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THE SPLIT COMMON FIXED POINT PROBLEM WITH MULTIPLE OUTPUT SETS FOR STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. In this paper, we investigate a split common fixed point problem with multiple output sets within a more general framework and we present a novel iterative method that boasts the advantage of the step size calculation that is independent of the norm of linear mappings. We prove the weak convergence of the method and the strong convergence of its variants under certain conditions. Furthermore, we apply our main results to the split feasibility problem with multiple output sets. Our numerical results indicate that our method is an effective approach to this problem.

Keywords. Multiple output sets; Strict pseudo-contraction; Split common fixed point problem; Split feasibility problem.

1. INTRODUCTION

Since its inception in 1994, the split feasibility problem (SFP) [1] has been a topic of much interest due to its applications in signal processing, image reconstruction, and machine learning [2–5]. In particular, it has presented significant progress in intensity-modulated radiation therapy [6, 7].

The SFP aims to identify a point $\hat{x}$ that satisfies two conditions: it belongs to a nonempty, convex and closed subset C of a Hilbert space H_0, and its image under a linear bounded operator A belongs to another nonempty, convex, and closed subset Q of a Hilbert space H_1. In 2002, Byrne [8] proposed a CQ method to solve the SFP. This method generates an iterative sequence as follows:

$$x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), \tag{1.1}$$

where I represents the identity mapping, P_C is the metric projection, and A^* denotes the conjugate of the mapping A. It has been proven that if the step size satisfies $0 < \tau < \frac{2}{\|A\|^2}$, then the iterative method above converges weakly to a solution of the corresponding problem. Although the original formulation of the SFP was in finite-dimensional Euclidean spaces, recent research has shifted towards studying the problem in infinite-dimensional Hilbert spaces; see, e.g., [9–17], and the references therein.

In the literature, there exist several generalizations of the SFP. Two of these extensions are the split common fixed point problem (SCFP) [18] and the split feasibility problem with multiple

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output sets (SFPMOS) [19]. Let $T_0 : H \to H$ and $T_1 : H_1 \to H_1$ be two nonlinear mappings. The SCFP [18] aims to find $x^\dagger \in H$ that satisfies the following condition:

$$x^\dagger \in \mathrm{Fix}(T_0) \bigcap A^{-1}(\mathrm{Fix}(T_1)), \tag{1.2}$$

where $\mathrm{Fix}(T_0)$ is the fixed point set of T_0, and $A^{-1}(\mathrm{Fix}(T_1)) = \{x \in H : Ax \in \mathrm{Fix}(T_1)\}$. On the other hand, the objective of the SFPMOS is to find an element $x^\dagger \in H$ that satisfies:

$$x^\dagger \in C \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(Q_i) \right), \tag{1.3}$$

where A_i is a bounded linear mapping from H to H_i, and Q_i is a nonempty, convex and closed subset of H_i for each $i = 1, \ldots, N$.

Assume that all the problems discussed in this paper have nonempty solution sets. In 2009, Censor and Segal [18] extended the CQ method to solve the SCFP (1.2) for firmly quasi-nonexpansive mappings. To approximate its solution, they proposed the following iterative scheme:

$$x_{n+1} = T_0(x_n - \tau A^*(I - T_1)Ax_n). \tag{1.4}$$

It has been shown that if the step size satisfies $0 < \tau < \frac{2}{\|A\|^2}$, then their iterative method converges weakly to a solution of the corresponding problem.

Recently, Reich, Truong, and Mai [19] proposed the following iterative method for this problem:

$$x_{n+1} = P_C\left[x_n - \tau \sum_{i=1}^{N} A_i^*(I - P_{Q_i})A_i x_n \right]. \tag{1.5}$$

It can be shown that method (1.5) converges weakly to a solution of problem (1.3) if τ is chosen such that:

$$0 < \tau < \frac{2}{N \max_{1 \le i \le N} \|A_i\|^2}. \tag{1.6}$$

In this paper, we aim to investigate the split common fixed point problem with multiple output sets (SCFPMOS) [20] within a more general framework. The SCFPMOS seeks to find $x^\dagger \in H$ that satisfies the following condition:

$$x^\dagger \in \mathrm{Fix}(T_0) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(\mathrm{Fix}(T_i)) \right). \tag{1.7}$$

In this work, we extend the corresponding nonlinear mapping from nonexpansive mappings to strictly pseudo-contractive mappings. Additionally, we extend the number of corresponding convex and closed sets from two to multiple output sets. Furthermore, we utilize the results obtained to solve the SFPMOS and develop two new iterative methods to approximate its solution.

