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EKELAND'S VARIATIONAL PRINCIPLE WITH A SCALARIZATION TYPE WEIGHTED SET ORDER RELATION

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Abstract. In this paper, a weighted set order relation given by the oriented distance function is introduced, which does not need any convexity and is applicable even the ordering cone has an empty interior. The Ekeland's variational principle, Caristi's fixed point theorem, and Takahashi's minimization theorem associated with the introduced weighted set order relation are constructed, and the equivalences among them are deduced. As an application, the existence of solutions to a set optimization problem is examined to verify the validity of the results obtained.
Keywords. Ekeland's variational principle; Set order; Scalarization function; Set optimization.

1. INTRODUCTION

Set optimization problem, which depends on comparisons among values of the objective mapping to obtain its minimal solution, has attracted many scholars' attentions. Set order relations play an important role in set optimization. The firstly emerged set relations are upper or lower set orders introduced by Kuroiwa [1]. Since it can not fully achieve the comparison of two sets if only one of them is used, then the set less order relation was proposed by Jahn and Ha [2]. However, the set less order relation is just a simple union of upper set order and lower set order, and a seamless transition between two relations is not considered, which sometimes leads to the set of optimal solutions become empty. In view of this issue, the weighted set order relation started to come into focus [3, 4, 5]. Scalarization is a powerful tool to deal with the weighted set order relation. Chen et al. [3] proposed a weighted order relation by Gerstewitz's function under the assumption that the ordering cone has a nonempty interior. Köbis and Köbis [5] defined another weighted order relation by linear functions under the conditions of cone convexity.

The oriented distance function and its generalization have been successfully applied to set optimization problems; see, e.g., [6, 7, 8, 9]. Recently, Ha [10] presented a generalized oriented distance function, termed as the set scalarization. Jiménez et al. [11] explored the properness and boundedness of Ha's function, and employed it to characterize the lower set order relation; Han et al. [12] extended it to weak lower set order relation, and investigated the case of Ha's function with negative values; and Jiménez et al. [13] developed it to upper set order relation,

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and discussed positively homogeneity and monotonicity of Ha's function. One of the purposes of this paper is to supplement some special properties to this set scalarization function, such as triangle inequality property and translation property, and to construct a new weighted set order relation. In particular, compared with [3, 5], it will be illustrated that the new weighted set relation does not require any convexity and is still valid for the cones with empty interior.

Ekeland's variational principle is one of the significant contents in optimization theory, which has been generalized to the case of set-valued maps; see, e.g., [14, 15, 16]. Recently, several scholars made great efforts to develop Ekeland's variational principles involving various set order relations. For instance, Qiu and He [17] constructed an Ekeland's variational principle of set-valued maps related to lower set order relation. Sach and Le [18] obtained an Ekeland's principle in the sense of upper set order and lower set order relations. Zhang and Huang [19] presented an Ekeland's principle in terms of strict lower set order relation. An Ekeland's variational principle with respect to weighted set relations defined by Gerstewitz's function was discussed in [4]. In this paper, we attempt to establish an Ekeland's variational principle, Caristi's fixed point theorem, and Takahashi's minimization theorem associated with the newly introduced weighted set order relation. As an application of Ekeland's variational principle of this work, we also explore the existence of solution to a set optimization problem.

The paper is organized as follows: Section 2 provides some basic notations, definitions and lemmas. Section 3 presents a new weighted set order relation by using set scalarization function, and explores its some properties. Section 4 constructs Ekeland's variational principle, Caristi's fixed point theorem, and Takahashi's minimization theorem with introduced weighted set relation, and proves their equivalences. Section 5 applies the obtained results to examine the existence of solution to a set optimization problem. Section 6, the last section, ends this manuscript with conclusions.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $\mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, ..., n\}$, and $\mathbb{R}^n_{++} = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, ..., n\}$. Let X, Y be Banach spaces, and let Y^* be the dual of Y. For any $y_1, y_2 \in Y$, the distance between y_1 and y_2 is given by $d(y_1, y_2) = \|y_1 - y_2\|$. It is always assume that $K \subset Y$ is a proper pointed closed and convex cone. $K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0,\ \forall y \in K\}$ represents the dual cone of K. The cone K induces the following order relation on Y:

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K,\ \forall y_1, y_2 \in Y.$$

For a nonempty set $A \subset Y$, $intA$, ∂A, clA, and $Y \backslash A$ stand for the interior, the boundary, the closure, and the complement of A, respectively. It is said that A is K-proper if $A + K \neq Y$, A is K-closed if $A + K$ is closed, A is K-bounded if there exists a nonempty bounded set $M \subset Y$ such that $A \subset M + K$, and A is K-compact if any cover of A of the form $\{U_\alpha + K : \alpha \in I,\ U_\alpha \text{ are open}\}$ admits a finite subcover. It is called that A is $\pm K$-proper (resp. $\pm K$-bounded, $\pm K$-compact) if A is K-proper (resp. K-bounded, K-compact) and $-K$-proper (resp. $-K$-bounded, $-K$-compact). In particular, A is K-compact implies that A is K-bounded and K-closed (see [11]).

The family of all nonempty subsets in Y is denoted as $P(Y)$. Let $A, B \in P(Y)$, the sum and difference of A and B are defined by

$$A + B := \{a + b : a \in A, b \in B\},\ A - B := \{a - b : a \in A, b \in B\}.$$

Definition 2.1. [20] Let $M \in P(Y)$. The oriented distance function $\triangle_M : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as $\triangle_M(y) = d_M(y) - d_{Y\setminus M}(y)$ for all $y \in Y$, where $d_M(y) = \inf_{m\in M} \|y-m\|$ is the distance function from y to M.

It was pointed out in [7, 20] that function $\triangle_M$ can be rewritten as

$$\triangle_M(y) = \begin{cases} d_M(y), & y \in Y\setminus M, \\ -d_{Y\setminus M}(y), & y \in M, \end{cases}$$

and if M is a convex cone with $intM \neq \emptyset$, then

$$\triangle_M(y) = \sup_{y^* \in S(M^*)} \langle -y^*, y\rangle, \ \forall y \in Y,$$

where $S(M^*) := \{y^* \in M^* : \|y^*\| = 1\}$.

Some properties of the oriented distance function are summarized in the following.

Lemma 2.1. [20, 21] *Let $M \in P(Y)$ with $M \neq Y$. The following assertions hold:*

(i) $\triangle_M$ is real-valued and 1-Lipschitzian.

(ii) $clM = \{y \in Y : \triangle_M(y) \leq 0\}$, $\partial M = \{y \in Y : \triangle_M(y) = 0\}$, $Y\setminus M = \{y \in Y : \triangle_M(y) \geq 0\}$. If $intM \neq \emptyset$, then $intM = \{y \in Y : \triangle_M(y) < 0\}$.

(iii) $\triangle_M(-y) = \triangle_{-M}(y)$ for all $y \in Y$.

(iv) If M is a cone, then $\triangle_M$ is positively homogeneous.

(v) If M is a closed convex cone, then $\triangle_{-M}(y_1 + y_2) \leq \triangle_{-M}(y_1) + \triangle_{-M}(y_2)$ for all $y_1, y_2 \in Y$.

Next, we give the notions of two set scalarization functions.

