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A NEW INERTIAL RELAXED TSENG EXTRGRADIENT METHOD FOR SOLVING QUASI-MONOTONE BILEVEL VARIATIONAL INEQUALITY PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce an inertial relaxed Tseng extragradient method involving only a single projection for solving bilevel variational inequality problems with Lipschitz continuous and quasimonotone mapping in Hilbert spaces. Under some mild standard assumptions, we obtain a strong convergence result for solving bilevel quasimonotone variational inequality problems. The main advantages of the proposed iterative method are that it requires only one projection onto the feasible set and the use self adaptive step-size rule based on operator knowledge rather than a Lipschitz constant or some line search method. Moreover, some interesting preliminary numerical experiments and comparisons were presented.

Keywords. Bilevel variational inequality problem; Inertial method; Quasimonotone operator; Tseng extragradient method.

1. Introduction

Let C be a closed and convex subset of a real Hilbert space H with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let $A,F : H \to H$ be two single-valued mappings. Our interest in this paper is to study the bilevel variational inequality problem (BVIP):

$$\text{Find } x^* \in VI(C,A) \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \ \forall x \in VI(C,A), \tag{1.1}$$

where $VI(C,A)$ denotes the set of all solutions of the classical variational inequality problem (VIP) given as follows:

$$\text{Find } y^* \in C \text{ such that } \langle A(y^*), z - y^* \rangle \geq 0, \ \forall z \in C. \tag{1.2}$$

Let S_D be the solution set of the following problem,

$$\text{Find } x^* \in C \text{ such that } \langle A(y), y - x^* \rangle \geq 0, \ \forall y \in C.$$

It is known that S_D is a closed and convex set (possibly empty) and if A is continuous and C is convex, then $S_D \subseteq VI(C,A)$. If A is a pseudomonotone and continuous mapping, then $VI(C,A) = S_D$ [1]. The inclusion $VI(C,A) \subset S_D$ is false if A is a quasi monotone and continuous mapping [2].

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Recently, many authors introduced and studied various iterative algorithms for the approximation of the solutions of variational inequality problem (1.2); see, e.g., [3, 4, 5, 6, 7] and the references therein. One of the oldest iterative method, which is also simplest, for solving (1.2) is the gradient projection method as follows: $x_{n+1} = P_C(x_n - \lambda A(x_n))$ for all $n \geq 0$, where $\lambda > 0$ is a suitable parameter and P_C denotes the metric projection from H onto C. It is known that the sequence generated by the gradient projection method converges to an element of $VI(C,A)$ if A is L-Lipschitz continuous and α-strongly monotone and $\lambda \in (0, \frac{2\alpha}{L^2})$. The assumption that A is strongly monotone (or inverse strongly monotone) is a necessary condition to ensure the convergence of the gradient projection method. If the condition of strong monotonicity is relaxed to monotonicity, the gradient method can diverge. For example, take A to be the rotation operator in a plane [8].

A famous method, which was recently used in the approximation of the solutions of the variational inequality problem (1.2) such that the assumption that the strongly monotone (or inverse strongly monotone) cost function A could be relaxed to being just monotone and the convergence is still guaranteed, is the extragradient method, introduced by Korpelevich [9] (also independently by Antipin [10]). The extragradient method generates two sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$, L is the Lipschitz constant of A, and P_C is the metric projection from H onto C. If the solution set $VI(C,A)$ is nonempty, cost function A is monotone and L-Lipschitz continuous, then the sequence $\{x_n\}$ generated by iterative algorithm (1.3) converges weakly to an element in $VI(C,A)$. In recent years, the extragradient method has received great attention. For related results on the extragradient method and its modifications for monotone and Lipschitz continuous operators; we refer to [11, 12, 13, 14].

It can easily be seen that the extragradient method needs to calculate two orthogonal projections onto the feasible set C per iteration. The orthogonal projection onto a closed and convex set C is related to an optimization problem:minimum distance problem and if C is general closed and convex set, the performance of the extragradient method suffers a setback. One of the iterative methods introduced to improve the extragradient method by reducing the number of projections onto the feasible set is the subgradient extragradient method [15, 16, 17]. The subgradient extragradient method is defined as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ T_n = \{x \in H : \langle x_n - \lambda A(x_n) - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \end{cases} \tag{1.4}$$

where $\lambda \in (0, \frac{1}{L})$. This method replaces two projections onto C by one projection onto C and one onto a half-space which can be computed more easily. Another algorithm that improves the extragradient method is the Tseng's extragradient method [18], which uses only one projection in each iteration

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = y_n + \lambda(A(x_n) - A(y_n)), \end{cases} \tag{1.5}$$

where $\lambda \in (0, \frac{1}{L})$. In recent years, the Tseng's extragradient method for solving the problem (VIP) (1.2) has received great attention; see, e.g., [19, 20, 21] and the references therein.

Methods (1.3), (1.4), and (1.5) require a prior knowledge of the Lipschitz constant of the operator A. This is a source of concern for the use of these methods because the Lipschitz constant is often unknown or difficult to approximate. Authors adopted the linesearch procedure to overcome this problem but it is known that a linesearch is an inner loop running at each outer iteration until some finite stopping criterion is satisfied. Thus a method with a linesearch can be time consuming because it requires many extra computations. Yang et al. [22, 23, 24] proposed modifications of gradient methods for solving variational inequality problems with the self adaptive step size rules. But the step sizes are non-increasing and the algorithms may depend on the choice of the initial step-size. Recently, Liu and Yang [25] introduced a type of Tseng's extragradient algorithm with non-monotonic step sizes for solving quasimonotone variational inequalities (or without monotonicity).

It is now wildly known that the problem (BVIP (1.1)) includes several classes of mathematical programs with equilibrium constraints, bilevel minimization problems, variational inequalities, minimum-norm problems with the solution set of variational inequalities, bilevel convex programming models, and bilevel linear programming [14, 26, 27, 28]. Therefore, developing modified iterative methods for solving bilevel variational inequality problem (1.1) is a good area of research interest and should be given adequate attention.

Thong and Hieu [29] proposed a modified subgradient algorithm for solving BVIP (1.1) as follows: Choose a sequence $\alpha_n \in (0,1)$ with the following properties:

$$(C1) \lim_{n\to\infty} \alpha_n = 0, \ (C2) \sum_{n=1}^{\infty} \alpha_n = +\infty.$$

Algorithm 1.1. Initialization: Give $\tau \in (0, \frac{1}{L_1}), \alpha \in (0,2)$ and $0 < \gamma < \frac{2\lambda}{L_2^2}$ (L_1 is the Lipschitz constant of A, λ is the modulus of the strong monotonicity of F, and L_2 is the Lipschitz constant of F). Let $x_0 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows

Step 1. Compute $y_n = P_C(x_n - \tau A(x_n))$ for all $n \geq 0$.