2. Preliminaries

In what follows, "$\to$" stands for strong convergence, and "$\rightharpoonup$" weak convergence. Let now T be a mapping from H into itself.

Definition 2.1. [21] Suppose that there is $k<1$ so that, for each $x,y\in H$,

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2,$$

where I stands for the identity mapping. Then T is called firmly nonexpansive if $k=-1$; nonexpansive if $k=0$; k-strictly pseudo-contractive (k-spc) if $k\in(0,1)$.

Definition 2.2. [22] Suppose that $\mathrm{Fix}(T)\neq\varnothing$ and there is $k<1$ so that, for each $(x,z)\in H\times\mathrm{Fix}(T)$, $\|Tx-z\|^2\le\|x-z\|^2+k\|x-Tx\|^2$. Then T is called firmly quasi-nonexpansive if $k=-1$; quasi-nonexpansive if $k=0$; k-demicontractive if $k\in(0,1)$.

Strict pseudo-contractive mappings constitute a broad class of nonlinear mappings, encompassing nonexpansive and firmly nonexpansive mappings as special cases. However, it should be noted that the converse is not necessarily true, as demonstrated by the following example.

Example 2.1. Let $T:\ell_2\to\ell_2$ be a mapping defined by

$$Tx = 4x - \frac{3x}{\max(1,\|x\|)}.$$

It is easy to check that T is $\frac{1}{2}$-spc, but T is not nonexpansive. Indeed, set $x=(1,0,0,\cdots)$ and $y=(1,1,0,\cdots)$. A simple calculation shows that $\|Tx-Ty\|\approx 2.074>\|x-y\|=1$.

Lemma 2.1. *[21] Let T be k-spc. Then the following assertions hold.*

(1) $\mathrm{Fix}(T)$ *is closed and convex.*
(2) T *is* $(1+\sqrt{k})/(1-\sqrt{k})$*-Lipschitz continuous.*
(3) *For each* $(x,z)\in H\times\mathrm{Fix}(T)$*, it follows that*

$$\langle x-Tx, x-z\rangle \ge \frac{1-k}{2}\|x-Tx\|^2.$$

(4) T *is demiclosed at the origin, which means that, for any sequence* $\{x_n\}\subseteq H$ *and* $x^\dagger\in H$*, the following holds:*

$$\left.\begin{array}{c} x_n \rightharpoonup x^\dagger \\ x_n - Tx_n \to 0 \end{array}\right] \Longrightarrow Tx^\dagger = x^\dagger.$$

Suppose now that $C\subseteq H$ is a nonempty, convex and closed subset. A typical example of firmly nonexpansive mappings is the metric projection P_C from H onto C defined by

$$P_C x = \arg\min_{y\in C}\|x-y\|, x\in H.$$

Let $x\in H$. Then $y=P_Cx$ if and only if $y\in C$ and $\langle x-y, z-y\rangle\le 0$ for all $z\in C$.

Definition 2.3. [23] A sequence $\{x_n\}\subseteq H$ is said to be quasi Fejér monotone with regards to C if there is a real sequence such that $\sum_{n=0}^{\infty}\varepsilon_n<\infty$ and

$$\|x_{n+1}-z\|^2 \le \|x_n-z\|^2+\varepsilon_n, \quad \forall n\ge 0, \forall z\in C.$$

The following lemmas are very useful in the convergence analysis.