Definition 2.2. [10, 13] Let $A, B \in P(Y)$. The set scalarization functions $\mathbb{D}_K, \mathscr{D}_K : P(Y) \times P(Y) \to \mathbb{R} \cup \{\pm\infty\}$ are defined as

$$\mathbb{D}_K(A,B) = \sup_{b\in B} \inf_{a\in A} \triangle_{-K}(a-b),$$

and

$$\mathscr{D}_K(A,B) = \sup_{a\in A} \inf_{b\in B} \triangle_{-K}(a-b).$$

It is obvious that $\mathscr{D}_K(A,B) = \mathbb{D}_{-K}(B,A)$ (see [13]). The following lemmas collect some properties of functions $\mathbb{D}_K$ and $\mathscr{D}_K$.

Lemma 2.2. [11, 13] *Let $A, B \in P(Y)$. The following statements hold:*

(i) If A is $\pm K$-proper, then $\mathbb{D}_K(A,A) = \mathscr{D}_K(A,A) = 0$.

(ii) If A is K-proper and B is K-bounded, then $\mathbb{D}_K(A,B) \in \mathbb{R}$.

(iii) If B is $-K$-proper and A is $-K$-bounded, then $\mathscr{D}_K(A,B) \in \mathbb{R}$.

(iv) For all $\alpha > 0$, $\mathbb{D}_K(\alpha A, \alpha B) = \alpha \mathbb{D}_K(A,B)$, $\mathscr{D}_K(\alpha A, \alpha B) = \alpha \mathscr{D}_K(A,B)$.

Lemma 2.3. [13, 21] *Let $A, B \in P(Y)$ and $y \in Y$.*

(i) If A is K-compact, then there exists $\bar{a} \in A$ such that $\triangle_{-K}(\bar{a} - y) = \inf_{a\in A} \triangle_{-K}(a-y)$.

(ii) If B is $-K$-compact, then there exists $\bar{b} \in B$ such that $\triangle_{-K}(y - \bar{b}) = \inf_{b\in B} \triangle_{-K}(y-b)$.

The next proposition indicates that $\mathbb{D}_K$ and $\mathscr{D}_K$ also possess the properties of triangle inequality under conditions of compactness.

Proposition 2.1. *Let $A,B,C \in P(Y)$ be $\pm K$-proper and $\pm K$-compacted sets. Then*
(i) $\mathbb{D}_K(A,C) \leq \mathbb{D}_K(A,B)+\mathbb{D}_K(B,C)$.
(ii) $\mathscr{D}_K(A,C) \leq \mathscr{D}_K(A,B)+\mathscr{D}_K(B,C)$.

Proof. We only prove that (i) holds, since the proof of (ii) is similar to (i). It follows from Lemma 2.2 that $\mathbb{D}_K(A,C)$, $\mathbb{D}_K(A,B)$ and $\mathbb{D}_K(B,C)$ are finite. By Lemma 2.3, choose $\bar{b} \in B$ such that $\sup_{c\in C} \triangle_{-K}(\bar{b}-c) = \sup_{c\in C} \inf\limits_{b\in B} \triangle_{-K}(b-c)$, then

$$\begin{aligned}
\mathbb{D}_K(A,C) &= \sup_{c\in C}\inf_{a\in A} \triangle_{-K}(a-c) \\
&= \sup_{c\in C}\inf_{a\in A} \triangle_{-K}(a-\bar{b}+\bar{b}-c) \\
&\leq \sup_{c\in C}\inf_{a\in A} [\triangle_{-K}(a-\bar{b})+\triangle_{-K}(\bar{b}-c)] \\
&= \inf_{a\in A} \triangle_{-K}(a-\bar{b})+\sup_{c\in C} \triangle_{-K}(\bar{b}-c) \\
&\leq \sup_{b\in B}\inf_{a\in A} \triangle_{-K}(a-b)+\sup_{c\in C}\inf_{b\in B} \triangle_{-K}(b-c) \\
&= \mathbb{D}_K(A,B)+\mathbb{D}_K(B,C).
\end{aligned}$$

□

A function $f: Y \to \mathbb{R}$ is said to satisfy the translation property with respect to $e \in intK$ if $f(y+te) = f(y)+t$ for all $y \in Y$ and $t \in \mathbb{R}$ (see [22]). In general, the oriented distance function has no translation property. Some authors [6, 7] pointed out that it holds under the assumption of $d_A(-e) = d_{Y\setminus(A)}(e) = 1$, where $e \in A$ and A is a proper pointed convex cone in Y. In fact, $\mathbb{D}_K$ and $\mathscr{D}_K$ also have the translation properties in the similar conditions.

Proposition 2.2. *Let $A,B \in P(Y)$ be $\pm K$-proper and $\pm K$-bounded sets. Assume that $e \in K$, $d_{-K}(e) = d_{Y\setminus(-K)}(-e) = 1$ and $t \in \mathbb{R}$. Then*
(i) $\mathscr{D}_K(A+te,B) = \mathscr{D}_K(A,B)+t$.
(ii) $\mathscr{D}_K(A,B+te) = \mathscr{D}_K(A,B)-t$.
(iii) $\mathbb{D}_K(A+te,B) = \mathbb{D}_K(A,B)+t$.
(iv) $\mathbb{D}_K(A,B+te) = \mathbb{D}_K(A,B)-t$.

Proof. We only prove (i) holds, the similar proofs remain valid for (ii), (iii), and (iv). Since K is a proper pointed closed and convex cone, we obtain from Lemma 2.1 that $\triangle_{-K}$ is a sublinear function. Due to $e \in K$, we have $-e \in -K$. If $t \geq 0$, it follows from Definition 2.2 that

$$\begin{aligned}
\mathscr{D}_K(A+te,B) &= \sup_{a\in A}\inf_{b\in B} \triangle_{-K}(a+te-b) \\
&\leq \sup_{a\in A}\inf_{b\in B} [\triangle_{-K}(a-b)+\triangle_{-K}(te)] \\
&= \sup_{a\in A}\inf_{b\in B} \triangle_{-K}(a-b)+\triangle_{-K}(te) \\
&= \mathscr{D}_K(A,B)+\triangle_{-K}(te) \qquad (2.1) \\
&= \mathscr{D}_K(A,B)+t(d_{-K}(e)) = \mathscr{D}_K(A,B)+t.
\end{aligned}$$

In addition, we obtain

$$\begin{aligned}\mathscr{D}_K(A,B) &= \sup_{a\in A}\inf_{b\in B}\triangle_{-K}(a-b)\\ &= \sup_{a\in A}\inf_{b\in B}\triangle_{-K}(a+te-te-b)\\ &\leq \sup_{a\in A}\inf_{b\in B}\triangle_{-K}(a+te-b)+\triangle_{-K}(-te)\\ &= \mathscr{D}_K(A+te,B)+\triangle_{-K}(-te) \qquad (2.2)\\ &= \mathscr{D}_K(A+te,B)+t(-d_{Y\setminus(-K)}(-e))=\mathscr{D}_K(A+te,B)-t.\end{aligned}$$