Step 2. Compute $z_n = P_{T_n}(x_n - \alpha\tau\eta_n d_n)$ for all $n \geq 0$, where

$$T_n = \{x \in H : \langle x_n - \tau A(x_n) - y_n, x - y_n\rangle \leq 0\},$$

$$d_n := x_n - y_n - \tau(A(x_n) - A(y_n)) \ \forall n \geq 0,$$

and

$$\eta_n = \begin{cases} \frac{\langle x_n - y_n, d_n\rangle}{\|d_n\|^2}, \text{ if } d_n \neq 0 \\ 0, \text{ if } d_n = 0. \end{cases}$$

Step 3. Compute $x_{n+1} = z_n - \alpha_n \gamma F(z_n)$ for all $n \geq 0$.

Recently, Thong et. al. [30] proposed the following iterative algorithm: Choose a sequence $\alpha_n \in (0,1)$ with the following properties:

$$(C1) \lim_{n\to\infty} \alpha_n = 0, \ (C2) \sum_{n=1}^{\infty} \alpha_n = +\infty.$$

Algorithm 1.2. Initialization: Let $\tau_0 > 0, \alpha \in (0,2)$, and $0 < \gamma < \frac{2\lambda}{L_2^2}$ (λ is the modulus of the strong monotonicity of F and L_2 is the Lipschitz constant of F). Let $x_0 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows
Step 1. Compute $y_n = P_C(x_n - \tau_n A(x_n))$ for all $n \geq 0$.
Step 2. Compute $z_n = x_n - \alpha\eta_n d_n$ for all $n \geq 0$, where

$$d_n := x_n - y_n - \tau_n(A(x_n) - A(y_n)),\ \forall n \geq 0$$

and

$$\eta_n = \begin{cases} \frac{\langle x_n - y_n, d_n\rangle}{\|d_n\|^2},\ \text{if } d_n \neq 0 \\ 0,\ \text{if } d_n = 0. \end{cases}$$

Step 3. Compute $x_{n+1} = z_n - \alpha_n\gamma F(z_n)$ for all $n \geq 0$ and update

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|A(x_n) - A(y_n)\|}, \tau_n\right\} \text{ if } A(x_n) \neq A(y_n) \\ \tau_n,\ otherwise. \end{cases}$$

Motivated by [25] and the ongoing research in this direction, we propose an inertial relaxed Tseng extragradient iterative algorithm with a non-monotonic self adaptive step sizes such that no pre-knowledge of the Lipschitz constant of the cost function is required for the solving the quasimonotone bilevel variational inequality problem.

2. Preliminaries

Definition 2.1. Let H be a real Hilbert space. A mapping $F : H \to H$ is said to be:
(a) strongly monotone on H if there exists a constant $\lambda > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \lambda\|x - y\|^2, \forall x, y \in H;$$

(b) monotone on H if $\langle F(x) - F(y), x - y\rangle \geq 0$ for all $x, y \in H$;
(c) pseudo-monotone if $\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0$ for all $x, y \in H$;
(d) quasimonotone on H if $\langle F(x), y - x\rangle > 0 \Rightarrow \langle F(y), y - x\rangle \geq 0$ for all $x, y \in H$;
(e) L-Lipschitz-continuous on H if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \forall x, y \in H.$$

From the above definitions, we see that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$, but the converses are not always true.

Definition 2.2. Let C be a nonempty, closed, and convex subset of a Hilbert space H. The projection mapping from H onto C is denoted by P_C and defined by $P_C(x) = \arg\min_{y\in C}||x - y||$ for all $x \in H$.

It is well known that the projection mapping is firmly nonexpansive and is characterized by the following variational inequality: $\langle P_C(x) - x, P_C(x) - y\rangle \leq 0$ for all $y \in C$.

Lemma 2.1. [31] *Let $\{a_n\}$ be a positive real sequence, $\{b_n\}$ a real sequence, and $\{\alpha_n\}$ a real sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$. Assume that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n$ for all $n \geq 1$. If* $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, *then* $\lim a_n = 0$.

Lemma 2.2. [32] *Let α be a real number in $(0,1]$ and let γ be a positive real number. Let $F : H \to H$ be an L-Lipschitz continuous and λ-strongly monotone mapping. For any nonexpansive mapping T on H, define a mapping T_γ on H by $T_\gamma(x) = (I - \alpha\gamma F)(T(x))$ for all $x \in H$. If $\gamma < \frac{2\lambda}{L^2}$, then T_γ is a contraction, that is,*

$$||T_\gamma(x) - T_\gamma(y)|| \leq (1 - \alpha\eta)||x - y||, \forall x, y \in H, \tag{2.1}$$

where

$$\eta = 1 - \sqrt{1 - \gamma(2\lambda - \gamma L^2)} \in (0,1).$$

3. Main Results

In this section, we give an inertial Tseng Type iterative algorithms for solving bilevel quasimonotone variational inequality problems and analyse their convergence to a solution of the bilevel variational inequality problem with Lipschitz quasimonotone operators.

In this paper, we assume that the following conditions hold:
(A1) $\hat{x} \in S_D \neq \emptyset$, where $\hat{x}$ is the unique solution of the problem (BVIP) (1.1),
(A2) mapping A is Lipschitz-continuous with constant $L_1 > 0$,
(A3) mapping A is sequentially weakly continuous, i.e., for each sequence $\{x_n\}; \{x_n\}$ converging weakly to x implies that $\{A(x_n)\}$ converges weakly to $A(x)$,
(A4) mapping A is quasimonotone on H;
(A5) mpping $F : H \to H$ is λ-strongly monotone and Lipschitz-continuous with constant $L_2 > 0$.

