\mathrm{d}x\\&=\ \int_{\mathbb{R}^3}f(x,u^\pm)u^\pm\mathrm{d}x\\&\leq\ \frac{\gamma_0}{2}\|u^\pm\|_2^2+C_1\|u^\pm\|_{2_s^*}^{2_s^*}\\&\leq\ \frac{\gamma_0}{2}\|u^\pm\|^2_{H^s(\mathbb{R}^3)}+C_{2_s^*}\|u^\pm\|^{2_s^*}_{H^s(\mathbb{R}^3)}.\end{aligned}\tag{2.8}$$

We can then deduce that there exists a constant $\mu > 0$ independent of b such that

$$\|u^{\pm}\| \geq \sqrt{\gamma_0}\|u^{\pm}\|_{H^s(\mathbb{R}^3)} \geq \mu, \quad \forall u \in \mathscr{M}_b. \tag{2.9}$$

Similarly, there exists a constant $\mu_0 > 0$ independent of b such that $\|u\| \geq \sqrt{\gamma_0}\|u\|_{H^s(\mathbb{R}^3)} \geq \mu_0$, $\forall u \in \mathscr{N}_b$. Since $\mathscr{M}_b \subset \mathscr{N}_b$, we have $m_b \geq c_b$. Note that

$$I_b(u) \geq I_b(tu) + \frac{1-t^4}{4}\langle I_b'(u), u\rangle + \frac{a(1-t^2)^2}{4}\|(-\Delta)^{\frac{s}{2}}u\|_2^2, \ \forall u \in H, t \geq 0.$$

With $t = 0$ (see [20, Lemma 2.3]), one has

$$I_b(u) = I_b(u) - \frac{1}{4}\langle I_b'(u), u\rangle \geq \frac{a}{4}\|(-\Delta)^{\frac{s}{2}}u\|_2^2, \quad \forall u \in \mathscr{N}_b, \tag{2.10}$$

which implies $c_b = \inf_{u\in\mathscr{N}_b} I_b(u) \geq 0$.

We now demonstrate that $c_b > 0$. To this end, we choose $\{u_n\} \subset \mathscr{N}_b$ such that $I_b(u_n) \to c_b$. There are two possible cases: (1) $\inf_{n\in\mathbb{N}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2 > 0$ and (2) $\inf_{n\in\mathbb{N}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2 = 0$.

Case 1. $\inf_{n\in\mathbb{N}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2 = \mu_1 > 0$.

In this case, we conclude from (2.10) that $c_b + o(1) = I_b(u_n) \geq \frac{a}{4}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 \geq \frac{a}{4}\mu_1^2$.

Case 2. $\inf_{n\in\mathbb{N}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2 = 0$.

Since $\|u_n\|^2 \geq \mu_0^2 > 0$, up to a subsequence, one has

$$\|(-\Delta)^{\frac{s}{2}}u_n\|_2 \to 0, \ \int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x \geq \mu_2 > 0 \text{ for some constant } \mu_2 > 0. \tag{2.11}$$

Let $t_n = \left[\int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x\right]^{-\frac{1}{2}}$. It follows from (2.11) that $t_n \leq \mu_2^{-\frac{1}{2}}$. By (2.6), (2.7), and the Sobolev inequality, we obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} F(x, t_nu_n)\mathrm{d}x\right| &\leq \int_{\mathbb{R}^3}\left[\frac{\gamma_0}{4}t_n^2u_n^2 + C_3|t_nu_n|^{2_s^*}\right]\mathrm{d}x \\ &\leq \frac{t_n^2}{4}\|u_n\|^2 + C_3|t_n|^{2_s^*}S_s^{-\frac{2_s^*}{2}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^{2_s^*},\end{aligned} \tag{2.12}$$

where

$$S_s = \inf_{u\in\mathscr{D}^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}}}.$$

Since $u_n \in \mathscr{N}_b$, it follows from (1.5), (2.11), (2.12), and [20, Corollary 2.4] that

$$\begin{aligned} c_b + o(1) &= I_b(u_n) \geq I_b(t_nu_n) \\ &= \frac{at_n^2}{2}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^3}V(x)u_n^2\mathrm{d}x + \frac{bt_n^4}{4}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^4 - \int_{\mathbb{R}^3}F(x,t_nu_n)\mathrm{d}x \\ &\geq \frac{t_n^2}{4}\int_{\mathbb{R}^3}V(x)u_n^2\mathrm{d}x - C_3|t_n|^{2_s^*}S_s^{-\frac{2_s^*}{2}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^{2_s^*} = \frac{1}{4} + o(1).\end{aligned}$$

Case 1 and 2 imply that $c_b = \inf_{u\in\mathscr{N}_b} I_b(u) > 0$. Therefore, $m_b \geq c_b > 0$.

