J. Nonlinear Var. Anal. 7 (2023), No. 3, pp. 409-420
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.3.06

A REMARK ON CHERN-SIMONS-SCHRÖDINGER EQUATIONS WITH HARTREE TYPE NONLINEARITY

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Abstract. This paper is devoted to studying the following Chern-Simons-Schrödinger equation with Hartree type nonlinearity:

$$-\frac{1}{2m}\Delta\psi+\omega\psi+\frac{2e^4}{m\kappa^2}\left(\int_{|x|}^{+\infty}\frac{a(\tau)}{\tau}\psi^2(\tau)\mathrm{d}\tau+\frac{a^2(|x|)}{|x|^2}\right)\psi=(R_\alpha*F(\psi))F'(\psi)\quad \text{in } \mathbb{R}^2,$$

where $e>0$ is a parameter, $m,\omega,\kappa>0$ are constants, $a(\tau)=\frac{1}{2}\int_0^\tau s\psi^2(s)\mathrm{d}s$, and $F\in C^1(\mathbb{R},\mathbb{R})$. By using variational methods and perturbation arguments, the existence of positive solutions for the above equation is derived. In addition, the asymptotic behavior of solutions with regard to the parameter e is also considered.

Keywords. Asymptotic behavior; Chern-Simons-Schrödinger equation; Hartree type nonlinearity; Ground state solution; Variational methods.

1. Introduction

In this paper, we focus on the following Chern-Simons-Schrödinger equation with Hartree type general nonlinearities:

$$-\frac{1}{2m}\Delta\psi+\omega\psi+\frac{2e^4}{m\kappa^2}\left(\int_{|x|}^{+\infty}\frac{a(\tau)}{\tau}\psi^2(\tau)\mathrm{d}\tau+\frac{a^2(|x|)}{|x|^2}\right)\psi=(R_\alpha*F(\psi))F'(\psi)\quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $m,\omega,\kappa>0$ are constants, $e>0$ is a parameter, $a(\tau)=\frac{1}{2}\int_0^\tau s\psi^2(s)\mathrm{d}s$, $R_\alpha:\mathbb{R}^2\to\mathbb{R}$ is the Riesz potential of order $\alpha\in(0,2)$ defined for all $x\in\mathbb{R}^2\setminus\{0\}$ by

$$R_\alpha(x)=\frac{C_\alpha}{|x|^{2-\alpha}},\quad \text{where } C_\alpha=\frac{\Gamma(\frac{2-\alpha}{2})}{\pi 2^\alpha\Gamma(\frac{\alpha}{2})},$$

Γ is the Gamma function, and $*$ denotes the standard convolution in $\mathbb{R}^2$.

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Received July 20, 2022; Accepted April 24, 2023.

The inspiration for studying (1.1) derives from the following Chern-Simons-Schrödinger system

$$\begin{cases} iD_t\phi + \frac{1}{2m}(D_1D_1+D_2D_2)\phi + g(\phi) = 0, \\ \frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x_1} = -\frac{e}{m\kappa}\mathrm{Im}(\bar{\phi}D_2\phi), \\ \frac{\partial A_2}{\partial t} - \frac{\partial A_0}{\partial x_2} = \frac{e}{m\kappa}\mathrm{Im}(\bar{\phi}D_1\phi), \\ \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} = \frac{e}{2\kappa}|\phi|^2, \end{cases} \tag{1.2}$$

in which i denotes the imaginary unit, ϕ is a complex scalar field, $A_k = A_k(t,x_1,x_2) : \mathbb{R}^3 \to \mathbb{R}$ is the gauge field, and $D_k = \partial_k + ieA_k$ is the covariant derivative for $k = 0,1,2$. System (1.2) appeared firstly in [10] comprising the Schrödinger equation augmented by the gauge field A_k. The Chern-Simons-Schrödinger system has been investigated extensively due to its close connection in applications. For example, it has been applied in high-temperature superconductivity, quantum Hall effect and the second quantized N body anyon problem, etc. For more detailed physical background of the system, we refer the readers to [7]. For system (1.2), if we look for standing wave solutions, namely, the solutions to (1.2) in the form:

$$\begin{cases} \phi(t,x) = e^{i\omega t}\psi(|x|), \quad A_0(t,x) = A_0(|x|), \\ A_1(t,x) = \frac{e}{\kappa}\frac{x_2}{|x|^2}a(|x|), \quad A_2(t,x) = -\frac{e}{\kappa}\frac{x_1}{|x|^2}a(|x|), \end{cases} \tag{1.3}$$

where $\omega > 0$ is a given frequency, $\psi(x), A_0(x)$ and $a(x)$ are real valued functions, substituting (1.3) into system (1.2) with $g(\psi) = \lambda|\psi|^{p-2}\psi$, we deduce the following Chern-Simons-Schrödinger equation:

$$-\frac{1}{2m}\Delta\psi + \omega\psi + \frac{2e^4}{m\kappa^2}\left(\zeta + \int_{|x|}^{+\infty}\frac{a(\tau)}{\tau}\psi^2(\tau)\mathrm{d}\tau + \frac{a^2(|x|)}{|x|^2}\right)\psi = \lambda|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^2, \tag{1.4}$$

where $a(\tau) = \frac{1}{2}\int_0^\tau s\psi^2(s)\mathrm{d}s$ and ζ is an integration constant of $A_0(|x|)$. In what follows we can take $\zeta = 0$.

When $2 < p < 4$, the existence and nonexistence results of (1.4) for different value of $\omega > 0$ were proved through investigating the geometry of the energy functional in [21] by Pomponio and Ruiz. In [25], Yuan, with the variational methods, acquired the multiplicity results for the L^2-normalized solutions to (1.4) for $p > 2, p \neq 4$. When $p > 6$, in [11], the authors obtained the existence of least energy sign-changing solutions for the equation

$$-\Delta\psi + \omega\psi + \mu\left(\int_{|x|}^{+\infty}\frac{a(\tau)}{\tau}\psi^2(\tau)\mathrm{d}\tau + \frac{a^2(|x|)}{|x|^2}\right)\psi = |\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^2, \tag{1.5}$$

where $\omega, \mu > 0$. In [24], Xia obtained the existence, nonexistence, and multiplicity of solutions to (1.5) for $2 < p < 4$ by using the fibering method. Huh, in [9], showed concern for the existence of infinitely many solutions to (1.5) for $p > 6$. More results on the Chern-Simons-Schrödinger equations were present in [2, 7, 8, 18] and the references therein.