Algorithm 3.1. Initialization: Take $\theta > 0, \mu \in (0,1), \tau_0 > 0, \beta \in (0,1]$, and a nonnegative sequence $\{\rho_n\}$ such that $\sum_{n=1}^{\infty} \rho_n < +\infty$ and $\gamma \in \left(0, \frac{2\lambda}{L_2^2}\right)$. Choose a positive sequence $\{\varepsilon_n\}$ such that $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Given x_{n-1} and $x_n (n \geq 1)$, calculate x_{n+1} as follows:
Step 1: Compute: $w_n = x_n + \theta_n(x_n - x_{n-1})$, where

$$\theta_n = \begin{cases} \min\left\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\} \text{ if } x_n \neq x_{n-1} \\ \theta, \ Otherwise. \end{cases} \tag{3.1}$$

Step 2: $y_n = P_C(w_n - \tau_n A(x_n))$,
Step 3: $z_n = (1-\beta)w_n + \beta(y_n + \tau_n(A(w_n) - A(y_n)))$,
Step 4: $x_{n+1} = z_n - \alpha_n \gamma F(z_n)$,
and update:

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|A(w_n) - A(y_n)\|}, \tau_n + \rho_n\right\} \text{ if } A(w_n) \neq A(y_n) \\ \tau_n + \rho_n, \ Otherwise. \end{cases}$$

Set $n := n+1$ and go back to Step 1.

Remark 3.1. We observe that inertial interpolation term (3.1) is easy to implement since $||x_n - x_{n-1}||$ is known before calculating θ_n. Furthermore, it follows from (3.1) and the conditions on α_n that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0.$$

Indeed, we obtain $\theta_n||x_n - x_{n-1}|| \le \varepsilon_n$ for all n, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \le \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0.$$

Lemma 3.1. *([25], Lemma 3.1) Let $\{\tau_n\}$ be generated as in Algorithm 3.1. Then* $\lim_{n\to\infty} \tau_n = \tau$ *and $\tau \in [\min\{\rho_0, \frac{\mu}{L_1}\}; \rho_0 + \rho]$, where $\rho = \sum_{n=1}^{\infty} \rho_n$.*

Lemma 3.2. *Let $\{w_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by Algorithm 3.1. Assume that A is quasi monotone and L_1-Lipschitz continuous and $S_D \neq \emptyset$. Then, for all $x^* \in S_D$,*

$$||z_n - x^*||^2 \le ||w_n - x^*||^2 - \beta\left[2 - \beta - 2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}} - \beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right]||w_n - y_n||^2. \tag{3.2}$$

Proof. Notice that

$$\begin{aligned} ||z_n - x^*||^2 &= (1-\beta)^2||w_n - x^*||^2 + \beta^2||y_n - x^*||^2 + \beta^2\tau_n^2||A(w_n) - A(y_n)||^2 \\ &\quad + 2\beta(1-\beta)\langle w_n - x^*, y_n - x^*\rangle + 2\tau_n\beta(1-\beta)\langle w_n - x^*, A(w_n) - A(y_n)\rangle \\ &\quad + 2\tau_n\beta^2\langle y_n - x^*, A(w_n) - A(y_n)\rangle \end{aligned} \tag{3.3}$$

and

$$2\langle w_n - x^*, y_n - x^*\rangle = ||w_n - x^*||^2 + ||y_n - x^*||^2 - ||w_n - y_n||^2. \tag{3.4}$$

Substituting (3.4) into (3.3), we have

$$\begin{aligned} ||z_n - x^*||^2 &= (1-\beta)^2||w_n - x^*||^2 + \beta^2||y_n - x^*||^2 + \beta^2\tau_n^2||A(w_n) - A(y_n)||^2 \\ &\quad + \beta(1-\beta)[||w_n - x^*||^2 + ||y_n - x^*||^2 - ||w_n - y_n||^2] \\ &\quad + 2\tau_n\beta(1-\beta)\langle w_n - x^*, A(w_n) - A(y_n)\rangle + 2\tau_n\beta^2\langle y_n - x^*, A(w_n) - A(y_n)\rangle \\ &= (1-\beta)||w_n - x^*||^2 + \beta||y_n - x^*||^2 - \beta(1-\beta)||w_n - y_n||^2 \\ &\quad + \beta^2\tau_n^2||A(w_n) - A(y_n)||^2 + 2\tau_n\beta(1-\beta)\langle w_n - x^*, A(w_n) - A(y_n)\rangle \\ &\quad + 2\tau_n\beta^2\langle y_n - x^*, A(w_n) - A(y_n)\rangle. \end{aligned} \tag{3.5}$$

Since $x^* \in S_D \subset VI(C,A) \subset C$, we have from the definition of y_n that

$$\langle y_n - (w_n - \tau_n A(w_n)), y_n - x^*\rangle \le 0. \tag{3.6}$$

Consequently,

$$2\langle w_n - y_n, y_n - x^*\rangle - 2\tau_n\langle A(w_n) - A(y_n), y_n - x^*\rangle - 2\tau_n\langle A(y_n), y_n - x^*\rangle \ge 0. \tag{3.7}$$

Moreover,

$$2\langle w_n - y_n, y_n - x^*\rangle = ||w_n - x^*||^2 - ||w_n - y_n||^2 - ||y_n - x^*||^2. \tag{3.8}$$

Again, by (3.6), we have $\langle y_n - w_n, y_n - x^*\rangle \le -\tau_n\langle A(w_n), y_n - x^*\rangle$. Furthermore, since $y_n \in C$ and $x^* \in S_D$, we obtain

$$\langle A(y_n), y_n - x^*\rangle \ge 0, \forall n \ge 0. \tag{3.9}$$

From (3.7), (3.8), and (3.9), we reach

$$||y_n - x^*||^2 \le ||w_n - x^*||^2 - ||w_n - y_n||^2 - 2\tau_n\langle A(w_n) - A(y_n), y_n - x^*\rangle. \tag{3.10}$$

Thus, from (3.5) and (3.10), we obtain

$$\begin{aligned}
||z_n - x^*||^2 &\leq (1-\beta)||w_n - x^*||^2 + \beta[||w_n - x^*||^2 - ||w_n - y_n||^2 \\
&\quad -2\tau_n\langle A(w_n) - A(y_n), y_n - x^*\rangle] - \beta(1-\beta)||w_n - y_n||^2 \\
&\quad +\beta^2\tau_n^2||A(w_n) - A(y_n)||^2 + 2\tau_n\beta(1-\beta)\langle w_n - x^*, A(w_n) - A(y_n)\rangle \\
&\quad +2\tau_n\beta^2\langle y_n - x^*, A(w_n) - A(y_n)\rangle \\
&= ||w_n - x^*||^2 - \beta(2-\beta)||w_n - y_n||^2 + \beta^2\tau_n^2||A(w_n) - A(y_n)||^2 \\
&\quad +2\tau_n\beta(1-\beta)\langle w_n - y_n, A(w_n) - A(y_n)\rangle \\
&\leq ||w_n - x^*||^2 - \beta(2-\beta)||w_n - y_n||^2 + \beta^2\tau_n^2\frac{\mu^2}{\tau_{n+1}^2}||w_n - y_n||^2 \\
&\quad +2\beta\tau_n(1-\beta)\frac{\mu}{\tau_{n+1}}||w_n - y_n||^2 \\
&= ||w_n - x^*||^2 - \beta[2 - \beta - 2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}} - \beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}]||w_n - y_n||^2.
\end{aligned}$$