If $t<0$, according to (2.1), one has

$$\begin{aligned}\mathscr{D}_K(A+te,B) &\leq \mathscr{D}_K(A,B)+\triangle_{-K}(te)\\ &= \mathscr{D}_K(A,B)-t\triangle_{-K}(-e)\\ &= \mathscr{D}_K(A,B)-t(-d_{Y\setminus(-K)}(-e))=\mathscr{D}_K(A,B)+t.\end{aligned}$$

In view of (2.2), it yields

$$\begin{aligned}\mathscr{D}_K(A+te,B) &\geq \mathscr{D}_K(A,B)-\triangle_{-K}(-te)\\ &= \mathscr{D}_K(A,B)+t\triangle_{-K}(e)\\ &= \mathscr{D}_K(A,B)+t(d_{-K}(e))=\mathscr{D}_K(A,B)+t.\end{aligned}$$

Therefore, $\mathscr{D}_K(A+te,B)=\mathscr{D}_K(A,B)+t$ for all $t\in\mathbb{R}$. The proof is concluded. □

3. Weighted Set Order Relation

In this section, we introduce a new weighted set order relation via the set scalarization functions $\mathbb{D}_K$ and $\mathscr{D}_K$ formulated in Section 2, and discuss its some properties. Let us recall some known set order relations on $P(Y)$.

Definition 3.1. [2] Let $A,B\in P(Y)$ be arbitrary chosen sets.

(i) The lower set less order relation, denoted by $\preceq^l_K$, is defined as

$$A\preceq^l_K B\Longleftrightarrow B\subset A+K.$$

(ii) The upper set less order relation, denoted by $\preceq^u_K$, is defined as

$$A\preceq^u_K B\Longleftrightarrow A\subset B-K.$$

(iii) The set less order relation, denoted by $\preceq^s_K$, is defined as

$$A\preceq^s_K B\Longleftrightarrow A\subset B-K \text{ and } B\subset A+K.$$

For convenience, the notation $\npreceq^*_K$ indicates that the considered relation $\preceq^*_K$ does not hold, where $*\in\{u,l,s\}$.

Lemma 3.1. [13] *Let $A,B\in P(Y)$.*

(i) If A is K-closed, then $A\preceq^l_K B\Longleftrightarrow \mathbb{D}_K(A,B)\leq 0$.

(ii) If B is $-K$-closed, then $A\preceq^u_K B\Longleftrightarrow \mathscr{D}_K(A,B)\leq 0$.

Although the order relations mentioned in Definition 3.1 can achieve the comparison between sets, there are some practical situations that can not be described by them, as seen in the following example. To overcome this defect, we put forward the following weighted set order relation.

Definition 3.2. Let $\lambda \in [0,1]$, $A,B \in P(Y)$ be $\pm K$-proper and $\pm K$-compact sets. The weighted set order relation with respect to K, denoted by $\preceq_K^\lambda$, is defined by

$$A \preceq_K^\lambda B \Longleftrightarrow \lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B) \leq 0. \tag{3.1}$$

Let $W_K^\lambda(A,B) = \lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B)$. Then, (3.1) can be restated as

$$A \preceq_K^\lambda B \Longleftrightarrow W_K^\lambda(A,B) \leq 0.$$

Remark 3.1. Obviously, it follows from Lemma 3.1 that if $\lambda = 0$ or $\lambda = 1$, then $A \preceq_K^\lambda B$ reduces to $A \preceq_K^l B$ or $A \preceq_K^u B$, respectively. In addition, it is clear that if both $A \preceq_K^l B$ and $A \preceq_K^u B$ are satisfied, then $A \preceq_K^\lambda B$ is valid for any $\lambda \in [0,1]$. Conversely, it may not be true.

Example 3.1. Let $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $A = [3,6]$, and $B = [1,9]$. It is easy to check that $K^* = \mathbb{R}_+$ and $S(K^*) = \{1\}$. Due to

$$\triangle_{-K}(a-b) = \triangle_K(b-a) = \sup_{y^* \in S(K^*)} \langle -y^*, b-a \rangle = a-b,$$

we derive

$$\mathscr{D}_K(A,B) = \sup_{a\in A} \inf_{b \in B} \triangle_{-K}(a-b) = -3,\ \mathbb{D}_K(A,B) = \sup_{b\in B} \inf_{a\in A} \triangle_{-K}(a-b) = 2.$$

Similarly, we have

$$\mathscr{D}_K(B,A) = \sup_{b\in B} \inf_{a\in A} \triangle_{-K}(b-a) = 3,\ \mathbb{D}_K(B,A) = \sup_{a\in A} \inf_{b\in B} \triangle_{-K}(b-a) = -2.$$

By Lemma 3.1, we obtain that $A \preceq_K^u B$, but $A \npreceq_K^l B$, and $B \preceq_K^l A$, but $B \npreceq_K^u A$. In this case, a decision-maker does not know how to make the choice. In addition, it follows from Definition 3.1 that $A \npreceq_K^s B$ and $B \npreceq_K^s A$, which leads to the set of optimal solutions is empty. However, for any $\lambda \in [\frac{2}{5},1]$, it yields $\lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B) \leq 0$. For all $\lambda \in [0,\frac{2}{5}]$, we have $\lambda \mathscr{D}_K(B,A) + (1-\lambda)\mathbb{D}_K(B,A) \leq 0$. Therefore, $A \preceq_K^\lambda B$ for all $\lambda \in [\frac{2}{5},1]$ and $B \preceq_K^\lambda A$ for all $\lambda \in [0,\frac{2}{5}]$. This indicates that the weighted set relation $\preceq_K^\lambda$ is valid.

Proposition 3.1. *Let $\lambda \in [0,1]$, $A,B,C \in P(Y)$ be $\pm K$-proper and $\pm K$-compact sets. The following statements hold:*

(i) $\preceq_K^\lambda$ is reflexive (i.e., $A \preceq_K^\lambda A$) and transitive (i.e., if $A \preceq_K^\lambda B$ and $B \preceq_K^\lambda C$, imply $A \preceq_K^\lambda C$).

(ii) For all $\alpha > 0$, $A \preceq_K^\lambda B \Longrightarrow \alpha A \preceq_K^\lambda \alpha B$.

Proof. (i) Firstly, we verify that $\preceq_K^\lambda$ is reflexive. For any $A \in P(Y)$, since A is $\pm K$-proper, it follows from Lemma 2.2 (i) that $\mathbb{D}_K(A,A) = \mathscr{D}_K(A,A) = 0$. Consequently,

$$\lambda \mathscr{D}_K(A,A) + (1-\lambda)\mathbb{D}_K(A,A) = 0 \leq 0,\ \forall \lambda \in [0,1].$$

This means that $A \preceq_K^\lambda A$. Hence, $\preceq_K^\lambda$ is reflexive.