Besides, in many physical applications, the Hartree-type nonlinearities appear naturally, that is, $g(x,u) = (w(x) * F(u))f(u)$, where $F \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ and $f = F'$. In [13], Lieb proved the existence and uniqueness (up to translations) of the ground state solution to the following equation

$$-\Delta u + u = (|x|^{-1} * |u|^2)u, \quad u \in H^1(\mathbb{R}^3). \tag{1.6}$$

Equation (1.6) is usually called Choquard equation, which arises in various branches of mathematical physics, such as the quantum theory of large systems of nonrelativistic bosonic atoms and molecules, physics of multiple-particle systems, and so on; see, e.g., [14]. Lions [15] obtained the existence of a sequence of radially symmetric solutions for (1.6) by using variational methods. Recently, Ma and Zhao [17] considered the generalized Choquard equation

$$-\Delta u+u=\big(|x|^{-\alpha}*|u|^{p}\big)|u|^{p-2}u,\quad u\in H^{1}(\mathbb{R}^{N}),\tag{1.7}$$

where $\alpha\in(0,N)$. Under some assumptions on N,α and p, by using an integral version of the moving planes method, they certified that every positive solution of (1.7) is radially symmetric and monotone decreasing about some point. Moroz and Van Schaftingen [19] obtained the regularity, positivity, and radial symmetry of the ground state solutions, as well as the decay asymptotics at infinity for these ground state solutions. Further results for related problems, we refer to [1, 5, 6, 16, 20] and the references therein.

Stimulated by the above papers, in this paper, we study the Chern-Simons-Schrödinger equation with Hartree type general nonlinearities. We assume that the nonlinearity F satisfies the following hypotheses:

$(\mathbb{F}_1)$ $F\in C^1(\mathbb{R},\mathbb{R})$ and $\lim\limits_{t\to0}\frac{F(t)}{|t|^{1+\frac{\alpha}{2}}}=0$;

$(\mathbb{F}_2)$ for each $\theta>0$, there exists $C(\theta)>0$ such that $|F'(t)|\le C(\theta)\min\{1,|t|^{\frac{\alpha}{2}}\}e^{\theta|t|^2}$ for any $t>0$;

$(\mathbb{F}_3)$ there exists $t_0\in\mathbb{R}$ such that $F(t_0)\neq0$.

Our result is the following:

Theorem 1.1. *If $(\mathbb{F}_1)-(\mathbb{F}_3)$ hold, then there exists $e^*>0$ such that, for any $e\in(0,e^*)$, equation (1.1) has a positive solution $\psi^e\in H^1(\mathbb{R}^2)$. Moreover, up to a subsequence, $\psi^e\to\psi^\star$ strongly in $H^1(\mathbb{R}^2)$ as $e\to0^+$, where $\psi^\star$ is a ground state solution to the equation*

$$-\frac{1}{2m}\Delta\psi+\omega\psi=(R_\alpha*F(\psi))F'(\psi)\quad in\ \mathbb{R}^2.\tag{1.8}$$

Remark 1.1. The existence of the ground states solutions to equation (1.8) was achieved in [3]. More precisely, if $(\mathbb{F}_1)-(\mathbb{F}_3)$ hold, then equation (1.8) has a ground state solution $\psi_0\in H^1(\mathbb{R}^2)$. It is well known that $(\mathbb{F}_1)-(\mathbb{F}_3)$ are almost necessary for the existence of solutions of (1.8).

We note that Chern-Simons-Schrödinger equation (1.1) is doubly nonlocal, and it is not a pointwise identity since the appearance of the Chern-Simons term $\Big(\int_{|x|}^{+\infty}\frac{a(\tau)}{\tau}\psi^2(\tau)\mathrm{d}\tau+\frac{a^2(|x|)}{|x|^2}\Big)\psi$ and the Hartree term $(R_\alpha*F(\psi))F'(\psi)$. The two nonlocal terms give rise to some mathematical difficulties and make the problem more interesting.

2. Preliminaries

First, we give some notations:

- $H^1(\mathbb{R}^2)$ is the usual Sobolev space endowed with norm $\|w\|_{H^1}=\big(\int_{\mathbb{R}^2}(|\nabla w|^2+w^2)\mathrm{d}x\big)^{\frac12}$. For fixed $m,\omega>0$, we also use the notation $\|w\|=\big(\int_{\mathbb{R}^2}(\frac{1}{2m}|\nabla w|^2+\omega w^2)\mathrm{d}x\big)^{\frac12}$ which is a norm equivalent to $\|w\|_{H^1}$.
- For any $1\le p<\infty$, we denote by $\|w\|_{L^p(\mathbb{R}^2)}$ the standard norm of $L^p(\mathbb{R}^2)$.
- $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence.

Now we introduce the following Trudinger-Moser inequality in $\mathbb{R}^2$.

Lemma 2.1. *(see [3]) If $u \in H^1(\mathbb{R}^2)$, then $\int_{\mathbb{R}^2}(e^{\beta|u|^2}-1)\mathrm{d}x < +\infty$ for any $\beta > 0$. Moreover, if $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$, and $\beta \in (0,4\pi)$, then there exists a constant $C(\beta)$, which depends only on β such that*

$$\int_{\mathbb{R}^2} \min\{1,|u|^2\} e^{\beta|u|^2}\mathrm{d}x \le C(\beta)\int_{\mathbb{R}^2}|u|^2\mathrm{d}x.$$

The following Hardy-Littlewood-Sobolev inequality is required to deal with the Hartree non-local term.