□

Lemma 3.3. *Assume that conditions (A1)-(A4) hold and $\{x_n\}$ is the sequence generated by Algorithm 3.1. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to $z \in H$ and $||x_{n_k} - y_{n_k}|| \to 0$ as $n \to \infty$, then $z \in S_D$ or $A(z) = 0$.*

Proof. Since $\{x_{n_k}\}$ converges weakly to $z \in H$ and $\lim_{k\to\infty}||x_{n_k} - y_{n_k}|| = 0$, then $y_{n_k} \rightharpoonup z$ and $z \in C$. If $\limsup_{n\to\infty}||A(y_{n_k})|| = 0$, then

$$\lim_{k\to\infty}||A(y_{n_k})|| = \liminf_{k\to\infty}||A(y_{n_k})|| = 0.$$

Since A is sequentially weakly continuous on C and $\{y_{n_k}\}$ converges weakly to $z \in C$, we have that $\{A(y_{n_k})\}$ converges weakly to $A(z)$. Therefore, it follows from the sequentially weakly semi-continuity of the norm operator that

$$0 \leq ||A(z)|| \leq \limsup_{k\to\infty}||A(y_{n_k})|| = 0.$$

Thus $A(z) = 0$. Now, if $\limsup_{k\to\infty}||A(y_{n_k})|| > 0$, without loss of generality, we take $\lim_{k\to\infty}||A(y_{n_k})|| = D > 0$ (otherwise, we take a subsequence of $\{y_{n_k}\}$). Then there exists a $K \in \mathbb{N}$ such that $||A(y_{n_k})|| > \frac{M}{2}$ for all $k \geq K$. Moreover, we have $\langle y_{n_k} - w_{n_k} + \tau_{n_k}A(w_{n_k}), \bar{x} - y_{n_k}\rangle \geq 0$ for all $\bar{x} \in C$. which implies

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, \bar{x} - y_{n_k}\rangle \leq \langle A(w_{n_k}), \bar{x} - y_{n_k}\rangle, \forall \bar{x} \in C.$$

Therefore,

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, \bar{x} - y_{n_k}\rangle - \langle A(w_{n_k}) - A(y_{n_k}), \bar{x} - y_{n_k}\rangle \leq \langle Ay_{n_k}, \bar{x} - y_{n_k}\rangle, \forall \bar{x} \in C.$$

Substituting $w_{n_k} = x_{n_k} + \theta_{n_k}(x_{n_k} - x_{n_k-1})$, we have

$$\begin{aligned}
&\frac{1}{\tau_{n_k}}\langle x_{n_k} - y_{n_k}, \bar{x} - y_{n_k}\rangle + \frac{1}{\tau_{n_k}}\alpha_{n_k}\left\langle \frac{\theta_{n_k}}{\alpha_{n_k}}(x_{n_k} - x_{n_k-1}, \bar{x} - y_{n_k}\right\rangle - \langle A(w_{n_k}) - A(y_{n_k}), \bar{x} - y_{n_k}\rangle \\
&\leq \langle Ay_{n_k}, \bar{x} - y_{n_k}\rangle, \forall \bar{x} \in C.
\end{aligned} \tag{3.11}$$

Since $\lim_{k\to\infty}||x_{n_k}-y_{n_k}||=0$ and A is L_1-Lipschitz continuous on H, we have

$$\begin{aligned}
\lim_{k\to\infty}||A(w_{n_k})-A(y_{n_k})|| &\leq \lim_{k\to\infty}L_1|||w_{n_k}-y_{n_k}|| \\
&= L_1\lim_{k\to\infty}||x_{n_k}-y_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1})|| \\
&\leq L_1\lim_{k\to\infty}[||x_{n_k}-y_{n_k}||+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||]=0.
\end{aligned}$$

Fixing $\bar{x}\in C$, and then letting $k\to\infty$ in (3.11), we find from $\lim_{k\to\infty}\tau_{n_k}=\tau>0$ that

$$0\leq \liminf_{k\to\infty}\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle \leq \limsup_{k\to\infty}\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle<+\infty. \tag{3.12}$$

If $\limsup_{k\to\infty}\langle Ay_{n_k},\bar{x}-y_{n_k}\rangle>0$, thenthere exists a subsequence $\{y_{n_{k_i}}\}$ such that $\lim_{i\to\infty}\langle Ay_{n_{k_i}},\bar{x}-y_{n_{k_i}}\rangle>0$. Thus there exists a natural number i_0 such that $\langle A(y_{n_{k_i}}),\bar{x}-y_{n_{k_i}}\rangle>0$ for all $i\geq i_0$. Now, since A is quasimonotone, we have, for all $i\geq i_0$, $\langle A(\bar{x}),\bar{x}-y_{n_{k_i}}\rangle\geq 0$. Letting $i\to\infty$, we obtain $z\in S_D$. However, if $\limsup_{k\to\infty}\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle=0$, we obtain from (3.12) that

$$\lim_{k\to\infty}\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle=\limsup_{k\to\infty}\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle=\liminf_{k\to\infty}\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle=0.$$