Next, we prove that $\preceq_K^\lambda$ is transitive. Assume that $A \preceq_K^\lambda B$ and $B \preceq_K^\lambda C$. Then

$$\lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B) \leq 0,$$

and

$$\lambda \mathscr{D}_K(B,C) + (1-\lambda)\mathbb{D}_K(B,C) \leq 0.$$

Hence

$$\lambda\left(\mathscr{D}_K(A,B) + \mathscr{D}_K(B,C)\right) + (1-\lambda)\left(\mathbb{D}_K(A,B) + \mathbb{D}_K(B,C)\right) \leq 0.$$

It follows from Proposition 2.1 that

$$\mathscr{D}_K(A,C) \leq \mathscr{D}_K(A,B) + \mathscr{D}_K(B,C),\ \mathbb{D}_K(A,C) \leq \mathbb{D}_K(A,B) + \mathbb{D}_K(B,C).$$

Thus $\lambda \mathscr{D}_K(A,C) + (1-\lambda)\mathbb{D}_K(A,C) \leq 0$, which implies that $A \preceq_K^\lambda C$. Therefore, $\preceq_K^\lambda$ is transitive.

(ii) Since $A \preceq_K^\lambda B$, it leads to $\lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B) \leq 0$. For all $\alpha > 0$, in view of Lemma 2.2 (iv), one has

$$\begin{aligned}\lambda \mathscr{D}_K(\alpha A, \alpha B) + (1-\lambda)\mathbb{D}_K(\alpha A, \alpha B) &= \alpha\lambda \mathscr{D}_K(A,B) + \alpha(1-\lambda)\mathbb{D}_K(A,B) \\ &= \alpha\left[\lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B)\right] \leq 0.\end{aligned}$$

Hence, $\alpha A \preceq_K^\lambda \alpha B$ is true. □

Remark 3.2. In [5], a weighted set order relation introduced by linear functional is defined as

$$A \preceq_K^\lambda B \iff \forall k^* \in K^* \backslash \{0\},\ \lambda\left(\sup_{a\in A}\inf_{b\in B} k^*(a-b)\right) + (1-\lambda)\left(\sup_{b\in B}\inf_{a\in A} k^*(a-b)\right) \leq 0,$$

where $\lambda \in [0,1]$, $A, B \in P(Y)$ are compact and $A+K$, $B-K$, $B+K$ are closed and convex. We would like to mention that the present weighted relation does not require any convexity.

Example 3.2. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = [(-2,-1),(-1,-1)] \cup [(-1,-1),(-1,-2)]$, and $B = \{(0,2),(2,0)\}$, where the notation $[(\cdot,\cdot),(\cdot,\cdot)]$ indicates the direct connection line between the points.

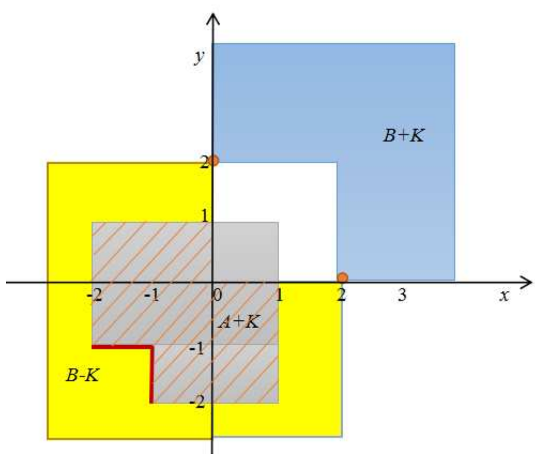

Fig 1. Illustrations of A, B, $A+K$, $B-K$ and $B+K$.

As demonstrated in Fig 1, we have that $A+K$, $B-K$ and $B+K$ are not convex, where the blue area represents $B+K$, the yellow area represents $B-K$, the gray area represents $A+K$, and the shaded area represents the common parts of $A+K$ and $B-K$. Now, we verify that $A \preceq_K^\lambda B$ is applicable for all $\lambda \in [0,1]$. In fact, since $A - B \subset -K$, by Lemma 2.1, we derive $\triangle_{-K}(a-b) \leq 0$ for all $a \in A$ and $b \in B$. Since A and B are compact, it yields $\mathbb{D}_K(A,B) \leq 0$ and $\mathscr{D}_K(A,B) \leq 0$. For all $\lambda \in [0,1]$, one has

$$W_K^\lambda(A,B) = \lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B) \leq 0.$$

Thus, $A \preceq_K^\lambda B$ holds.

Remark 3.3. Let $A,B \in P(Y)$ be closed and bounded and $e \in intK \neq \emptyset$. The Gerstewitz's function $\varphi_{K,e} : Y \to \mathbb{R} \cup \{\pm\infty\}$ is given by $\varphi_{K,e}(y) = \inf\{t \in \mathbb{R} : y \in te - K\}$, $y \in Y$. Chen et al. [3] proposed a weighted set order relation via $\varphi_{K,e}$ as follows:

$$A \preceq_K^\lambda B \Longleftrightarrow \lambda\Big(\sup_{a\in A}\inf_{b\in B}\varphi_{K,e}(a-b)\Big) + (1-\lambda)\Big(\sup_{b\in B}\inf_{a\in A}\varphi_{K,e}(a-b)\Big) \leq 0,\ \lambda \in [0,1].$$

We emphasize that $intK \neq \emptyset$ is not necessary in Definition 3.2 of this paper.

Example 3.3. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}_+ \times \{0\}$, $A = [(-2,1),(-1,1)]$ and $B = [(1,1),(3,1)]$, where $[(a,b),(c,d)]$ is the line segment between (a,b) and (c,d).

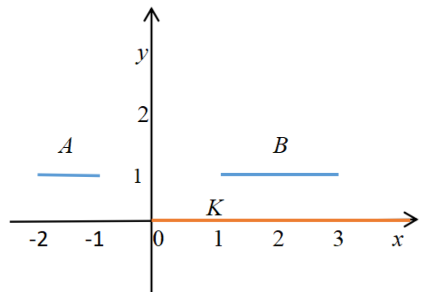

Fig 2. Illustrations of A, B and K.

From Fig 2, we can see that $intK = \emptyset$. Next, we prove $A \preceq_K^\lambda B$ is valid for all $\lambda \in [0,1]$. Indeed, due to $A - B \subset -\partial K$, it follows from Lemma 2.1 that $\triangle_{-K}(a-b) = 0$ for all $a \in A$ and $b \in B$. Since A and B are compact, we deduce $\mathbb{D}_K(A,B) = 0$ and $\mathscr{D}_K(A,B) = 0$. Thus $W_K^\lambda(A,B) = 0$ for all $\lambda \in [0,1]$, which means that $A \preceq_K^\lambda B$ is valid.

Now, let us explore the properties of function W_K^λ.

Proposition 3.2. *Assume that $A,B,C \in P(Y)$ are $\pm K$-proper and $\pm K$-compact sets. Then, for $\lambda \in [0,1]$ and $t \in \mathbb{R}$, the following assertions hold:*

(i) $W_K^\lambda(A,B) \in \mathbb{R}$. In particular, if $A = B$, then $W_K^\lambda(A,B) = 0$.

(ii) $W_K^\lambda(\alpha A, \alpha B) = \alpha W_K^\lambda(A,B)$ for all $\alpha > 0$.

(iii) $W_K^\lambda(A,C) \leq W_K^\lambda(A,B) + W_K^\lambda(B,C)$.