Lemma 2.2. *(see [12], Theorem 4.3) For any $p \in [1,\frac{2}{\alpha})$ and $f \in L^p(\mathbb{R}^2)$, there exists a constant $C(\alpha,p)$ such that*

$$\|R_\alpha * f\|_{L^{\frac{2p}{2-\alpha p}}(\mathbb{R}^2)} \le C(\alpha,p)\|f\|_{L^p(\mathbb{R}^2)}.$$

Lemma 2.3. *(see [3]) For any $p \in [1,\frac{2}{\alpha})$, $q \in (\frac{2}{\alpha},+\infty)$, and $f \in L^p(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)$, there exists a constant $C(\alpha,p,q)$ such that*

$$\|R_\alpha * f\|_{L^\infty(\mathbb{R}^2)} \le C(\alpha,p,q)\left(\|f\|_{L^p(\mathbb{R}^2)}+\|f\|_{L^q(\mathbb{R}^2)}\right).$$

The corresponding energy functional of problem (1.1) is defined by

$$\begin{aligned}\mathscr{E}_e(\psi) = &\frac{1}{4m}\|\nabla\psi\|^2_{L^2(\mathbb{R}^2)} + \frac{\omega}{2}\|\psi\|^2_{L^2(\mathbb{R}^2)} + \frac{e^4}{4m\kappa^2}\int_{\mathbb{R}^2}\frac{\psi^2(x)}{|x|^2}\left(\int_0^{|x|}\tau\psi^2(\tau)\mathrm{d}\tau\right)^2\mathrm{d}x\\ &-\frac{1}{2}\int_{\mathbb{R}^2}(R_\alpha * F(\psi))F(\psi)\mathrm{d}x. \end{aligned} \tag{2.1}$$

Under our assumptions, using the Lemmas 2.1 and 2.2, it is easy to check that energy functional $\mathscr{E}_e$ is well-defined and a C^1 functional, and its critical point ψ is a weak solution to (1.1).

Since we are interested in the positive solutions of (1.1), from now on, we assume that $F(t) = 0$ for all $t \le 0$. If $e = 0$, equation (1.1) becomes $-\frac{1}{2m}\Delta\psi + \omega\psi = (R_\alpha * F(\psi))F'(\psi)$ in $\mathbb{R}^2$, which will be referred as the limit problem of (1.1). We use the notations

$$\mathscr{E}_0(\psi) = \frac{1}{4m}\|\nabla\psi\|^2_{L^2(\mathbb{R}^2)} + \frac{\omega}{2}\|\psi\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{2}\int_{\mathbb{R}^2}(R_\alpha * F(\psi))F(\psi)\mathrm{d}x.$$

Let us list some properties of $\mathscr{E}_0$ (see, e.g., [3]).

Lemma 2.4. *Let F satisfy $(\mathbb{F}_1)-(\mathbb{F}_3)$. Then the following properties hold:*

(i) *there exist $\eta,\theta > 0$ such that $\mathscr{E}_0(\psi) \ge \theta$ for $\|\psi\| = \eta$, and there exists $e_0 \in H^1(\mathbb{R}^2)$ such that $\|e_0\| > \eta$ and $\mathscr{E}_0(e_0) < 0$;*

(ii) *there exists a critical point $\psi_0 \in H^1(\mathbb{R}^2)$ of $\mathscr{E}_0$ such that*

$$\mathscr{E}_0(\psi_0) = c_0 := \inf_{\gamma\in\Lambda}\max_{0\le t\le 1}\mathscr{E}_0(\gamma(t)),$$

where $\Lambda = \{\gamma \in C([0,1],H^1(\mathbb{R}^2)) : \gamma(0) = 0, \gamma(1) = e_0\}$;

(iii) *$c_0 = \inf\{\mathscr{E}_0(\psi) : \mathscr{E}_0'(\psi) = 0, 0 \ne \psi \in H^1(\mathbb{R}^2)\}$;*

(iv) *the set $\mathscr{A} := \{\psi \in H^1(\mathbb{R}^2) : \mathscr{E}_0'(\psi) = 0, \mathscr{E}_0(\psi) = c_0\}$ is compact in $H^1(\mathbb{R}^2)$.*

(v) *there exists a path $\gamma_0(t) \in \Lambda$ passing through ψ_0 at $t = \frac{1}{2}$ and satisfying*

$$\mathscr{E}_0(\psi_0) > \mathscr{E}_0(\gamma_0(t)),\ \forall\, t \in [0,1]\backslash\{\frac{1}{2}\}.$$

From Lemma 2.4, when $e>0$ small enough, $\mathscr{E}_e(e_0)<0$. Thus $\mathscr{E}_e$ has the mountain pass geometry and we can define

$$c_e := \inf_{\gamma\in\Lambda}\max_{0\le t\le 1}\mathscr{E}_e(\gamma(t)),$$

where

$$\Lambda = \{\gamma\in C([0,1],H^1(\mathbb{R}^2)) : \gamma(0)=0, \gamma(1)=e_0\}.$$

Furthermore, there exists a $(PS)_{c_e}$ sequence $\{\psi_n\}$ for $\mathscr{E}_e$, that is, $\mathscr{E}_e(\psi_n)\to c_e$ and $\mathscr{E}_e'(\psi_n)\to 0$ as $n\to\infty$. However, under conditions $(\mathbb{F}_1)-(\mathbb{F}_3)$, it is not easy to prove the $(PS)_{c_e}$ sequence $\{\psi_n\}$ is bounded. To deal with this obstacle, we define a modified mountain pass energy level of $\mathscr{E}_e$ by

$$m_e := \inf_{\gamma\in\Lambda_L}\max_{0\le t\le 1}\mathscr{E}_e(\gamma(t)), \tag{2.2}$$

where

$$\Lambda_L = \big\{\gamma\in\Lambda : \sup_{0\le t\le 1}\|\gamma(t)\|\le L\big\},\ \ L=2\max\big\{\sup_{\psi\in\mathscr{A}}\|\psi\|, \sup_{0\le t\le 1}\|\gamma_0(t)\|\big\}. \tag{2.3}$$

By the choice of L, one easily checks that $\gamma_0(t)\in\Lambda_L$. From Lemma 2.4 (ii) and (v), we infer that

$$c_0 = m_0 = \inf_{\gamma\in\Lambda_L}\max_{0\le t\le 1}\mathscr{E}_0(\gamma(t)).$$

However, since $\Lambda_L\subsetneq\Lambda$, the standard mountain pass theorem cannot be directly applicable, so other arguments are indispensable for showing that m_e is a critical value.