Lemma 2.2. *[23] Let $\{x_n\}$ be quasi Fejér monotone with regard to C. Then $\{x_n\}$ converges weakly to an element in C if and only if each of its weak cluster points belongs to C.*

Lemma 2.3. *[24] Suppose that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \rho_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$. Then the sequence $\{a_n\}$ converges to 0 provided that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \overline{\lim_{n\to\infty}}\, \delta_n \leq 0, \sum_{n=0}^{\infty} \rho_n < \infty.$$

Lemma 2.4. *[25] Suppose that $\{a_n\}$ and $\{\rho_n\}$ are two sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} \rho_n a_n < \infty, \sum_{n=0}^{\infty} \rho_n^2 < \infty$ and $\sum_{n=0}^{\infty} \rho_n = \infty$. If there is $M > 0$ such that $|a_{n+1} - a_n| \leq M\rho_n$, then $\{a_n\}$ converges to 0.*

3. Iterative Methods with Variable Step Sizes

For convenience, let us define Λ as the set $\{0,1,2,\ldots,N\}$ and assume that $k_i \in (0,1)$ for all $i \in \Lambda$. We denote by A_0 the identity mapping and $\mathscr{S}$ the solution set of the SCFPMOS (1.7). In our subsequent analysis, we make use of the following lemma.

Lemma 3.1. *Let $T_i : H_i \to H_i$ be k_i-spc for each $i \in \Lambda$. If the real sequence $\{\|\sum_{i=0}^{N} A_i^*(I - T_i)A_i x_n\|\}$ converges to 0, then each weak custer point of $\{x_n\}$ belongs to $\mathscr{S}$.*

Proof. Fix any $z \in \mathscr{S}$. By the property of strict pseudo-contractions, one has

$$\begin{aligned}\sum_{i=0}^{N}(1-k_i)\|A_i x_n - T_i(A_i x_n)\|^2 &\leq 2\sum_{i=0}^{N}\langle A_i x_n - T_i(A_i x_n), A_i x_n - A_i z\rangle \\ &= 2\sum_{i=0}^{N}\langle A_i^*(I-T_i)A_i x_n, x_n - z\rangle \\ &\leq 2\left\|\sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n\right\| \cdot \|x_n - z\|.\end{aligned}$$

Thus $\{\|A_i x_n - T_i(A_i x_n)\|\}$ converges to 0 for each $i \in \Lambda$. Let x^* be any weak cluster point of $\{x_n\}$ and choose a subsequence $\{x_{n_k}\}$ that converges weakly to x^*. Since A_i is linear, it is straightforward to verify that $\{A_i x_{n_k}\}$ converges weakly to $A_i x^*$. Moreover, since T_i satisfies the demiclosedness principle by our assumption, we conclude that $A_i x^* \in \mathrm{Fix}(T_i)$ for each $i \in \Lambda$. Therefore, any weak cluster point of $\{x_n\}$ belongs to solution set $\mathscr{S}$. □

Note that the selection of the step size in (1.1) requires the norm of the linear mapping, which could be difficult to obtain in practice. To overcome this challenge, Yang [26] proposed a novel iterative method to approximate a solution of the SFP. The iterative method is as follows:

$$x_{n+1} = P_C(x_n - \tau_n A^*(Ax_n - P_Q(Ax_n))), \tag{3.1}$$

where $\tau_n = \rho_n / \|A^*(Ax_n - P_Q(Ax_n))\|$ and $\{\rho_n\}$ is a positive real sequence satisfying

$$\sum_{n=0}^{\infty} \rho_n = \infty, \sum_{n=0}^{\infty} \rho_n^2 < \infty. \tag{3.2}$$

This selection of step sizes does not require the value of $\|A\|$. Moreover, if Q is bounded and A is of full column rank, then method (3.1) converges to a solution of the SFP. It is worth noting that the assumptions about A and Q could be completely removed [27]. Recently, method (3.1) has been extended to the SCFP (1.2) for firmly nonexpansive mappings [28].

As seen in (1.5), determining the value of $\max_{1\leq i\leq N}\|A_i\|$ in advance is often difficult in practice. To address this issue, we follow (3.1) and adopt a variable step size that is ultimately independent of $\max_{1\leq i\leq N}\|A_i\|$.