(iv) If $e \in K$ and $d_{-K}(e) = d_{Y\setminus(-K)}(-e) = 1$, then

$$W_K^\lambda(A+te,B) = W_K^\lambda(A,B) + t,\ W_K^\lambda(A,B+te) = W_K^\lambda(A,B) - t.$$

Proof. (i) By Lemma 2.2 (ii) and (iii), we have $\mathscr{D}_K(A,B) \in \mathbb{R}$ and $\mathbb{D}_K(A,B) \in \mathbb{R}$. Thus, $W_K^\lambda(A,B) \in \mathbb{R}$. Moreover, if $A = B$, by Lemma 2.2 (i), then $\mathscr{D}_K(A,B) = \mathbb{D}_K(A,B) = 0$. Hence, $W_K^\lambda(A,B) = 0$.

(ii) For all $\alpha > 0$, combining with Lemma 2.2 (iv) yields

$$\begin{aligned} W_K^\lambda(\alpha A, \alpha B) &= \lambda \mathscr{D}_K(\alpha A, \alpha B) + (1-\lambda)\mathbb{D}_K(\alpha A, \alpha B) \\ &= \alpha\lambda \mathscr{D}_K(A,B) + \alpha(1-\lambda)\mathbb{D}_K(A,B) \\ &= \alpha\big[\lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B)\big] = \alpha W_K^\lambda(A,B). \end{aligned}$$

(iii) According to Proposition 2.1, we have

$$\begin{aligned} W_K^\lambda(A,C) &= \lambda \mathscr{D}_K(A,C) + (1-\lambda)\mathbb{D}_K(A,C) \\ &\leq \lambda[\mathscr{D}_K(A,B) + \mathscr{D}_K(B,C)] + (1-\lambda)[\mathbb{D}_K(A,B) + \mathbb{D}_K(B,C)] \\ &= [\lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B)] + [\lambda \mathscr{D}_K(B,C) + (1-\lambda)\mathbb{D}_K(B,C)] \\ &= W_K^\lambda(A,B) + W_K^\lambda(B,C). \end{aligned}$$

(iv) By means of Proposition 2.2, it yields

$$\mathbb{D}_K(A+te,B) = \mathbb{D}_K(A,B) + t,\ \mathscr{D}_K(A+te,B) = \mathscr{D}_K(A,B) + t.$$

Thus

$$\begin{aligned} W_K^\lambda(A+te,B) &= \lambda \mathscr{D}_K(A+te,B) + (1-\lambda)\mathbb{D}_K(A+te,B) \\ &= \lambda(\mathscr{D}_K(A,B) + t) + (1-\lambda)(\mathbb{D}_K(A,B) + t) \\ &= \lambda \mathscr{D}_K(A,B) + (1-\lambda)\mathbb{D}_K(A,B) + t \\ &= W_K^\lambda(A,B) + t. \end{aligned}$$

By a similar calculation, we can derive $W_K^\lambda(A,B+te) = W_K^\lambda(A,B) - t$. This completes the proof. □

Proposition 3.3. *Let $\lambda \in [0,1]$, $A,B \in P(Y)$ be $\pm K$-proper and $\pm K$-compact sets. Assume that $e \in K$ and $d_{-K}(e) = d_{Y\setminus(-K)}(-e) = 1$. Then, the following statements hold:*

(i) If $r \geq 0$ and $A + re \preceq_K^\lambda B$, then $A \preceq_K^\lambda B$.

(ii) If $0 \leq r_2 \leq r_1$ and $A + r_1 e \preceq_K^\lambda B$, then $A + r_2 e \preceq_K^\lambda B$.

(iii) If $p,q \in \mathbb{R}$ such that $A + pe \preceq_K^\lambda B$ and $A + qe \not\preceq_K^\lambda B$, then $p < q$.

Proof. (i) Let $A + re \preceq_K^\lambda B$, by Definition 3.2 and Proposition 3.2 (iv), we obtain

$$W_K^\lambda(A+re,B) = W_K^\lambda(A,B) + r \leq 0.$$

Due to $r \geq 0$, we have $W_K^\lambda(A,B) \leq 0$. Hence, $A \preceq_K^\lambda B$.

(ii) If $A + r_1 e \preceq_K^\lambda B$, then $W_K^\lambda(A + r_1 e, B) = W_K^\lambda(A,B) + r_1 \leq 0$. Since $0 \leq r_2 \leq r_1$, it leads to $W_K^\lambda(A,B) + r_2 \leq 0$. By Proposition 3.2 (iv), we derive $A + r_2 e \preceq_K^\lambda B$.

(iii) Since $A + pe \preceq_K^\lambda B$, it yields $W_K^\lambda(A + pe, B) = W_K^\lambda(A,B) + p \leq 0$. Moreover, we obtain from $A + qe \not\preceq_K^\lambda B$ that $W_K^\lambda(A+qe,B) = W_K^\lambda(A,B) + q \not\leq 0$. Hence

$$W_K^\lambda(A,B) + p < W_K^\lambda(A,B) + q.$$

Note that $W_K^\lambda(A,B) \in \mathbb{R}$. Therefore, we have $p < q$. The proof is completed. □

4. Ekeland's Variational Principle

This section focuses on establishing an Ekeland's variational principle for set-valued maps related to the weighted set order relation defined in Definition 3.2. Throughout the rest of paper, we always assume that $e \in K\setminus\{0\}$, $d_{-K}(e) = d_{Y\setminus(-K)}(-e) = 1$, $S \subset X$ is a nonempty subset and $H : S \rightrightarrows Y$ is a set-valued map with nonempty, $\pm K$-proper and $\pm K$-compact values. We denote by $H(S) = \bigcup_{x\in S} H(x)$.

The following lemma plays a crucial role in the proof of Ekeland's variational principle.

Lemma 4.1. [18, 23] *Let $Q : S \rightrightarrows S$ be a set-valued map. If the following conditions hold:*

(I) for all $x \in S$, $Q(x)$ is a closed set;

(II) for all $x \in S$, $x \in Q(x)$;

(III) for all $x, y \in S$, $y \in Q(x) \Rightarrow Q(y) \subset Q(x)$;

(IV) for all sequences $x_1, x_2, \ldots, x_n, \ldots$ in S, that are generalized Picard-iterations starting from x_1, i.e., fulfill $x_2 \in Q(x_1), x_3 \in Q(x_2), \ldots, x_n \in Q(x_{n-1}), \ldots$, the distances $d(x_n, x_{n+1})$ tend to 0 as $n \to \infty$.

Then, there exists a point $\bar{x} \in Q(x_1)$ such that $Q(\bar{x}) = \{\bar{x}\}$.

Assumption 4.1. Let $\lambda \in [0,1]$ and $x, y, z \in S$.

(a) The set $\{y \in S : H(y) + d(x,y)e \preceq_K^\lambda H(x)\}$ is closed;

(b) $H(x) \preceq_K^\lambda H(x)$;

(c) If $H(y)+d(x,y)e \preceq_K^\lambda H(x)$ and $H(z)+d(y,z)e \preceq_K^\lambda H(y)$, then $H(z)+[d(x,y)+d(y,z)]e \preceq_K^\lambda H(x)$.

Now, we establish the Ekeland's variational principle with respect to $\preceq_K^\lambda$.