We choose a sequence $\{\xi_k\}$ of positive numbers decreasing with $\lim_{k\to\infty}\xi_k=0$. Let $\zeta_k=|\langle A(y_{n_k}),\bar{x}-y_{n_k}\rangle|+\xi_k>0$. Let $\Upsilon_{n_k}=\frac{A(y_{n_k})}{||A(y_{n_k})||^2}$ for all $k\geq K$, we obtain $\langle A(y_{n_k}),\Upsilon_{n_k}\rangle=1$. Therefore, we have, for all $k\geq K$, $\langle A(y_{n_k}),\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}\rangle>0$. Hence, since A is quasimonotone, we have, $k>K$, $\langle A(\bar{x}+\zeta_k\Upsilon_{n_k}),\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}\rangle\geq 0$, which implies that, for all $k\geq K$,

$$\begin{aligned}
\langle A(\bar{x}),\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}\rangle &= \langle A(\bar{x})-A(\bar{x}+\zeta_k\Upsilon_{n_k}),\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}\rangle \\
&\quad +\langle A(\bar{x}+\zeta_k\Upsilon_{n_k}),\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}\rangle \\
&\geq \langle A(\bar{x})-A(\bar{x}+\zeta_k\Upsilon_{n_k}),\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}\rangle \\
&\geq -||A(\bar{x})-A(\bar{x}+\zeta_k\Upsilon_{n_k})||||\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}|| \\
&\geq -\zeta_kL_1||\Upsilon_{n_k}||||\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}|| \\
&= -\zeta_k\frac{L_1}{||A(y_{n_k})||}||\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}|| \\
&\geq -\zeta_k\frac{2L_1}{M}||\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}||.
\end{aligned} \tag{3.13}$$

Since $\{||\bar{x}+\zeta_k\Upsilon_{n_k}-y_{n_k}||\}$ is bounded and $\lim_{k\to\infty}\zeta_k=0$,, then taking limits as $k\to\infty$ in (3.13), we have $\langle A(\bar{x}),\bar{x}-z\rangle\geq 0$ for all $\bar{x}\in C$. Thus we have that $z\in S_D$. □

Theorem 3.1. *Assume that conditions (A1)-(A5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $\hat{x}$, where $\hat{x}$ is the unique solution to problem (BVIP) (1.1).*

Proof. Since $\tau_n\to\tau$, $\mu\in(0,1)$, and $\beta\in(0,1]$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right] \\
&=[2-\beta-\beta\mu^2-2\mu(1-\beta)] \\
&=(1-\mu)(2-\beta+\beta\mu)>0.
\end{aligned}$$

Therefore, there exists $n_0 \geq 1$ such that

$$\beta\Big[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\Big]>0, \forall n \geq n_0.$$

From (3.2) and $\hat{x} \in S_D$, we have that $||z_n-\hat{x}|| \leq ||w_n-\hat{x}||$ for all $n \geq n_0$, which together with (2.1) follows that, for each $n \geq n_0$,

$$\begin{aligned} ||x_{n+1}-\hat{x}|| &= ||(I-\alpha_n\gamma F)(z_n)-(I-\alpha_n\gamma F)(\hat{x})-\alpha_n\gamma F(\hat{x})|| \\ &\leq ||(I-\alpha_n\gamma F)(z_n)-(I-\alpha_n\gamma F)(\hat{x})||+\alpha_n\gamma||F(\hat{x})|| \\ &\leq (1-\alpha_n\eta)||z_n-\hat{x}||+\alpha_n\gamma||F(\hat{x})|| \\ &\leq (1-\alpha_n\eta)||w_n-\hat{x}||+\alpha_n\eta\frac{\gamma}{\eta}||F(\hat{x})||, \end{aligned} \tag{3.14}$$

where $\eta = 1-\sqrt{1-\gamma(2\lambda-\gamma L_2^2)} \in (0,1)$. But

$$||w_n-\hat{x}|| \leq ||x_n-\hat{x}||+\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||. \tag{3.15}$$

From Remark 3.1, we have $\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| \to 0$ as $n \to \infty$. Therefore, there is a constant $Q_1 > 0$ that satisfies

$$\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| \leq Q_1, \forall n \geq 1. \tag{3.16}$$

Therefore, from (3.15) and (3.16), we obtain

$$||z_n-\hat{x}|| \leq ||w_n-\hat{x}|| \leq ||x_n-\hat{x}||+\alpha_n Q_1, \forall n \geq n_0. \tag{3.17}$$

Using (3.14) and (3.17), we have

$$\begin{aligned} ||x_{n+1}-\hat{x}|| &\leq (1-\alpha_n\eta)||x_n-\hat{x}||+(1-\alpha_n\eta)\alpha_n Q_1+\alpha_n\eta\frac{\gamma}{\eta}||F(\hat{x})|| \\ &\leq (1-\alpha_n\eta)||x_n-\hat{x}||+\alpha_n\eta\frac{1}{\eta}Q_1+\alpha_n\eta\frac{\gamma}{\eta}||F(\hat{x})|| \\ &= (1-\alpha_n\eta)||x_n-\hat{x}||+\alpha_n\eta\Big[\frac{Q_1+\gamma||F(\hat{x})||}{\eta}\Big] \\ &\leq \max\Big\{||x_n-\hat{x}||, \Big[\frac{Q_1+\gamma||F(\hat{x})||}{\eta}\Big]\Big\} \\ &\quad\vdots \\ &\leq \max\Big\{||x_0-\hat{x}||, \Big[\frac{Q_1+\gamma||F(\hat{x})||}{\eta}\Big]\Big\}. \end{aligned}$$

Hence, sequence $\{x_n\}$ is bounded.

$$\begin{aligned} ||x_{n+1}-\hat{x}||^2 &= ||z_n-\alpha_n\gamma F(z_n)-\hat{x}||^2 \\ &= ||(I-\alpha_n\gamma F)(z_n)-(I-\alpha_n\gamma F)(\hat{x})-\alpha_n\gamma F(\hat{x})||^2 \\ &\leq ||(I-\alpha_n\gamma F)(z_n)-(I-\alpha_n\gamma F)(\hat{x})||^2-2\alpha_n\gamma\langle F(\hat{x}),x_{n+1}-\hat{x}\rangle \\ &\leq (1-\alpha_n\eta)^2||z_n-\hat{x}||^2+2\alpha_n\gamma\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle \\ &\leq ||z_n-\hat{x}||^2+\alpha_n Q_2, \text{ (for some } Q_2>0). \end{aligned} \tag{3.18}$$

From (3.2) and (3.18), we have

$$\begin{aligned}
||x_{n+1}-\hat{x}||^2 &\leq ||w_n-\hat{x}||^2-\beta\left[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right]||w_n-y_n||^2+\alpha_nQ_2\\
&\leq ||x_n-\hat{x}||^2+2\alpha_nQ_1||x_n-\hat{x}||+\alpha_n^2Q_1^2\\
&\quad -\beta\left[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right]||w_n-y_n||^2+\alpha_nQ_2\\
&\leq ||x_n-\hat{x}||^2-\beta\left[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right]||w_n-y_n||^2\\
&\quad +\alpha_n[2Q_1||x_n-\hat{x}||+\alpha_nQ_1^2+Q_2]\\
&\leq ||x_n-\hat{x}||^2-\beta\left[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right]||w_n-y_n||^2\\
&\quad +\alpha_nQ_3,(Q_3=\sup\{2Q_1||x_n-\hat{x}||+\alpha_nQ_1^2+Q_2\}),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\beta\left[2-\beta-2\mu(1-\beta)\frac{\tau_n}{\tau_{n+1}}-\beta\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right]||w_n-y_n||\\
&\leq ||x_n-\hat{x}||^2-||x_{n+1}-\hat{x}||^2+\alpha_nQ_3.
\end{aligned}\tag{3.19}$$