Algorithm 3.1. *Choose a real sequence $\{\rho_n\}$ and an initial guess $x_0\in H$. At each iteration, given the current iterate x_n, check if it satisfies the condition $\|\sum_{i=0}^N A_i^*(I-T_i)A_ix_n\|=0$. If it does, then we stop the method, as x_n is a solution to problem (1.7). Otherwise, we update the next iterate x_{n+1} as follows:*

$$x_{n+1}=x_n-\tau_n\sum_{i=0}^N A_i^*(I-T_i)A_ix_n, \tag{3.3}$$

where

$$\tau_n=\frac{\rho_n}{\|\sum_{i=0}^N A_i^*(I-T_i)A_ix_n\|}.$$

It is worth noting that $\{x_n\}$ is assumed to be infinite, as the method does not terminate finitely when condition $\|\sum_{i=0}^N A_i^*(I-T_i)A_ix_n\|=0$ is not satisfied.

Theorem 3.1. *Assume that T_i is k_i-spc for each $i\in\Lambda$, and that condition (3.2) is satisfied. If problem (1.7) is consistent, then the sequence $\{x_n\}$ generated by method 3.1 converges weakly to an element of $\mathscr{S}$.*

Proof. First, we prove that $\{x_n\}$ is quasi Fejér monotone. In fact, fix any $z\in\mathscr{S}$, and set $y_n=\sum_{i=0}^N A_i^*(I-T_i)A_ix_n$ and

$$\tau=\min_{0\leq i\leq N}\frac{1-k_i}{(N+1)\|A_i\|^2}.$$

It then follows from Lemma 2.1 that

$$\begin{aligned}2\langle y_n,x_n-z\rangle&=2\sum_{i=0}^N\langle (I-T_i)A_ix_n,A_ix_n-A_iz\rangle\geq\sum_{i=0}^N(1-k_i)\|(I-T_i)A_ix_n\|^2\\&\geq\sum_{i=0}^N\frac{1-k_i}{\|A_i\|^2}\|A_i^*(I-T_i)A_ix_n\|^2\geq(N+1)\tau\sum_{i=0}^N\|A_i^*(I-T_i)A_ix_n\|^2\\&\geq\tau\left(\sum_{i=0}^N\|A_i^*(I-T_i)A_ix_n\|\right)^2\geq\tau\left\|\sum_{i=0}^N A_i^*(I-T_i)A_ix_n\right\|^2.\end{aligned}$$

Hence $2\langle y_n,x_n-z\rangle\geq\tau\|y_n\|$. This combined with (3.3) yields

$$\begin{aligned}\|x_{n+1}-z\|^2&=\|x_n-z\|^2-2\tau_n\langle y_n,x_n-z\rangle+\tau_n^2\|u_n\|^2\\&\leq\|x_n-z\|^2-\tau\tau_n\|y_n\|+\tau_n^2\|u_n\|^2\\&=\|x_n-z\|^2-\tau\rho_n\|y_n\|+\rho_n^2.\end{aligned}\tag{3.4}$$

In particular, for all $n\geq 0$, we have

$$\|x_{n+1}-z\|^2\leq\|x_n-z\|^2+\rho_n^2,$$

so $\{x_n\}$ is quasi Fejér monotone with regard to $\mathscr{S}$.

Next, we show that any weak cluster point of the sequence $\{x_n\}$ is a solution to problem (1.7). In fact, according to formula (3.4), we deduce from the recurrence relation that

$$\sum_{n=0}^{\infty} \rho_n \|y_n\| < \infty. \tag{3.5}$$

Let $T_i' = I - T_i$. Then it is easy to check that T_i' is $2/(1-\sqrt{k_i})$-Lipschitz continuous. From this, it then follows that

$$\begin{aligned}
\big|\|y_{n+1}\| - \|y_n\|\big| &\le \left\| \sum_{i=0}^{N} A_i^*(T_i' A_i x_n - T_i' A_i x_{n+1}) \right\| \\
&\le \sum_{i=0}^{N} \|A_i^* T_i'(A_i x_n - A_i x_{n+1})\| \\
&\le \sum_{i=0}^{N} \|A_i^*\| \|T_i'(A_i x_n - A_i x_{n+1})\| \\
&\le \sum_{i=0}^{N} \frac{2\|A_i\|}{1-\sqrt{k_i}} \|A_i(x_n - x_{n+1})\| \\
&\le \sum_{i=0}^{N} \frac{2\|A_i\|^2}{1-\sqrt{k_i}} \|x_n - x_{n+1}\| \\
&= \left(\sum_{i=0}^{N} \frac{2\|A_i\|^2}{1-\sqrt{k_i}} \right) \rho_n.
\end{aligned}$$