Theorem 4.1. *Suppose that Assumption 4.1 holds, and, for all $x_1 \in S$ and $r > 0$*

$$H(y) + re \npreceq_K^\lambda H(x_1), \ \forall y \in S. \tag{4.1}$$

Then, there exists $\bar{x} \in S$ such that

(i) $d(x_1, \bar{x}) < r$;

(ii) $H(\bar{x}) + d(x_1, \bar{x})e \preceq_K^\lambda H(x_1)$;

(iii) $H(y) + d(\bar{x}, y)e \npreceq_K^\lambda H(\bar{x})$, $\forall y \in S,\ y \neq \bar{x}$.

Proof. Let $Q(x) = \{y \in S : H(y) + d(x,y)e \preceq_K^\lambda H(x)\}$ for all $x \in S$. It follows from Assumption 4.1 (a) and (b) that $Q(x)$ is closed and $x \in Q(x)$ for all $x \in S$. This implies conditions (I) and (II) of Lemma 4.1. Next, we verify that $Q(x)$ satisfies condition (III). Taking $x, y \in S$ with $y \in Q(x)$, we have

$$H(y) + d(x,y)e \preceq_K^\lambda H(x). \tag{4.2}$$

For all $z \in Q(y)$, it yields

$$H(z) + d(y,z)e \preceq_K^\lambda H(y). \tag{4.3}$$

From (4.2), (4.3), and Assumption 4.1 (c), we obtain

$$H(z) + [d(x,y) + d(y,z)]e \preceq_K^\lambda H(x).$$

Due to $d(x,z) \leq d(x,y) + d(y,z)$, by Proposition 3.3 (ii), we have $H(z) + d(x,z)e \preceq_K^\lambda H(x)$, which indicates that $z \in Q(x)$ for all $z \in Q(y)$. Thus $Q(y) \subset Q(x)$. The condition (III) of Lemma 4.1 is satisfied.

Now, we consider an arbitrary sequence of Picard-iterations starting from x_1. That is, for all $n \geq 2$ fulfill $x_n \in Q(x_{n-1})$. Next, we claim that

$$H(x_n) + [d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)]e \preceq_K^\lambda H(x_1). \tag{4.4}$$

It is proved by induction. By $x_2 \in Q(x_1)$, we obtain $H(x_2) + d(x_1,x_2)e \preceq_K^\lambda H(x_1)$. By $x_3 \in Q(x_2)$, we have $H(x_3) + d(x_2,x_3)e \preceq_K^\lambda H(x_2)$. In view of Assumption 4.1 (c), we derive

$$H(x_3) + [d(x_1,x_2) + d(x_2,x_3)]e \preceq_K^\lambda H(x_1).$$

Thus (4.4) holds for $n = 2,3$. Assume that (4.4) is valid for all $n = k-1$ and $k > 4$. We need to prove it holds for $n = k$. Indeed, for $n = k-1$, we have

$$H(x_{k-1}) + [d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{k-2},x_{k-1})]e \preceq_K^\lambda H(x_1). \tag{4.5}$$

Due to $x_k \in Q(x_{k-1})$, we arrive at

$$H(x_k) + d(x_{k-1},x_k)e \preceq_K^\lambda H(x_{k-1}). \tag{4.6}$$

Combined with (4.5), (4.6), and Assumption 4.1 (c), it yields

$$H(x_k) + [d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{k-1},x_k)]e \preceq_K^\lambda H(x_1).$$

Hence, (4.4) is satisfied for all $n \geq 2$. Let $q_n = d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)$. (4.4) can be rewritten as $H(x_n) + q_n e \preceq_K^\lambda H(x_1)$. By (4.1) and Proposition 3.3 (iii), we have $q_n < r$. Consequently, sequence q_n is convergent, and $d(x_n,x_{n+1}) = q_{n+1} - q_n$ tend to 0 as $n \to \infty$. This implies that the condition (IV) is satisfied. It follows from Lemma 4.1 that there exists $\bar{x} \in Q(x_1)$ such that $Q(\bar{x}) = \{\bar{x}\}$. The $\bar{x} \in Q(x_1)$ means that $H(\bar{x}) + d(x_1,\bar{x})e \preceq_K^\lambda H(x_1)$. Furthermore, together with (4.1) and Proposition 3.3 (iii), we obtain $d(x_1,\bar{x}) < r$. The results (i) and (ii) are proved. Assume that result (iii) is not true. Then there exists $\hat{y} \in S$ with $\hat{y} \neq \bar{x}$ such that $H(\hat{y}) + d(\bar{x},\hat{y})e \preceq_K^\lambda H(\bar{x})$. Thus, $\hat{y} \in Q(\bar{x})$, which contradicts to $Q(\bar{x}) = \{\bar{x}\}$. Hence, the result (iii) holds. The proof is completed. □

We provide a concrete example to illustrate Theorem 4.1.

Example 4.1. Let $X = Y = \mathbb{R}$, $K = \mathbb{R}_+$, $e = 1$, $S = [0,1]$, and

$$H(x) = \begin{cases} [-1,4], & x = 1, \\ [0,2x], & x \neq 1. \end{cases}$$

It is obtained that $d_{-K}(e) = d_{Y\setminus(-K)}(-e) = 1$, and

$$\mathbb{D}_K(H(x),H(1)) = 1,\ \mathscr{D}_K(H(x),H(1)) = 2x - 4,\ \forall x \in S\setminus\{1\}.$$

Letting $\lambda = \frac{1}{2}$, we have

$$W_K^{\frac{1}{2}}(H(x),H(1)) = \frac{1}{2}(2x-4) + \frac{1}{2} = x - \frac{3}{2} \leq 0,\ \forall x \in S\setminus\{1\}.$$

Hence,

(a) $Q(1) = \{y \in S : H(y) + d(1,y)e \preceq_K^{\frac{1}{2}} H(1)\} = [0,1]$ is closed;

(b) $H(1) \preceq_K^{\frac{1}{2}} H(1)$;

(c) for all $y,z \in S\setminus\{1\}$ with $z \leq y$, since

$$W_K^{\frac{1}{2}}(H(y) + d(1,y)e, H(1)) = y - \frac{3}{2} + |1-y| = -\frac{1}{2} \leq 0,$$

$$W_K^{\frac{1}{2}}(H(z) + d(y,z)e, H(y)) = (z-y) + |z-y| = 0 \leq 0,$$

it leads to

$$H(y) + d(1,y)e \preceq_K^{\frac{1}{2}} H(1),\ H(z) + d(y,z)e \preceq_K^{\frac{1}{2}} H(y).$$

Due to

$$y - \frac{3}{2} + |1-y| + (z-y) + |z-y| = z - \frac{3}{2} + |1-y| + |z-y| \leq 0,$$

we derive $W_K^{\frac{1}{2}}(H(z)+d(1,y)+d(y,z),H(1)) \leq 0$. Thus $H(z)+[d(1,y)+d(y,z)]e \preceq_K^{\frac{1}{2}} H(1)$. Furthermore, for all $x_1 \in S$ and $r > x_1+y+1$, we obtain $H(y)+re \npreceq_K^{\frac{1}{2}} H(x_1)$ for all $y \in S$. Therefore, all the conditions of Theorem 4.1 are satisfied. There exists $\bar{x} = 0 \in S$ such that $d(x_1,0) = x_1 < r$, and $H(0)+d(x_1,0)e \preceq_K^{\frac{1}{2}} H(x_1)$. For all $y \in S$ with $y \neq 0$, considering the following two cases: If $y = 1$, then $W_K^{\frac{1}{2}}(H(1)+d(0,1)e,H(0)) = 2 \nleq 0$. If $y \neq 1$, then $W_K^{\frac{1}{3}}(H(y)+d(0,y)e,H(0)) = 2y \nleq 0$. Consequently, $H(y)+d(0,y)e \npreceq_K^{\frac{1}{2}} H(0)$ for all $y \in S$ with $y \neq 0$. Hence, $\bar{x} = 0$ fulfills all the results of Theorem 4.1.