Next, we prove that m_b can be achieved. Let $\{u_n\} \subset \mathscr{M}_b$ be a minimizing sequence such that $I_b(u_n) \to m_b$. Then, (2.10) implies that

$$m_b + o(1) \geq I_b(u_n) - \frac{1}{4}\langle I_b'(u_n), u_n\rangle \geq \frac{a}{4}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2.$$

So $\{\|(-\Delta)^{\frac{s}{2}}u_n\|_2\}$ is bounded. In order to obtain the boundedness of $\{u_n\}$, we have to prove that $\int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x$ is bounded. By contradiction, we assume that $\int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x \to \infty$. Let

$$t_n = \frac{2(m_b+1)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x\right)^{\frac{1}{2}}}.$$

Then $t_n \to 0$, and (2.12) still holds. Using (1.5), (2.12), and [20, Corollary 2.4], we have that

$$\begin{aligned} m_b + o(1) &= I_b(u_n) \geq I_b(t_nu_n) \\ &= \frac{at_n^2}{2}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x + \frac{bt_n^4}{4}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^4 - \int_{\mathbb{R}^3} F(x,t_nu_n)\mathrm{d}x \\ &\geq \frac{t_n^2}{4}\int_{\mathbb{R}^3} V(x)u_n^2\mathrm{d}x - C_3|t_n|^{2_s^*}S_s^{-\frac{2_s^*}{2}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^{2_s^*} \\ &= m_b + 1 + o(1). \end{aligned} \tag{2.13}$$

This contradiction demonstrates that $\{u_n\}$ is bounded in H. Up to a subsequence, we have $u_n^\pm \rightharpoonup u_b^\pm$ weakly in H and $u_n^\pm \to u_b^\pm$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 2_s^*)$. By [23, Lemma 2.4], we have

$$\int_{\mathbb{R}^3} F(x,u_n^\pm)\mathrm{d}x = \int_{\mathbb{R}^3} F(x,u_n^\pm - u_b^\pm)\mathrm{d}x + \int_{\mathbb{R}^3} F(x,u_b^\pm)\mathrm{d}x + o(1). \tag{2.14}$$

Using (2.6), we obtain

$$\left|\int_{\mathbb{R}^3} F(x,u_n^\pm - u_b^\pm)\mathrm{d}x\right| \leq \varepsilon\int_{\mathbb{R}^3} |u_n^\pm - u_b^\pm|^2\mathrm{d}x + C_\varepsilon|u_n^\pm - u_b^\pm|^{2_s^*}\mathrm{d}x = \varepsilon J_1 + C_\varepsilon J_2,$$

where $J_1 = \int_{\mathbb{R}^3} |u_n^\pm - u_b^\pm|^2\mathrm{d}x$ and $J_2 = \int_{\mathbb{R}^3} |u_n^\pm - u_b^\pm|^{2_s^*}\mathrm{d}x$. Since $\|u_n\|$ is bounded, in connection with Minkowski inequality, one has $|J_1| \leq C_1$ and $|J_2| \leq C_1$, where $C_1 > 0$. So, $\int_{\mathbb{R}^3} F(x,u_n^\pm - u_b^\pm)\mathrm{d}x \to 0$ as $n \to \infty$. Furthermore, it follows from (2.14) that $\int_{\mathbb{R}^3} F(x,u_n^\pm)\mathrm{d}x = \int_{\mathbb{R}^3} F(x,u_b^\pm)\mathrm{d}x + o(1)$, which implies

$$\int_{\mathbb{R}^3} f(x,u_n^\pm)u_n^\pm\mathrm{d}x = \int_{\mathbb{R}^3} f(x,u_b^\pm)u_b^\pm\mathrm{d}x + o(1). \tag{2.15}$$

From (2.8), (2.9), and (2.15), we deduce that

$$0 < \mu^2 \leq \|u_n^\pm\|^2 \leq \int_{\mathbb{R}^3} f(x,u_n^\pm)u_n^\pm\mathrm{d}x = \int_{\mathbb{R}^3} f(x,u_b^\pm)u_b^\pm\mathrm{d}x + o(1),$$

which yields $u_b^\pm \neq 0$. From (2.15), [20, Lemma 2.1], the weak semicontinuity of norm, and the Fatou's Lemma, we conclude that