Lemma 2.5. *If $(\mathbb{F}_1)-(\mathbb{F}_2)$ hold, $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^2)$, and $\psi_n\rightharpoonup\psi$ weakly in $H^1(\mathbb{R}^2)$ as $n\to\infty$, then there hold:*

(i) $(R_\alpha * F(\psi_n))F'(\psi_n)\rightharpoonup(R_\alpha * F(\psi))F'(\psi)$ *in $L^q_{loc}(\mathbb{R}^2)$ for all $q\ge 1$ as $n\to\infty$;*
(ii) $\int_{\mathbb{R}^2}(R_\alpha * F(\psi_n))F(\psi_n)\mathrm{d}x\to\int_{\mathbb{R}^2}(R_\alpha * F(\psi))F(\psi)\mathrm{d}x$ *as $n\to\infty$.*

Proof. Since $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^2)$, by $(\mathbb{F}_1)-(\mathbb{F}_2)$, we have that $\{F(\psi_n)\}$ is bounded in $L^p(\mathbb{R}^2)$ for every $p\ge\frac{4}{2+\alpha}$. Moreover, up to subsequences, we can assume that $\psi_n\to\psi$ almost everywhere as $n\to\infty$. By the continuity of the function F, we conclude $F(\psi_n)\to F(\psi)$ almost everywhere as $n\to\infty$. This implies that $F(\psi_n)\rightharpoonup F(\psi)$ weakly in $L^p(\mathbb{R}^2)$ for every $p\ge\frac{4}{2+\alpha}$ as $n\to\infty$. Notice that $\frac{2}{\alpha}>\frac{4}{2+\alpha}$. By Lemmas 2.2 and 2.3, we have

$$R_\alpha * F(\psi_n)\rightharpoonup(R_\alpha * F(\psi)) \text{ weakly in } L^{\frac{4}{2-\alpha}}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2) \text{ as } n\to\infty.$$

By condition $(\mathbb{F}_2)$ and Lemma 2.1, the sequence $\{F'(\psi_n)\}$ is bounded in $L^p(\mathbb{R}^2)$ for every $p\ge\frac{2}{\alpha}$, and by continuity $F'(\psi_n)\to F'(\psi)$ almost everywhere as $n\to\infty$. Therefore, $F'(\psi_n)\to F'(\psi)$ strongly in $L^q_{loc}(\mathbb{R}^2)$ for every $q\ge 1$ as $n\to\infty$. Hence $(R_\alpha * F(\psi_n))F'(\psi_n)\rightharpoonup(R_\alpha * F(\psi))F'(\psi)$ in $L^q_{loc}(\mathbb{R}^2)$ for all $q\ge 1$ as $n\to\infty$. □

Let $H^1_{rad}(\mathbb{R}^2)=\{w\in H^1(\mathbb{R}^2) : w \text{ is radially symmetric}\}$, we note that $H^1_{rad}(\mathbb{R}^2)$ is a natural constraint for $\mathscr{E}_e$, that is, a critical point of $\mathscr{E}_e|_{H^1_{rad}(\mathbb{R}^2)}$ is also a critical point of $\mathscr{E}_e$ (see [9]). Hence, we can consider the functional $\mathscr{E}_e$ restricted to $H^1_{rad}(\mathbb{R}^2)$. Let us recall the compactness lemma for radial functions from [22].

Lemma 2.6. *$H^1_{rad}(\mathbb{R}^2)$ is compactly embedded in $L^p(\mathbb{R}^2)$ for every $p\in(2,+\infty)$.*

For any $\psi \in H^1_{rad}(\mathbb{R}^2)$, let

$$\mathscr{T}(\psi) = \int_{\mathbb{R}^2} \frac{\psi^2(x)}{|x|^2} \left(\int_0^{|x|} \tau \psi^2(\tau) \mathrm{d}\tau \right)^2 \mathrm{d}x.$$

Note that

$$\int_0^s r\psi^2(r)\mathrm{d}r = \int_{B_s} \frac{1}{2\pi} \psi^2(y)\mathrm{d}y \le Cs^2.$$

Then

$$\mathscr{T}(\psi) \le C \int_{\mathbb{R}^2} \psi^2(x) \left(\int_{B_{|x|}} \psi^4(y)\mathrm{d}y \right) \mathrm{d}x \le C\|\psi\|^6. \tag{2.4}$$

Moreover, we can check that $\mathscr{T}(\psi) \in C^1(H^1_{rad}(\mathbb{R}^2), \mathbb{R})$, and for any $\chi \in H^1_{rad}(\mathbb{R}^2)$,

$$\begin{aligned} \langle \mathscr{T}'(\psi), \chi \rangle = {} & 2\int_{\mathbb{R}^2} \frac{\psi(x)\chi(x)}{|x|^2} \Big(\int_0^{|x|} \tau \psi^2(\tau)\mathrm{d}\tau \Big)^2 \mathrm{d}x \\ & + 4\int_{\mathbb{R}^2} \frac{\psi^2(x)}{|x|^2} \Big(\int_0^{|x|} \tau \psi^2(\tau)\mathrm{d}\tau \Big) \Big(\int_0^{|x|} \tau u(\tau)\chi(\tau)\mathrm{d}\tau \Big) \mathrm{d}x. \end{aligned}$$

Furthermore, for the Chern-Simons nonlocal term $\mathscr{T}(\psi)$, we have the following compactness properties, which can be found in [4] (see Lemma 3.2).

Lemma 2.7. *Assume that $\psi_n \rightharpoonup \psi$ weakly in $H^1_{rad}(\mathbb{R}^2)$. Then*

(i) $\lim_{n\to\infty} \mathscr{T}(\psi_n) = \mathscr{T}(\psi)$;
(ii) $\lim_{n\to\infty} \langle \mathscr{T}'(\psi_n), \psi_n \rangle = \langle \mathscr{T}'(\psi), \psi \rangle$;
(iii) $\lim_{n\to\infty} \langle \mathscr{T}'(\psi_n), \chi \rangle = \langle \mathscr{T}'(\psi), \chi \rangle$, $\forall\, \chi \in H^1_{rad}(\mathbb{R}^2)$.