The following is a Caristi's fixed point theorem for order relation $\preceq_K^{\lambda}$.

Theorem 4.2. *Suppose that Assumption 4.1 is fulfilled and $T : S \rightrightarrows S$ is a set-valued map with nonempty values such that for all $x \in S$ there exists $y \in T(x)$ satisfying*

$$H(y)+d(x,y)e \preceq_K^{\lambda} H(x). \tag{4.7}$$

Then, there exists $\bar{x} \in S$ such that $\bar{x} \in T(\bar{x})$. In particular, if for all $y \in T(x)$ such that (4.7) holds, then $T(\bar{x}) = \{\bar{x}\}$.

Proof. It follows from the proof of Theorem 4.1 that there exists $\bar{x} \in S$ such that

$$H(y)+d(\bar{x},y)e \npreceq_K^{\lambda} H(\bar{x}),\ \forall y \in S,\ y \neq \bar{x}.$$

Moreover, according to the assumption, there exists $y \in T(\bar{x})$ satisfying

$$H(y)+d(\bar{x},y)e \preceq_K^{\lambda} H(\bar{x}). \tag{4.8}$$

Thus $y = \bar{x}$ and $\bar{x} \in T(\bar{x})$. If for any $y \in T(\bar{x})$ satisfying (4.8), then $T(\bar{x}) = \{\bar{x}\}$. □

Now, we introduce the notion of minimal set with respect to $\preceq_K^{\lambda}$.

Definition 4.1. Let $\Lambda \subset P(Y)$, $\lambda \in [0,1]$. An element $\bar{A} \in \Lambda$ is called λ-minimal set of Λ if $A \in \Lambda$, $A \preceq_K^{\lambda} \bar{A} \Longrightarrow \bar{A} \preceq_K^{\lambda} A$. We denote the set of all λ-minimal sets of Λ by $\Omega(\Lambda, \preceq_K^{\lambda})$.

A Takahashi's minimization theorem is presented in the following.

Theorem 4.3. *If Assumption 4.1 holds, and for all $x,y \in S$ with $H(y) \preceq_K^{\lambda} H(x)$ and $H(x) \npreceq_K^{\lambda} H(y)$, there exists $z \in S$ with $z \neq x$ such that $H(z)+d(x,z)e \preceq_K^{\lambda} H(x)$, then there exists $\bar{x} \in S$ such that $H(\bar{x}) \in \Omega(H(S), \preceq_K^{\lambda})$.*

Proof. Let $Q(x) = \{y \in S : H(y)+d(x,y)e \preceq_K^{\lambda} H(x)\}$ for all $x \in S$. By the proof of Theorem 4.1, there exists $\bar{x} \in S$ such that $Q(\bar{x}) = \{\bar{x}\}$. Now, we prove that $H(\bar{x})$ is minimal set of $H(S)$. If not, it follows from Definition 4.1 that there exists $\hat{y} \in S$ such that $H(\hat{y}) \preceq_K^{\lambda} H(\bar{x})$ and $H(\bar{x}) \npreceq_K^{\lambda} H(\hat{y})$. According to the above condition, there exists $z \in S$ with $z \neq \bar{x}$ such that $H(z)+d(\bar{x},z)e \preceq_K^{\lambda} H(\bar{x})$. Hence, $z \in Q(\bar{x})$, which contradicts $Q(\bar{x}) = \{\bar{x}\}$. Therefore, $H(\bar{x}) \in \Omega(H(S), \preceq_K^{\lambda})$. □

Theorem 4.4. *Theorem 4.1, Theorem 4.2, and Theorem 4.3 are equivalent to each other.*

Proof. First, we prove Theorem 4.1 implies Theorem 4.2. It is obtained directly from Theorem 4.2.

Then, we clarify Theorem 4.2 implies Theorem 4.3. Let

$$T(x) = \{y \in S : H(y)+d(x,y)e \preceq_K^{\lambda} H(x)\},\ \forall x \in S.$$

By Assumption 4.1 (b), we have $x \in T(x)$ and $T(x) \neq \emptyset$. By virtue of Theorem 4.2, there exists $\bar{x} \in S$ such that $T(\bar{x}) = \{\bar{x}\}$. It follows from the definition of T that $H(\bar{x})$ is minimal set of $H(S)$. That is, $H(\bar{x}) \in \Omega(H(S), \preceq_K^\lambda)$.

Finally, we verify that Theorem 4.3 can deduce Theorem 4.1. Define

$$X_1 := \{y \in S : H(y) + d(x_1, y)e \preceq_K^\lambda H(x_1)\}, \ \forall x_1 \in S.$$

It follows from Assumption 4.1 (b) that $x_1 \in X_1$ and $X_1 \neq \emptyset$. Moreover, we get from Assumption 4.1 (a) that X_1 is closed and hence complete. Suppose that the result (iii) of Theorem 4.1 does not hold. Then for all $x \in X_1$, there exists $\hat{y} \in S$ with $\hat{y} \neq x$ such that

$$H(\hat{y}) + d(x, \hat{y})e \preceq_K^\lambda H(x). \tag{4.9}$$

By $x \in X_1$, we derive

$$H(x) + d(x_1, x)e \preceq_K^\lambda H(x_1). \tag{4.10}$$

From (4.9), (4.10), and Assumption 4.1 (c), we have

$$H(\hat{y}) + [d(x_1, x) + d(x, \hat{y})]e \preceq_K^\lambda H(x_1).$$

Due to $d(x_1, \hat{y}) \leq d(x_1, x) + d(x, \hat{y})$, we deduce from Proposition 3.3 (ii) that

$$H(\hat{y}) + d(x_1, \hat{y})e \preceq_K^\lambda H(x_1),$$

which means that $\hat{y} \in X_1$. In addition, by Theorem 4.3, there exists $\bar{x} \in X_1$ such that $H(\bar{x}) \in \Omega(H(S), \preceq_K^\lambda)$. Hence $H(\bar{x}) \preceq_K^\lambda H(\hat{y})$. However, from (4.9), we have

$$H(\hat{y}) + d(\bar{x}, \hat{y})e \preceq_K^\lambda H(\bar{x}).$$

According to Proposition 3.3 (i), $H(\hat{y}) \preceq_K^\lambda H(\bar{x})$. This leads to a contradiction. Consequently, the result (iii) of Theorem 4.1 holds. Furthermore, it is obtained from $\bar{x} \in X_1$ that

$$H(\bar{x}) + d(x_1, \bar{x})e \preceq_K^\lambda H(x_1),$$

which together with Proposition 3.3 (iii) yields $d(x_1, \bar{x}) < r$. Hence, the results (i) and (ii) of Theorem 4.1 are also hold. □

5. Applications to Set Optimization Problems

In this section, we investigate the existence theorem of solutions to a set optimization problem by applying Ekeland's variational principle obtained in Section 4.