Therefore, from Lemma 2.4, one has $\lim_n \|y_n\| = 0$. By Lemma 3.1, any weak cluster point of $\{x_n\}$ belongs to the solution set. We thus apply Lemma 2.2 to conclude that $\{x_n\}$ converges weakly to an element of $\mathscr{S}$. □

Remark 3.1. As a direct application, we present a new method for solving the SFPMOS. This method is given by the following formula:

$$x_{n+1} = x_n - \tau_n \sum_{i=0}^{N} A_i^*(I - P_{Q_i}) A_i x_n, \tag{3.6}$$

where

$$\tau_n = \frac{\rho_n}{\|\sum_{i=0}^{N} A_i^*(I-P_{Q_i})A_i x_n\|}.$$

This method is distinct from Reich, Truong, and Mai's method, and its step size selection does not depend on the value of $\max_{1\le i\le N} \|A_i\|$.

Remark 3.2. Consider the special case that $N = 1$ in (3.6). In this instance, we obtain a completely new method for solving the SFP. The method is given by the following formula:

$$x_{n+1} = x_n - \tau_n[(I - P_C)x_n + A^*(I - P_Q)Ax_n],$$

where $\tau_n = \rho_n \|(I - P_C)x_n + A^*(I - P_Q)Ax_n\|^{-1}$. This method is distinct from Yang's method, and its convergence does not require the condition that Q is bounded and A is of full column rank.

Remark 3.3. An important question to consider is whether the assumptions regarding strict pseudo-contractions can be relaxed to those of demicontractions. However, it can be observed from the above proof that Lipschitz continuity plays a critical role in the convergence analysis, while demicontractions are typically discontinuous.

As shown above, the sequence generated by our method exhibits only weak convergence in infinite dimensional Hilbert spaces. To ensure the strong convergence, we will need to modify the method.

Algorithm 3.2. *Choose a fixed element u, two real sequences* $\{\alpha_n\} \subseteq [0,1]$ *and* $\{\rho_n\} \subseteq (0,\infty)$, *and an arbitrary initial guess* $x_0 \in H$. *Given the current iteration* x_n, *if*

$$\left\| \sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n \right\| = 0,$$

then stop; otherwise, update the next iteration x_{n+1} *via*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau_n \sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n\right],$$

where

$$\tau_n = \frac{\rho_n}{\|\sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n\|}.$$

Theorem 3.2. *Assume that the parameters satisfy the following conditions:*

(c1) $\lim_n(\alpha_n/\rho_n) = 0$;
(c2) $\lim_n \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c3) $\sum_{n=0}^{\infty} \rho_n = \infty$ *and* $\sum_{n=0}^{\infty} \rho_n^2 < \infty$.

Under these conditions, if T_i *is* k_i*-spc for each* $i \in \Lambda$, *the sequence* $\{x_n\}$ *generated by method 3.2 converges strongly to z, which is the point in* $\mathscr{S}$ *nearest to the anchor u. Specifically,* $z = P_{\mathscr{S}}(u)$.

Proof. Set $y_n = \sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n$ and $z_n = x_n - \tau_n y_n$. Similarly, we can deduce that

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - \tau\rho_n\|y_n\| + \rho_n^2. \tag{3.7}$$

The definition of τ here is the same as the previous theorem.

In what follows we divide the proof into four steps.