Lemma 2.8. *Suppose that $(\mathbb{F}_1)-(\mathbb{F}_3)$ hold. Then there holds $\lim\limits_{e\to 0^+} m_e = c_0$, where m_e is defined in (2.2).*

Proof. It is easy to obtain that $m_e \ge c_0$ for all $e \ge 0$. On the other hand, by Lemma 2.4, there exists $\gamma_0(t) \in \Lambda$ satisfying

$$\max_{0\le t\le 1} \mathscr{E}_0(\gamma_0(t)) = \mathscr{E}_0(\psi_0) = c_0.$$

Then we obtain that

$$\begin{aligned} m_e \le \max_{0\le t\le 1} \mathscr{E}_e(\gamma_0(t)) = {} & \max_{0\le t\le 1} \left(\mathscr{E}_0(\gamma_0(t)) + \frac{e^4}{m\kappa^2} \int_{\mathbb{R}^2} \frac{|\gamma_0(t)(x)|^2}{|x|^2} \Big(\int_0^{|x|} \frac{\tau}{2} |\gamma_0(t)(\tau)|^2 \mathrm{d}\tau \Big)^2 \mathrm{d}x \right) \\ \le {} & c_0 + \frac{e^4}{m\kappa^2} \max_{0\le t\le 1} \int_{\mathbb{R}^2} \frac{|\gamma_0(t)(x)|^2}{|x|^2} \Big(\int_0^{|x|} \frac{\tau}{2} |\gamma_0(t)(\tau)|^2 \mathrm{d}\tau \Big)^2 \mathrm{d}x. \end{aligned} \tag{2.5}$$

Note that $\gamma_0(t) \in \Lambda_L$. Using (2.4), we have

$$\max_{0\le t\le 1} \int_{\mathbb{R}^2} \frac{|\gamma_0(t)(x)|^2}{|x|^2} \Big(\int_0^{|x|} \frac{\tau}{2} |\gamma_0(t)(\tau)|^2 \mathrm{d}\tau \Big)^2 \mathrm{d}x \le C \max_{0\le t\le 1} \|\gamma_0(t)\|^6 \le CL^6,$$

which yields

$$\frac{e^4}{m\kappa^2} \max_{0\le t\le 1} \int_{\mathbb{R}^2} \frac{|\gamma_0(t)(x)|^2}{|x|^2} \Big(\int_0^{|x|} \frac{\tau}{2} |\gamma_0(t)(\tau)|^2 \mathrm{d}\tau \Big)^2 \mathrm{d}x \to 0 \text{ as } e \to 0. \tag{2.6}$$

Therefore, by (2.5) and (2.6), we deduce that $m_e \le c_0 + o(1)$ as $e \to 0$, which implies that

$$\limsup_{e\to 0} m_e \le c_0. \tag{2.7}$$

This completes the proof. ☐

For $\rho > 0$, we use the notation $\mathscr{B}_\rho(\psi) := \{v \in H^1_{rad}(\mathbb{R}^2) : \|\psi - v\| \le \rho\}$. For any subset A of $H^1_{rad}(\mathbb{R}^2)$, we set

$$A^\rho := \bigcup_{\psi\in A} \mathscr{B}_\rho(\psi).$$

Lemma 2.9. *Suppose that* $(\mathbb{F}_1) - (\mathbb{F}_3)$ *hold. Let* $\rho > 0$ *be a fixed number and suppose that there exist sequences* $\{e_n\} \to 0$ *and* $\{\psi_n\} \subset \mathscr{A}^\rho$ *satisfying*

$$\lim_{n\to\infty} \mathscr{E}_{e_n}(\psi_n) \le c_0, \textit{ and } \lim_{n\to\infty} \mathscr{E}'_{e_n}(\psi_n) = 0.$$

Then there exists $\rho_0 > 0$ *such that for* $\rho \in (0, \rho_0)$*, up to a subsequence,* $\psi_n \to \psi \in \mathscr{A}$ *as* $n \to \infty$.

Proof. By the definition of $\mathscr{A}^\rho$ and Lemma 2.4(iv), we know that there is $z_n \in \mathscr{A}$ such that

$$\operatorname{dist}(\psi_n, \mathscr{A}) = \operatorname{dist}(\psi_n, z_n) \le \rho.$$

Then, passing to a subsequence, there exists $z \in \mathscr{A}$ such that $z_n \to z$. Hence, $\operatorname{dist}(\psi_n, z) \le 2\rho$ for n large enough. Therefore, $\{\psi_n\}$ is bounded, and then we may assume that $\psi_n \rightharpoonup \psi$ in $H^1_{rad}(\mathbb{R}^2)$. Note that $\mathscr{B}_{2\rho}(z)$ is weakly closed in $H^1_{rad}(\mathbb{R}^2)$. We infer that $\psi \in \mathscr{B}_{2\rho}(z) \subset \mathscr{A}^{2\rho}$, which implies $\psi \neq 0$ for $\rho > 0$ small enough.

Since $\mathscr{E}'_{e_n}(\psi_n) \to 0$ as $n \to \infty$ and $\{\psi_n\}$ is bounded, then by Lemma 2.5 and Lemma 2.7, for all $\chi \in C_0^\infty(\mathbb{R}^2)$, we obtain

$$\begin{aligned}\langle \mathscr{E}'_0(\psi), \chi\rangle &= \int_{\mathbb{R}^2} (\frac{1}{2m}\nabla\psi\nabla\chi + \omega\psi\chi)\mathrm{d}x - \int_{\mathbb{R}^2} (R_\alpha * F(\psi))F'(\psi)\chi\mathrm{d}x \\ &= \lim_{n\to\infty}\left(\int_{\mathbb{R}^2} (\frac{1}{2m}\nabla\psi_n\nabla\chi + \omega\psi_n\chi)\mathrm{d}x - \int_{\mathbb{R}^2} (R_\alpha * F(\psi_n))F'(\psi_n)\chi\mathrm{d}x\right) \\ &= \lim_{n\to\infty}\left(\langle \mathscr{E}'_{e_n}(\psi_n), \chi\rangle - e_n\langle \mathscr{T}'(\psi_n), \chi\rangle\right) = 0.\end{aligned}$$