Let $F : S \rightrightarrows Y$ be a set-valued map with nonempty, $\pm K$-proper, and $\pm K$-compact values. A set optimization problem is defined as

$$\text{(SOP)} \quad \min F(x), \quad \text{s.t. } x \in S.$$

Definition 5.1. Let $\lambda \in [0, 1]$. A feasible point $\bar{x} \in S$ is called λ-minimal solution of problem (SOP), if

$$x \in S, \ F(x) \preceq_K^\lambda F(\bar{x}) \Longrightarrow F(\bar{x}) \preceq_K^\lambda F(x).$$

The set of all λ-minimal solutions of problem (SOP) is denoted as Ω^λ.

Theorem 5.1. *In problem (SOP), assume that $\lambda \in [0,1]$ and the following conditions hold.*

(i) For all $x \in S$, $\{y \in S : F(y) + d(x,y)e \preceq_K^\lambda F(x)\}$ is closed.

(ii) For all $x \in S$, $F(x) \preceq_K^\lambda F(x)$.

(iii) For all $x,y,z \in S$, if $F(y) + d(x,y)e \preceq_K^\lambda F(x)$ and $F(z) + d(y,z)e \preceq_K^\lambda F(y)$, then $F(z) + [d(x,y) + d(y,z)]e \preceq_K^\lambda F(x)$.

(iv) For any $x,y \in S$ with $F(y) \preceq_K^\lambda F(x)$ and $F(x) \npreceq_K^\lambda F(y)$, there exists $\bar{z} \in S$ with $\bar{z} \neq x$ such that

$$F(\bar{z}) + d(x,\bar{z})e \preceq_K^\lambda F(x).$$

Then, $\Omega^\lambda \neq \emptyset$.

Proof. Let $Q(x) = \{y \in S : F(y) + d(x,y)e \preceq_K^\lambda F(x)\}$. By a proof similar to Theorem 4.1, there exists $\bar{x} \in S$ such that $Q(\bar{x}) = \{\bar{x}\}$. Now, we verify $\bar{x}$ is λ-minimal solution of problem (SOP). If not, there exists $\hat{y} \in S$ such that $F(\hat{y}) \preceq_K^\lambda F(\bar{x})$ and $F(\bar{x}) \npreceq_K^\lambda F(\hat{y})$. In view of condition (iv), there exists $\bar{z} \in S$ with $\bar{z} \neq \bar{x}$ such that $F(\bar{z}) + d(\bar{x},\bar{z})e \preceq_K^\lambda F(\bar{x})$. Thus, $\bar{z} \in Q(\bar{x})$. This is a contradiction to $Q(\bar{x}) = \{\bar{x}\}$. Hence, $\bar{x}$ is λ-minimal solution of problem (SOP). This is, $\Omega^\lambda \neq \emptyset$. □

Here is an example to interpret Theorem 5.1.

Example 5.1. In problem (SOP), let $X = Y = \mathbb{R}$, $K = \mathbb{R}_+$, $e = 1$ and $S = [0,1]$. We define

$$F(x) = \begin{cases} [-1,7], & x = 1, \\ [0, 3x+1], & x \neq 1. \end{cases}$$

By direct computation, we have $d_{-K}(e) = d_{Y\setminus(-K)}(-e) = 1$, and

$$\mathbb{D}_K(F(x),F(1)) = 1,\ \mathscr{D}_K(F(x),F(1)) = 3x-6,\ \forall x \in S\setminus\{1\}.$$

Letting $\lambda = \frac{1}{3}$, we obtain

$$W_K^{\frac{1}{3}}(F(x),F(1)) = x - \frac{4}{3} \leq 0,\ \forall x \in S\setminus\{1\}.$$

Thus,

(a) $Q(1) = \{y \in S : F(y) + d(1,y)e \preceq_K^{\frac{1}{3}} F(1)\} = [0,1]$ is closed;

(b) $F(1) \preceq_K^{\frac{1}{3}} F(1)$;

(c) For all $y,z \in S\setminus\{1\}$ with $z \leq y$, since

$$W_K^{\frac{1}{3}}(F(y) + d(1,y)e, F(1)) = (y - \frac{4}{3}) + |1-y| = -\frac{1}{3} \leq 0, \tag{5.1}$$

$$W_K^{\frac{1}{3}}(F(z) + d(y,z)e, F(y)) = (z-y) + |z-y| = 0, \tag{5.2}$$

we obtain

$$F(y) + d(1,y)e \preceq_K^{\frac{1}{3}} F(1),\ F(z) + d(y,z)e \preceq_K^{\frac{1}{3}} F(y).$$

It follows from (5.1) and (5.2) that

$$(y - \frac{4}{3}) + |1-y| + (z-y) + |z-y| = (z - \frac{4}{3}) + |1-y| + |z-y| \leq 0.$$

This is equivalent to

$$F(z) + [d(1,y) + d(y,z)]e \preceq_K^{\frac{1}{3}} F(1).$$

In addition, for all $y \in S\backslash\{1\}$, we have that $F(y) \preceq_K^{\frac{1}{3}} F(1)$ and $F(1) \not\preceq_K^{\frac{1}{3}} F(y)$, there exists $\bar{z} = 0$ such that

$$W_K^{\frac{1}{3}}(F(0)+d(1,0)e, F(1)) = -\frac{1}{3} \leq 0.$$

Thus $F(0)+d(1,0)e \preceq_K^{\frac{1}{3}} F(1)$. This means that all the conditions of Theorem 5.1 are satisfied. Further, due to

$$W_K^{\frac{1}{3}}(F(0),F(1)) = -\frac{4}{3} \leq 0,\ W_K^{\frac{1}{3}}(F(0),F(y)) = -y \leq 0,\ \forall y \in S\backslash\{1\},$$

we obtain $F(0) \preceq_K^{\frac{1}{3}} F(1)$ and $F(0) \preceq_K^{\frac{1}{3}} F(y)$ for all $y \in S\backslash\{1\}$. Hence $F(0) \preceq_K^{\frac{1}{3}} F(y)$ for all $y \in S$. This implies that $\bar{z} = 0$ is a $\frac{1}{3}$-minimal solution of this problem. i.e. $\Omega^{\frac{1}{3}} \neq \emptyset$.

6. Conclusions

We proposed a new weighted set order relation by utilizing the set scalarization function, which does not require any convexity assumption and is suitable for the cones with empty interior compared with [4, 5]. In addition, we established Ekeland's variational principle, Caristi's fixed point theorem, and Takahashi's minimization theorem for set-valued maps with respect to the introduced weighted set relation, and proved the equivalences between them. These results are different from those in [4] in terms of methods and assumptions. We also applied the results to explore the existence of solution for a set optimization problem.

It would be meaningful to further investigate the approximate minimality and variable domination structures for this new weighted set order relation. Indeed, Köbis and Köbis [24] discussed approximate minimality and variable domination structures for an upper set order relation. However, we obtained from Remark 3.1 that the new weighted set order relation can be reduced to the upper set order relation if $\lambda = 1$. Therefore, the further works seem feasible.

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