$$\begin{aligned} &a\|(-\Delta)^{\frac{s}{2}}(u_b^\pm)\|_2 + 2aP(u_b^+,u_b^-) + \int_{\mathbb{R}^3} V(x)(u_b^\pm)^2\mathrm{d}x + b\|(-\Delta)^{\frac{s}{2}}u_b\|_2^2\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}u_b^\pm\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\left[a\|(-\Delta)^{\frac{s}{2}}(u_n^\pm)\|_2 + 2aP(u_n^+,u_n^-) + \int_{\mathbb{R}^3} V(x)(u_n^\pm)^2\mathrm{d}x \right.\\ &\quad \left. + b\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}u_n^\pm\mathrm{d}x\right] \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^3} f(x,u_n^\pm)u_n^\pm\mathrm{d}x = \int_{\mathbb{R}^3} f(x,u_b^\pm)u_b^\pm\mathrm{d}x, \end{aligned}$$

which demonstrates that $\langle I_b'(u_b), u_b^{\pm}\rangle \leq 0$. Moreover, by (1.4), it is easy to verify that $u_b \in \mathscr{E}_b$. In STEP 1, there exist $\alpha_{u_b}, \beta_{u_b} > 0$ such that $\alpha_{u_b}u_b^+ + \beta_{u_b}u_b^- \in \mathscr{M}_b$. By (F_4), one has

$$\begin{aligned}&\frac{1-t^2}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{V(x)}{4}(1-t^2)^2\tau^2\\ &= \int_t^1 \left[\frac{f(x,\tau)-V(x)\tau}{\tau^3} - \frac{f(x,\alpha\tau)-V(x)\alpha\tau}{(\alpha\tau)^3}\right]\alpha^3\tau^4 \mathrm{d}\alpha \geq 0\end{aligned}$$

for all $t \geq 0$ and $\tau \in \mathbb{R}\backslash\{0\}$. Letting $t = 0$ in the equality above, we have $\frac{1}{4}f(x,\tau)\tau - F(x,\tau) + \frac{1}{4}V(x)\tau^2 \geq 0$, $x \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$. Thus, by (1.4), (1.5), [20, Lemmas 2.2 and 2.7], the weak semicontinuity of norm, and the Fatou's Lemma, we have

$$\begin{aligned} m_b &= \lim_{n\to\infty}\left[I_b(u_n) - \frac{1}{4}\langle I_b'(u_n), u_n\rangle\right]\\ &= \lim_{n\to\infty}\left\{\frac{a}{4}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,u_n)u_n - F(x,u_n) + \frac{1}{4}V(x)u_n^2\right]\mathrm{d}x\right\}\\ &\geq \frac{a}{4}\|(-\Delta)^{\frac{s}{2}}u_b\|_2^2 + \int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,u_b)u_b - F(x,u_b) + \frac{1}{4}V(x)u_b^2\right]\mathrm{d}x\\ &= I_b(u_b) - \frac{1}{4}\langle I_b'(u_b), u_b\rangle\\ &\geq \sup_{\alpha,\beta\geq 0}\left[I_b(\alpha u_b^+ + \beta u_b^-) + \frac{1-\alpha^4}{4}\langle I_b'(u_b), u_b^+\rangle + \frac{1-\beta^4}{4}\langle I_b'(u_b), u_b^-\rangle\right] - \frac{1}{4}\langle I_b'(u_b), u_b\rangle\\ &\geq \sup_{\alpha,\beta\geq 0} I_b(\alpha u_b^+ + \beta u_b^-)\\ &\geq I_b(\alpha_{u_b}u_b^+ + \beta_{u_b}u_b^-)\\ &\geq m_b,\end{aligned}$$

which implies that $I_b(u_b) = m_b$ and $u_b \in \mathscr{M}_b$. Similarly, we can prove that there exists $\bar{u}_b \in \mathscr{N}_b$ such that $I_b(\bar{u}_b) = c_b$. The proof of STEP 2 is complete.

STEP 3. Critical point of I_b.

Using [20, Lemma 2.9], we let $u_b \in \mathscr{M}_b$ and $\bar{u}_b \in \mathscr{N}_b$ satisfy $I_b(u_b) = m_b = \inf_{u\in\mathscr{M}_b} I_b(u)$ and $I_b(\bar{u}_b) = c_b = \inf_{u\in\mathscr{N}_b} I_b(u)$. So we prove that u_b and $\bar{u}_b$ are critical point of I_b. Moreover, u_b is a radial sign-changing solution to problem (1.1) and $\bar{u}_b$ is a radial solution to problem (1.1). □

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References

[1] V. Ambrosio V, T. Isernia, A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^N$ with a general nonlinearity, Commun. Contemp. Math. 20 (2018), 1750054.

[2] Z. Liu, M. Squassina, J. Zhang, Ground states for fractional Kirchhoff equations with critical nonlinearity in low dimension, Nonlinear Differential Equations Appl. NoDEA 24 (2017), 1-32.