Hence $\mathscr{E}'_0(\psi) = 0$. It indicates that ψ is a nontrivial critical point of $\mathscr{E}_0$. Moreover, since $\psi_n \in \mathscr{A}^\rho$, we can obtain

$$\begin{aligned}c_0 \le \mathscr{E}_0(\psi) \le \lim_{n\to\infty} \mathscr{E}_0(\psi_n) &= \lim_{n\to\infty} \mathscr{E}_0(\psi_n) + \lim_{n\to\infty} e_n\mathscr{T}(\psi_n) \\ &= \lim_{n\to\infty} \mathscr{E}_{e_n}(\psi_n) \le c_0.\end{aligned}$$

Thus $\mathscr{E}_0(\psi) = c_0$ and $\psi \in \mathscr{A}$. In addition, by the condition $\lim_{n\to\infty} \mathscr{E}_{e_n}(\psi_n) \le c_0$ and Lemma 2.5, we reach that

$$\begin{aligned}
\liminf_{n\to\infty} \|\psi_n\|^2 \ge \|\psi\|^2 =& 2\Big(\mathscr{E}_0(\psi) + \int_{\mathbb{R}^2} (R_\alpha * F(\psi))F(\psi)\mathrm{d}x\Big) \\
=& 2\Big(c_0 + \int_{\mathbb{R}^2} (R_\alpha * F(\psi))F(\psi)\mathrm{d}x\Big) \\
\ge& 2\Big(\lim_{n\to\infty} \mathscr{E}_{e_n}(\psi_n) + \int_{\mathbb{R}^2} (R_\alpha * F(\psi))F(\psi)\mathrm{d}x\Big) \\
=& \lim_{n\to\infty} \|\psi_n\|^2 + \lim_{n\to\infty} \frac{e_n}{4} \mathscr{T}(\psi_n) \\
& - 2\Big(\lim_{n\to\infty} \int_{\mathbb{R}^2} (R_\alpha * F(\psi_n))F(\psi_n)\mathrm{d}x - \int_{\mathbb{R}^2} (R_\alpha * F(\psi))F(\psi)\mathrm{d}x\Big) \\
=& \lim_{n\to\infty} \|\psi_n\|^2.
\end{aligned}$$

Therefore, $\psi_n \to \psi$ strongly in $H^1_{rad}(\mathbb{R}^2)$ as $n \to \infty$. This completes the proof. □

3. Proof of the Main Result

In what follows, we always support that $(\mathbb{F}_1) - (\mathbb{F}_3)$ hold. Set

$$\ell_e = \max_{0\le t\le 1} \mathscr{E}_e(\gamma_0(t)),$$

where $\gamma_0(t)$ is obtained in Lemma 2.4 (v). Note that $\gamma_0(t) \in \Lambda_L$. From the definition (2.2) of m_e, we have

$$m_e \le \max_{0\le t\le 1} \mathscr{E}_e(\gamma_0(t)) = \ell_e.$$

Moreover, arguing as the proof of Lemma 2.8, one has $\lim_{e\to 0} \ell_e \le c_0$, which together with the conclusion of Lemma 2.8 yields that

$$\lim_{e\to 0} m_e = \lim_{e\to 0} \ell_e = m_0 = c_0. \tag{3.1}$$

Also, we define

$$\mathscr{E}_e^{\ell_e} = \{\psi \in H^1(\mathbb{R}^2) : \mathscr{E}_e(\psi) \le \ell_e\}.$$

Then we have the following lemma.

Lemma 3.1. *Let ρ_1, ρ_2 be two numbers satisfying $0 < \rho_2 < \rho_1 < \rho_0$. Then there are constants $\sigma > 0$ and $e_0 > 0$ depending on ρ_1 and ρ_2 such that, for $e \in (0, e_0)$, there holds:*

$$\|\mathscr{E}_e'(\psi)\| \ge \sigma \ \textit{ for all } \ \psi \in \mathscr{E}_e^{\ell_e} \cap (\mathscr{A}^{\rho_1} \setminus \mathscr{A}^{\rho_2}).$$

Proof. Assume by contradiction that, for some $\rho_1, \rho_2 > 0$ satisfying $\rho_0 > \rho_1 > \rho_2$, there exist a sequence $\{e_n\}$ with $\lim_{n\to\infty} e_n = 0$ and a sequence of functions $\{\psi_n\} \subset \mathscr{A}^{\rho_1} \setminus \mathscr{A}^{\rho_2}$ such that

$$\lim_{n\to\infty} \mathscr{E}_{e_n}(\psi_n) \le c_0 \quad \text{and} \quad \lim_{n\to\infty} \mathscr{E}_{e_n}'(\psi_n) = 0.$$

Then by Lemma 2.9, one has that there exists $\psi \in \mathscr{A}$ such that $\psi_n \to \psi$ strongly in $H^1_{rad}(\mathbb{R}^2)$. Thus we obtain that $\mathrm{dist}(\psi_n, \mathscr{A}) \to 0$ as $n \to \infty$, contradicting the relation $\psi_n \notin \mathscr{A}^{\rho_2}$ for all $n \in \mathbb{N}^+$. □

Lemma 3.2. *Let $\rho > 0$ be a fixed real number. Then there exists $\delta > 0$ such that if $e > 0$ is small enough,*

$$t \in [0,1] \text{ and } \mathscr{E}_e(\gamma_0(t)) \geq m_e - \delta \text{ implies that } \gamma_0(t) \in \mathscr{A}^{\rho}.$$

Proof. We argue by contradiction and assume that, for some $\rho > 0$, there are sequences $\{\delta_n\} \to 0$, $\{e_n\} \to 0$ and $\{t_n\} \in [0,1]$ such that

$$\mathscr{E}_{e_n}(\gamma_0(t_n)) \geq m_{e_n} - \delta_n \quad \text{but} \quad \gamma_0(t_n) \notin \mathscr{A}^{\rho}.$$