Again, using (2.1) and (3.17), we obtain

$$\begin{aligned}
||x_{n+1}-\hat{x}||^2 &= ||(I-\alpha_n\gamma F)(z_n)-(I-\alpha_n\gamma F)(\hat{x})-\alpha_n\gamma F(\hat{x})||^2\\
&\leq ||(I-\alpha_n\gamma F)(z_n)-(I-\alpha_n\gamma F)(\hat{x})||^2-2\alpha_n\gamma\langle F(\hat{x}),x_{n+1}-\hat{x}\rangle\\
&\leq (1-\alpha_n\eta)^2||z_n-\hat{x}||^2+2\alpha_n\gamma\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle\\
&\leq (1-\alpha_n\eta)||w_n-\hat{x}||^2+2\alpha_n\gamma\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle\\
&\leq (1-\alpha_n\eta)(||x_n-\hat{x}||+\theta_n||x_n-x_{n-1}||)^2+2\alpha_n\gamma\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle\\
&\leq (1-\alpha_n\eta)\Big[||x_n-\hat{x}||^2+2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-\hat{x}||||x_n-x_{n-1}||\\
&\quad +\alpha_n^2\frac{\theta_n^2}{\alpha_n^2}||x_n-x_{n-1}||^2\Big]+2\alpha_n\gamma\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle\\
&\leq (1-\alpha_n\eta)||x_n-\hat{x}||^2+\alpha_n\left[D\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\alpha_n^2\frac{\theta_n^2}{\alpha_n^2}||x_n-x_{n-1}||^2\right]\\
&\quad +2\alpha_n\gamma\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle\\
&= (1-\alpha_n\eta)||x_n-\hat{x}||^2+\alpha_n\eta\Big[D\frac{\theta_n}{\alpha_n\eta}||x_n-x_{n-1}||+\alpha_n^2\frac{\theta_n^2}{\alpha_n^2\eta}||x_n-x_{n-1}||^2\\
&\quad +2\frac{\gamma}{\eta}\langle F(\hat{x}),\hat{x}-x_{n+1}\rangle\Big],
\end{aligned}\tag{3.20}$$

where $D=\sup\{||x_n-\hat{x}||\}$. To prove that $\{||x_n-\hat{x}||^2\}$ converges to zero, in view of Lemma 2.1 and Remark 3.1, we only need to demonstrate that $\limsup_{k\to\infty}\langle F(\hat{x}),\hat{x}-x_{n_k+1}\rangle\leq 0$ for any subsequence $\{||x_{n_k}-\hat{x}||\}$ of $\{||x_n-\hat{x}||\}$ satisfying $\liminf_{k\to\infty}(||x_{n_k+1}-\hat{x}||-||x_{n_k}-\hat{x}||)\geq 0$.

Suppose that $\{||x_{n_k} - \hat{x}||\}$ is a subsequence of $\{||x_n - \hat{x}||\}$ such that $\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0$. Then

$$\begin{aligned}&\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}||^2 - ||x_{n_k} - \hat{x}||^2)\\&= \liminf_{k\to\infty}[(||x_{n_k+1} - \hat{x}|| + ||x_{n_k} - \hat{x}||)(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||)] \geq 0.\end{aligned}$$

From (3.19), we have

$$\begin{aligned}&\limsup_{k\to\infty}\beta\left[2-\beta-2\mu(1-\beta)\frac{\tau_{n_k}}{\tau_{n_k+1}}-\beta\mu^2\frac{\tau_{n_k}^2}{\tau_{n_k+1}^2}\right]||w_{n_k}-y_{n_k}||\\&\leq \limsup_{k\to\infty}[||x_{n_k} - \hat{x}||^2 - ||x_{n_k+1} - \hat{x}||^2 + \alpha_{n_k}Q_3]\\&= -\liminf_{k\to\infty}[||x_{n_k+1} - \hat{x}||^2 - ||x_{n_k} - \hat{x}||^2] \leq 0,\end{aligned}$$

which gives $\lim_{k\to\infty}||w_{n_k} - y_{n_k}|| = 0$. Furthermore,

$$\begin{aligned}||z_{n_k} - w_{n_k}|| &= ||\beta(y_{n_k} - w_{n_k}) + \tau_{n_k}(Aw_{n_k} - Ay_{n_k})||\\&\leq (\beta + \tau_n L_1)||w_{n_k} - y_{n_k}|| \to 0, k \to \infty.\end{aligned}$$

Hence, $||z_{n_k} - y_{n_k}|| \leq ||z_{n_k} - w_{n_k}|| + ||w_{n_k} - y_{n_k}|| \to 0$ as $k \to \infty$. Moreover, using Remark 3.1 and the assumption on $\{\alpha_n\}$, we have

$$||x_{n_k} - w_{n_k}|| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| \to 0, k \to \infty$$

and $||x_{n_k+1} - z_{n_k}|| = \alpha_{n_k}\gamma||F(z_{n_k})|| \to 0, k \to \infty$. Thus, we obtain

$$||x_{n_k+1} - x_{n_k}|| \leq ||z_{n_k} - x_{n_k+1}|| + ||z_{n_k} - w_{n_k}|| + ||w_{n_k} - x_{n_k}|| \to 0, k \to \infty. \tag{3.21}$$

Since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $z \in H$ such that

$$\limsup_{k\to\infty}\langle F(\hat{x}), \hat{x} - x_{n_k}\rangle = \lim_{j\to\infty}\langle F(\hat{x}), \hat{x} - x_{n_{k_j}}\rangle = \langle F(\hat{x}), \hat{x} - z\rangle. \tag{3.22}$$

From $\lim_{k\to\infty}||x_{n_k} - y_{n_k}|| = 0$ and Lemma 3.3, we have $z \in VI(C,A)$. Since $\hat{x}$ is the unique solution to problem BVIP (1.1), it follows from (3.22) that

$$\limsup_{k\to\infty}\langle F(\hat{x}), \hat{x} - x_{n_k}\rangle = \langle F(\hat{x}), \hat{x} - z\rangle \leq 0. \tag{3.23}$$