[3] S. Baraket, G.M. Bisci, Multiplicity results for elliptic Kirchhoff-type problems, Adv. Nonlinear Anal. 6 (2017), 85–93.
[4] G.M. Bisci, Sequence of weak solutions for fractional equations, Math. Res. Lett. 21 (2014), 241-253.
[5] G.M. Bisci, D.D. Repoš, On doubly nonlocal fractional elliptic equations, Rend Lincei Mat Appl, 26 (2015), 161-176.
[6] G.M. Bisci, F. Tulone, An existence result for fractional Kirchhoff-type equations, Zeitschrift für Analysis und ihre Anwendungen 35 (2016), 181-197.
[7] G.M. Bisci, L. Vilasi, On a fractional degenerate Kirchhoff-type problem, Commun. Contemp. Math. 19 (2017), 1550088.
[8] A. Fiscella, P. Pucci, p-fractional Kirchhoff equations involving critical nonlinearities, Nonlinear Anal. Real World Appl. 35 (2017), 350-378.
[9] A. Fiscella, R. Servadei, E. Valdinoci, Density properties for fractional Sobolev spaces, Ann Acad Sci Fenn Math 40 (2015), 235-253.
[10] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, Nonlinear Anal. 94 (2014), 156-170.
[11] B. Ge, C. Zhang, Existence of a positive solution to Kirchhoff problems involving the fractional Laplacian, J. Anal. Appl. 34 (2015), 419-434.
[12] N. Nyamoradi, K. Teng, Existence of solutions for a Kirchhoff-type nonlocal operators of elliptic type, Commun. Pure Appl. Anal. 14 (2015), 361-371.
[13] N. Laskin, Fractional schrödinger equation, Phys. Rev. E 66 (2002), 056108.
[14] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, On some critical problem for the fractional Laplacian operator, J. Differential Equations 252 (2012), 6133-6162.
[15] X. Cabré, Y. Sire, Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions, Tran. Amer. Math. Soc. 367 (2011), 911-941.
[16] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Adv. Math. 224 (2010), 2052-2093.
[17] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equ. 32 (2007), 1245-1260.
[18] M. Fall, T. Weth, Nonexistence results for a class of fractional elliptic boundary value problems, J. Funct. Anal. 263 (2012), 2205-2227.
[19] K. Chang, Q. Gao, Sign-changing solutions for the stationary Kirchhoff problems involving the fractional Laplacian in $\mathbb{R}^N$, arXiv:1701.03862, 2017.
[20] S. Chen, X. Tang, F. Liao, Existence and asymptotic behavior of sign-changing solutions for fractional Kirchhoff-type problems in low dimensions, Nonlinear Differential Equations Appl. NoDEA 25 (2018), 1-23.
[21] P.L. Lions, Symètrie et compacitè dans les espaces de Sobolev, J. Funct. Anal. 49 (1982), 315-334.
[22] Y. Deng, S. Peng, W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$, J. Funct. Anal. 269 (2015), 3500-3527.
[23] H. Zhang, R. Zhang, Positive solutions to p-Kirchhoff-type elliptic equation with general subcritical growth, Bull. Korean Math. Soc. 54 (2017), 1023-1036.
[24] J. Huang, Q. Zhang, Existence of nonnegative solutions for fourth order elliptic equations of Kirchhoff type with general subcritical growth, Taiwanese J. Math. 24 (2020), 81-96.
[25] J. Liu, J. Liao, C. Tang, Positive solution for the Kirchhoff-type equations involving general subcritical growth, Commun. Pure Appl. Anal. 15 (2016), 445.
[26] W. Guan, H.F. Huo, Existence of ground state sign-changing solutions of fractional Kirchhoff-type equation with critical growth, Appl. Math. Optim. 84 (2021), 99-121.
[27] Y. Meng, X. Zhang, X. He, Ground state solutions for a class of fractional Schrodinger-Poisson system with critical growth and vanishing potentials, Adv. Nonlinear Anal. 10 (2021), 1328-1355.
[28] Y. Meng, X. Zhang, X. He, Least energy sign-changing solutions for a class of fractional Kirchhoff-Poisson system, J. Math. Phys. 62 (2021), 091508.
[29] T. Bartsch, Z. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations, Commun. Partial Differ. Equ. 29 (2004), 25-42.

[30] T. Bartsch, Z. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Commun. Partial Differ. Equ. 20 (1995), 1725-1741.
[31] T. Bartsch, M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 124 (1993), 261-276.
[32] E.S. Noussair, J. Wei, On the effect of the domain geometry on the existence and profile of nodal solution of some singularly perturbed semilinear Dirichlet problem, Indiana Univ. Math. J. 46 (1997), 1321-1332.
[33] C. Miranda, Un'osservazione su un teorema di Brouwer, Boll. Unione Mat. Ital. 3 (1940), 5-7.