Assuming $t_n \to t_0 \in [0,1]$, up to a subsequence, and by taking a limit, we have

$$c_0 \geq \mathscr{E}_0(\gamma_0(t_0)) \geq c_0 \quad \text{and} \quad \gamma_0(t_0) \notin \mathscr{A}^{\frac{\rho}{2}}.$$

On the other hand, by Lemma 2.4(iv), we obtain $\gamma_0(t_0) \in \mathscr{A}$. Thus we reach a contradiction. □

Lemma 3.3. *For any $\rho < \rho_0$ and sufficiently small $e > 0$ depending on ρ, there exists a sequence $\{\psi_n\} \subset \mathscr{E}_e^{\ell_e} \cap \mathscr{A}^{\rho}$ such that $\mathscr{E}'_e(\psi_n) \to 0$ as $n \to \infty$.*

Proof. To the contrary, we assume that there are $\rho < \rho_0$ and sequences $\{e_n\} \to 0$ and $\{d_n\} \subset (0,+\infty)$ such that

$$\|\mathscr{E}'_{e_n}(\psi)\| \geq d_n > 0 \text{ for all } \psi \in \mathscr{E}_{e_n}^{\ell_{e_n}} \cap \mathscr{A}^{\rho}.$$

Moreover, by Lemma 3.1, there exists constant $\sigma > 0$ such that

$$\|\mathscr{E}'_{e_n}(\psi)\| \geq \sigma > 0 \text{ for all } \psi \in \mathscr{E}_{e_n}^{\ell_{e_n}} \cap (\mathscr{A}^{\rho} \setminus \mathscr{A}^{\frac{\rho}{2}}). \tag{3.2}$$

Furthermore, there exists constant $M > 0$ such that

$$\|\mathscr{E}'_{e_n}(\psi)\| \leq M, \ \forall \ \psi \in \mathscr{A}^{\rho}.$$

Since ρ is a fixed number, by Lemma 3.2, we can take $\delta \in (0, \frac{2\sigma^2\rho}{M})$ small enough such that

$$t \in [0,1] \text{ and } \mathscr{E}_e(\varphi_0(t)) \geq m_e - \frac{\delta}{4} \ \Rightarrow \ \varphi_0(t) \in \mathscr{A}^{\frac{\rho}{2}}. \tag{3.3}$$

By (3.1) and $e_n \to 0$, we can take n large enough such that

$$\ell_{e_n} - m_{e_n} < \min\left\{\frac{\delta}{4}, \frac{\sigma^2\rho}{2M} - \frac{\delta}{4}\right\}. \tag{3.4}$$

Hereafter, we fix n so large that (3.3) and (3.4) hold. For the sake of simplicity, we denote e_n as e.

Now we are concerned with a pseudo-gradient vector field V_e of $\mathscr{E}_e$, and take a neighborhood $\mathscr{N}_e$ of $\mathscr{E}_e^{\ell_e} \cap \mathscr{A}^{\rho}$ satisfying $\mathscr{N}_e \subset \mathscr{B}_L(0)$ (see [23]), where L is defined in (2.3). Note that for any $\psi \in \mathscr{N}_e$ there hold

$$\|V_e(\psi)\| \leq 2\min\{1, \|\mathscr{E}'_e(\psi)\|\} \text{ and } \langle \mathscr{E}'_e(\psi), V_e(\psi)\rangle \geq \min\{1, \|\mathscr{E}'_e(\psi)\|\}\|\mathscr{E}'_e(\psi)\|.$$

We observe that $m_e < L$ for e small enough. Let $\eta_e \in C^{1,1}(H^1_{rad}(\mathbb{R}^2),[0,1])$ be defined by $\eta_e = 1$ on $\mathscr{E}_e^{\ell_e} \cap \mathscr{A}^{\rho}$ and $\eta_e = 0$ on $H^1_{rad}(\mathbb{R}^2) \setminus \mathscr{N}_e$. We also denote by ξ_e a $C^{1,1}(\mathbb{R},[0,1])$ function such that $\xi_e(\tau) = 1$ if $|\tau - m_e| \leq \frac{\delta}{2}$, $\xi_e(\tau) = 0$ if $|\tau - m_e| \geq \delta$. Considering the following Cauchy initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\Phi_e(\psi,t) = -\eta_e(\Phi_e(\psi,t))\xi_e(\mathscr{E}_e(\Phi_e(\psi,t)))V_e(\Phi_e(\psi,t))), \\ \Phi_e(\psi,0) = \psi, \end{cases} \tag{3.5}$$

we can assert that there exists a global solution $\Phi_e : H^1_{rad}(\mathbb{R}^2)\times\mathbb{R}\to H^1_{rad}(\mathbb{R}^2)$ of the above initial value problem (3.5).

Next, we claim that, for all $0\le t\le 1$, there is $t_e>0$ such that

$$\mathscr{E}_e(\Phi_e(\gamma_0(t),t_e))\le m_e-\frac{\delta}{4}.$$

Indeed, let $t_e:=\frac{\delta}{2\sigma^2}$ if, for any $0\le t\le 1$, there exists some $\tau_0\le t_e$ such that $\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau_0))\le m_e-\frac{\delta}{4}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_e(\Phi_e(\psi,t))\le-\eta_e(\Phi_e(\psi,t))\xi_e(\mathscr{E}_e(\Phi_e(\psi,t)))\|\mathscr{E}_e'(\Phi_e(\psi,t))\|^2, \tag{3.6}$$

which implies that $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau))\le 0$, then we obtain

$$\mathscr{E}_e(\Phi_e(\gamma_0(t),t_e))\le\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau_0))\le m_e-\frac{\delta}{4}.$$

This concludes the claim.