Combining (3.21) and (3.23), we have

$$\limsup_{k\to\infty}\langle F(\hat{x}), \hat{x} - x_{n_k+1}\rangle = \limsup_{k\to\infty}\langle F(\hat{x}), \hat{x} - x_{n_k}\rangle = \langle F(\hat{x}), \hat{x} - z\rangle \leq 0. \tag{3.24}$$

Thus, from Lemma 2.1, (3.20), and (3.24), we have $\lim_{n\to\infty}||x_n - \hat{x}|| = 0$. This completes the proof. $\square$

4. Numerical Examples

In this section, we give some numerical examples to demonstrate the applicability of the proposed algorithms and we show the efficiency of our proposed methods in comparison with some recent and notable methods in the literature. All the codes are written in MATLAB R2015a and run on HP Intel(R) Core(TM) i5 CPU M 520 @ 2.40GHz 2.40 GHz; 8.00 GB Ram laptop.

Example 4.1. Let $F:\mathbb{R}^m\to\mathbb{R}^m$ be the mapping defined by $F(x)=Mx+q$, where $M=NN^T+S+D$, N is an $m\times m$ matrix, S is an $m\times m$ skew symmetric matrix with entries being generated in $(-2,2)$, D is an $m\times m$ diagonal matrix, whose diagonal entries are positive in $(0,2)$, and q is a vector in $\mathbb{R}^m$. Observe that F is λ-strongly monotone with $\lambda=\min\{eig(M)\}$ and L_2-Lipschitz continuous with constant $L_2=\|M\|=\max\{eig(M)\}$. Consider the following quadratic fractional programming problem [19]:

$$\min\left\{f(x)=\frac{x^TQx+a^Tx+c}{b^Tx+d}\right\}$$

and

$$Q=\begin{pmatrix}5&-1&2&0&2\\-1&6&-1&3&0\\2&-1&3&0&1\\0&3&0&5&0\\2&0&1&0&4\end{pmatrix},a=\begin{pmatrix}1\\2\\-1\\-2\\1\end{pmatrix},b=\begin{pmatrix}1\\0\\-1\\0\\1\end{pmatrix},c=-2,d=20.$$

It was demonstrated [19] that f is pseudomontone on the open set

$$G=\{x\in\mathbb{R}^5:B^Tx+d=x_1-x_3+x_5+20>0\},$$

w hich implies that the mapping $A:\mathbb{R}^5\to\mathbb{R}^5$ defined by

$$A(x)=\nabla f(x)=\frac{(b^Tx+d)(2Qx+a)-b(x^TQx+a^Tx+c)}{(b^Tx+d)^2}$$

is pseudomonotone on G. We then take the feasibility set $C=\{x\in\mathbb{R}^5:1\le x_i\le 3, i=1,2,3,4,5\}$. Clearly, $C\subseteq G$. Therefore, A is pseudomonotone on C. Moreover, A is L_1-Lipschitz continuous on C with $L_1=148.68$ [19].

In this example, we choose stopping criterion $\|x_n-y_n\|\le 10^{-4}$ and the choices of the parameters are as follows: $\alpha=0.4,\mu=0.7,\tau_0=1,\gamma=\frac{\lambda}{||M||^2},\alpha_n=\frac{1}{n},\theta=0.001,\beta=0.7,\varepsilon_n=\frac{1}{n^2}=\rho_n$, and $\tau=\frac{1}{100L_1^2}$. We then consider two cases by interchanging the role of x_0 and x_1, i.e, Case 1: x_0 generated randomly in $(0,1)$ and $x_1=(0,0,0,0,0)$; Case 2: x_1 generated randomly in $(0,1)$ and $x_0=(0,0,0,0,0)$.

TABLE 1. Example 4.1

	x_1=(0,0,0,0,0) *iter cpu*	x_1=*random*(0,1) *iter cpu* .
Alg. 3.1	117 0.024283	124 0.021327
Alg 1.1	3915 1.1025	14642 3.9431
Alg 1.2	3379 0.66795	10461 2.3301

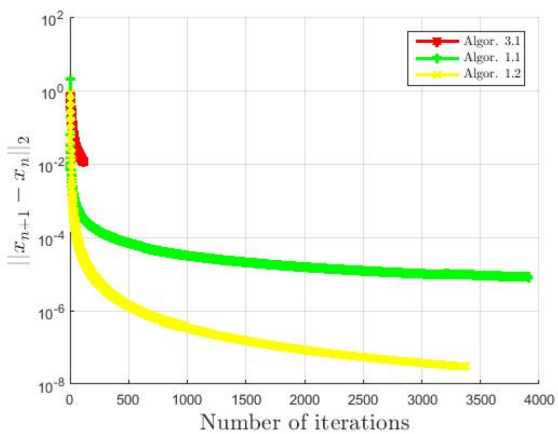

FIGURE 1. Example 4.1, $x_1 = (0,0,0,0,0)$

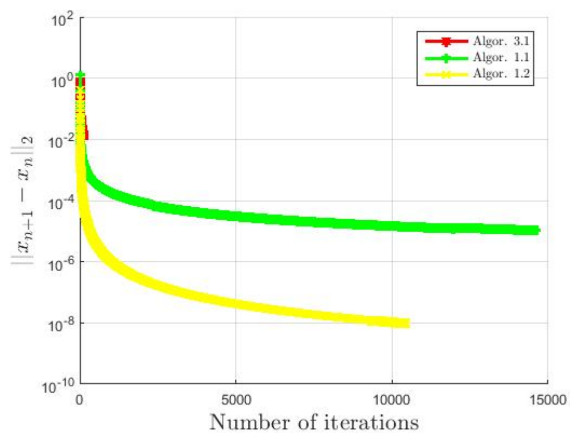

FIGURE 2. Example 4.1, $x_1 = random(0,1)$

Example 4.2. Take $H = L^2([0,1])$ with inner product $\langle x,y\rangle = \int_0^1 x(t)y(t)dt$ for all $x,y \in H$ and induced norm

$$||x|| = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}.$$

Let $F(x(t)) = \frac{4}{5}x(t)$. Then F is λ-strongly monotone and L_2-Lipschitz continuous with $\lambda = L_2 = \frac{4}{5}$. Let the operator $A : H \to H$ be define by

$$A(x(t)) = \max(0, x(t)) = \frac{x(t) + |x(t)|}{2}.$$

Then A is monotone and L_1-Lipschitz continuous with $L_1 = 1$. We take the feasible set C to be the unit ball, i.e.,

$$C = \{x \in H : ||x|| \leq 1\}.$$

Observe that the unique solution is 0. Take $x_0 = \cos \pi t$ and $x_1 = \sin \pi t$.