On the other hand, if

$$\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau)>m_e-\frac{\delta}{4},\ \ \forall\,\tau\in[0,t_e], \tag{3.7}$$

then we have by Lemma 3.2 that

$$\Phi_e(\gamma_0(t),0)=\gamma_0(t)\in\mathscr{A}^{\frac{\rho}{2}}\ \text{ and }\ \xi_e(\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau)))=1,\ \forall\,\tau\in[0,t_e].$$

(i) If $\Phi_e(\gamma_0(t),\tau))\in\mathscr{A}^{\rho}$ for all $\tau\in[0,t_e]$, then $\eta_e(\gamma_0(t),\tau))=1$ for all $\tau\in[0,t_e]$. Furthermore, from (3.2) and (3.6) one has $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau))\le-\sigma^2$. Thus we obtain

$$\begin{aligned}\mathscr{E}_e(\Phi_e(\gamma_0(t),t_e))=&\mathscr{E}_e(\gamma_0(t))+\int_0^{t_e}\frac{\partial}{\partial\tau}\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau))\mathrm{d}\tau\\ \le&\ell_e-\int_0^{t_e}\sigma^2\mathrm{d}\tau\\ =&\ell_e-\frac{\delta}{2}<m_e-\frac{\delta}{4}\ \ (\text{by (3.4)}),\end{aligned}$$

which is a contradiction to (3.7).

(ii) If there exists some $\tau\in[0,t_e]$ such that $\Phi_e(\gamma_0(t),\tau)\notin\mathscr{A}^{\rho}$, then there exist some $0\le\tau_1<\tau_2\le t_e$ such that $\Phi_e(\gamma_0(t),\tau_1)\in\partial\mathscr{A}^{\frac{\rho}{2}}$, $\Phi_e(\gamma_0(t),\tau_2)\in\partial\mathscr{A}^{\rho}$ and $\Phi_e(\gamma_0(t),\tau)\in\mathscr{A}^{\rho}\setminus\mathscr{A}^{\frac{\rho}{2}}$ for all $\tau\in(\tau_1,\tau_2)$. We notice that

$$\begin{aligned}\frac{\rho}{2}\le\|\Phi_e(\gamma_0(t),\tau_2)-\Phi_e(\gamma_0(t),\tau_1)\|=&\left\|\int_{\tau_1}^{\tau_2}\frac{\partial}{\partial\tau}\Phi_e(\gamma_0(t),\tau)\mathrm{d}\tau\right\|\\ \le&\int_{\tau_1}^{\tau_2}\left|\frac{\partial}{\partial\tau}\Phi_e(\gamma_0(t),\tau)\right|\mathrm{d}\tau\\ \le&\int_{\tau_1}^{\tau_2}M\mathrm{d}\tau=M(\tau_2-\tau_1).\end{aligned}$$

It follows that $\tau_2 - \tau_1 \geq \frac{\rho}{2M}$. Hence,

$$\begin{aligned}\mathscr{E}_e(\Phi_e(\gamma_0(t),t_e)) \leq & \mathscr{E}_e(\Phi_e(\gamma_0(t),\tau_2)) \\ = & \mathscr{E}_e(\gamma_0(t)) + \int_0^{\tau_2} \frac{\partial}{\partial \tau}\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau))\mathrm{d}\tau \\ \leq & \mathscr{E}_e(\gamma_0(t)) + \int_{\tau_1}^{\tau_2} \frac{\partial}{\partial \tau}\mathscr{E}_e(\Phi_e(\gamma_0(t),\tau))\mathrm{d}\tau \\ \leq & \ell_e - \sigma^2(\tau_2-\tau_1) \\ \leq & \ell_e - \frac{\sigma^2\rho}{2M} < m_e - \frac{\delta}{4} \quad (\text{by } (3.4)),\end{aligned}$$

which is also a contradiction to (3.7).

Finally, we set $\widetilde{\gamma}_0(t) = \Phi_e(\gamma_0(t),t_e)$. Then $\widetilde{\gamma}_0(t) \in \Lambda_L$ and $\mathscr{E}_e(\widetilde{\gamma}_0(t)) < m_e$ for all $0 \leq t \leq 1$, contradicting the definition of m_e. Thus we complete the proof. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.3, we know that there exists $e^* > 0$ such that, for $e \in (0,e^*)$, there exists $\{\psi_n^e\} \subset \mathscr{E}_e^{\ell_e} \cap \mathscr{A}^\rho$ such that $\mathscr{E}_e'(\psi_n^e) \to 0$ as $n \to \infty$. Since $\mathscr{A}$ is compact, it is easy to see that $\{\psi_n^e\}$ is bounded in $H^1_{rad}(\mathbb{R}^2)$. Passing to a subsequence, we may assume that

$$\psi_n^e \rightharpoonup \psi^e \in H^1_{rad}(\mathbb{R}^2).$$

Moreover, we can obtain that $\mathscr{E}_e'(\psi^e) = 0$. Hence ψ^e is a critical point of $\mathscr{E}_e$. Now we claim that $\psi^e \in \mathscr{A}^\rho$. Indeed, by the fact $\psi_n^e \in \mathscr{A}^{\frac{\rho}{2}}$, there is $v_n \in \mathscr{A}$ satisfying $\|v_n - \psi_n^e\| \leq \frac{\rho}{2}$. Then from the compactness of $\mathscr{A}$, there exists $v \in \mathscr{A}$ such that $v_n \to v$ in $\mathscr{A}$ as $n \to \infty$, which means that, for all n, $\psi_n \in \mathscr{B}_\rho(v)$, a weakly closed set in $H^1_{rad}(\mathbb{R}^2)$. Thus it follows that $\psi^e \in \mathscr{B}_\rho(v)$, by the choice of ρ, $\psi^e \neq 0$. Hence, ψ^e is the desired solution to (1.1).

Next, we consider the asymptotic behavior of ψ^e as $e \to 0$. For any sequence $\{e_n\} \subset (0,e^*)$ with $e_n \searrow 0$ as $n \to \infty$, let $\{\psi^{e_n}\} \subset H^1_{rad}(\mathbb{R}^2)$ be a sequence positive solutions obtained above. Note that, for any sequence $\{e_n\}$ converging to 0, the sequence $\{\psi^{e_n}\}$ satisfies all the assumptions of Lemma 2.9. Thus $\{\psi^{e_n}\}$ converges to some $\psi^\star \in \mathscr{A}$. This completes the proof. □

Acknowledgements

The authors are grateful to the reviewers for useful suggestions which improved the contents of this paper. This work was partially supported by NSFC (12126423).

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