In this example, we choose stopping criterion $\|x_n - y_n\| \leq 10^{-2}$ and the choice of the parameters areas follows: $\alpha = 0.4, \mu = 0.7, \tau_0 = 1, \gamma = \frac{\lambda}{L_2^2}, \alpha_n = \frac{2}{3n}, \theta = 0.001, \beta = 0.7, \varepsilon_n = \frac{1}{n^2} = \rho_n,$

and $\tau = \frac{5}{8}$. We then considered two cases with different initial points. In case one, we chose $x_0 = 2t, x_1 = t$ and in case two, we chose $x_0 = 0, x_1 = \sin t$.

TABLE 2. Example 4.2

	$x_0=2t, x_1=t$ $iter\ cpu$	$x_0=0, x_1=\sin t$ $iter\ cpu$.
Alg. 3.1	6 4.9791	6 20.8762
Alg 1.1	9 235.9209	8 81.6089
Alg 1.2	7 20.3717	7 33.6292

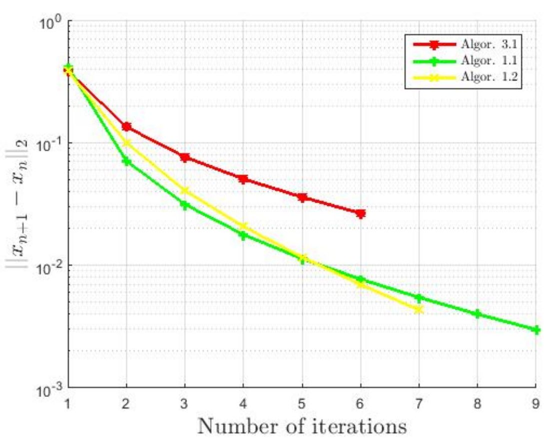

FIGURE 3. Example 4.2, $x_0 = 2t, x_1 = t$

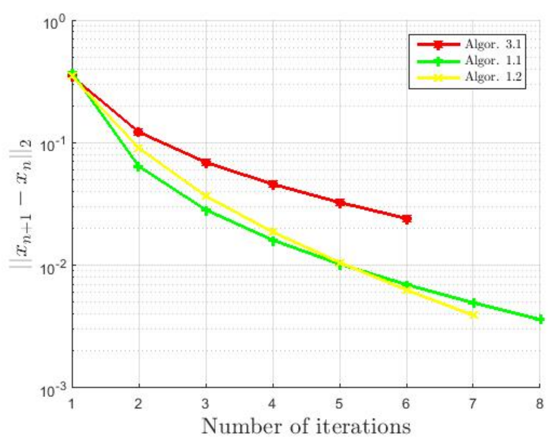

FIGURE 4. Example 4.2, $x_0 = 0, x_1 = \sin t$

Example 4.3. Let $H = \mathbb{R}$ and $C = [-1,1]$. Let $A : H \to H$ be defined by

$$A(x) = \begin{cases} 2x - 1, x > 1, \\ x^2, x \in [-1,1], \\ -2x - 1, x < -1, \end{cases}$$

Then A is quasimonotone and Lipschitz continuous. Moreover, $S_D = \{-1\}$ and $VI(C,A) = \{-1,0\}$. Again, let $F : H \to H$ be the mapping by $F(x) = x + 2$. Then F is strongly monotone and Lipschitz continuous. The unique solution of BVIP (1.1) is -1.

In this example, we choose stopping criterion $\|x_n - y_n\| \leq 10^{-4}$ and the choice of the parameters areas follows: $\alpha = 0.4, \mu = 0.7, \tau_0 = 1, \gamma = \frac{\lambda}{L_2^2}, \alpha_n = \frac{2}{3n}, \theta = 0.001, \beta = 0.7, \varepsilon_n = \frac{1}{n^2} = \rho_n$, and $\tau = \frac{3}{8}$. We then considered two cases by interchanging the role of x_0 and x_1, i.e, Case 1: $x_0 = -2, x_1 = 2$; Case 2: $x_0 = 2, x_1 = -2$.

TABLE 3. Example 4.3

	x_1=2	x_1=−2
	iter cpu	*iter cpu.*
Alg. 3.1	76 0.0011669	76 0.0009025
Alg 1.1	6668 0.92223	6668 0.92379
Alg 1.2	16669 1.113	16669 1.1167

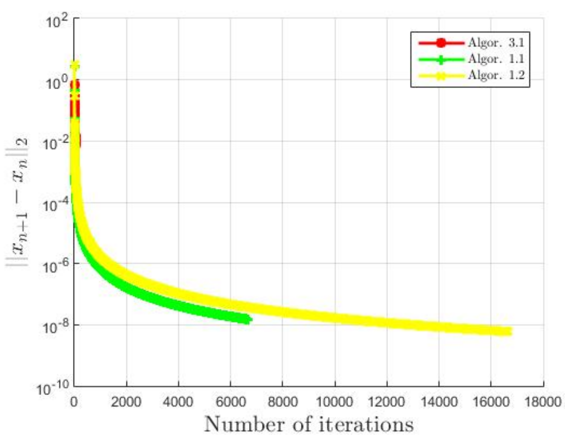

FIGURE 5. Example 4.3, $x_1 = 2$

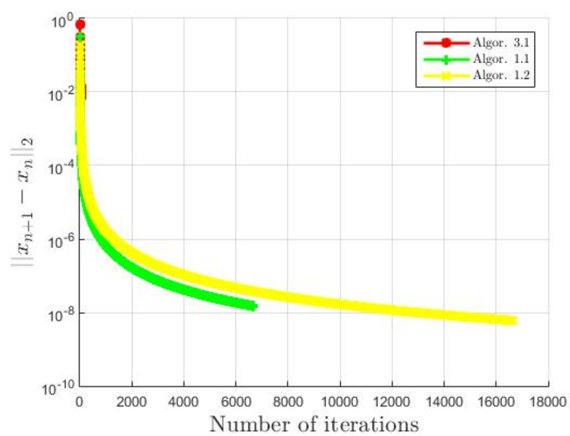

FIGURE 6. Example 4.3, $x_1 = -2$

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