Step 1. The sequence $\{x_n\}$ is bounded. To see this, we obtain from (3.7) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(u-z) + (1-\alpha_n)(z_n - z)\|^2 \\ &\le \alpha_n\|u-z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\le \alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 + \rho_n^2 \\ &\le \max\left\{\|x_n - z\|^2, \|u-z\|^2\right\} + \rho_n^2.\end{aligned}$$

By induction, we obtain

$$\|x_n - z\|^2 \le \max\left\{\|x_0 - z\|^2, \|u-z\|^2\right\} + \sum_{n=0}^{\infty} \rho_n^2$$

for all $n \ge 0$. Hence, by the condition (c3), $\{x_n\}$ is bounded.

Step 2. Set $a_n := \|x_n - z\|^2$ and

$$b_n := 2\langle u - z, x_{n+1} - z\rangle - \frac{\tau\rho_n}{\alpha_n}(1-\alpha_n)\|y_n\|.$$

Then there holds the inequality:

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n + \rho_n^2. \tag{3.8}$$

As a matter of fact, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1-\alpha_n)(z_n - z) + \alpha_n(u - z)\|^2 \\ &\le (1-\alpha_n)^2\|z_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\le (1-\alpha_n)\|z_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.9}$$

Substituting (3.7) into (3.9), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ - \tau(1-\alpha_n)\rho_n\|y_n\| + (1-\alpha_n)\rho_n^2, \end{aligned}$$

and (3.8) follows immediately.

Step 3. We claim that $\overline{\lim}_{n\to\infty} b_n \le 0$. First note that, since $\{b_n\}$ is bounded from above, $\overline{\lim}_n b_n$ is finite. Also from our conditions (c2) and (c3), it is easy to verify that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Based on this, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned} \overline{\lim_{n\to\infty}} b_n &= \lim_{k\to\infty} b_{n_k} \\ &= \lim_{k\to\infty}\left[2\langle u - z, x_{n_k+1} - z\rangle - \frac{\tau(1-\alpha_{n_k})}{\alpha_{n_k}}\rho_{n_k}\|y_{n_k}\|\right] \\ &= \lim_{k\to\infty}\left[2\langle u - z, x_{n_k} - z\rangle - \frac{\tau(1-\alpha_{n_k})}{\alpha_{n_k}}\rho_{n_k}\|y_{n_k}\|\right]. \end{aligned} \tag{3.10}$$

With no loss of generality, we may further assume that $\{x_{n_k}\}$ is weakly convergent to some point x^*. Thus

$$\lim_{k\to\infty}\langle u - z, x_{n_k} - z\rangle = \langle u - z, x^* - z\rangle. \tag{3.11}$$

A key ingredient of the proof of this step lies in proving that $x^* \in S$. To see this, we proceed as follows. Since $\alpha_{n_k} \to 0$, a consequence of (3.10) and (3.11) is that $\lim_{k\to\infty} \frac{\rho_{n_k}}{\alpha_{n_k}}\|y_{n_k}\|$ exists. In particular, $\{\frac{\rho_{n_k}}{\alpha_{n_k}}\|y_{n_k}\|\}$ is bounded. Therefore, using condition (c1), we have

$$\lim_{k\to\infty}\|y_{n_k}\| = \lim_{k\to\infty}\left(\frac{\alpha_{n_k}}{\rho_{n_k}}\cdot\frac{\rho_{n_k}}{\alpha_{n_k}}\|y_{n_k}\|\right) = 0.$$

That is

$$\lim_{k\to\infty}\left\|\sum_{i=0}^{N} A_i^*(I - T_i)A_i x_{n_k}\right\| = 0.$$

It then turns out from Lemma 3.1 that $x^* \in \mathscr{S}$. Combining (3.10) and (3.11) yields

$$\overline{\lim_{n\to\infty}} b_n \le \lim_{k\to\infty} 2\langle u - z, x_{n_k} - z\rangle = 2\langle u - P_{\mathscr{S}}u, x^* - P_{\mathscr{S}}u\rangle \le 0. \tag{3.12}$$

Step 4. We show the strong convergence of $\{x_n\}$. As a matter of fact, we thus apply Lemma 2.3 to (3.8) to obtain the desired result. □

Remark 3.4. The real sequences given below satisfy conditions (c1)-(c3):

$$\rho_n = 1/(n+1)^r,\ \alpha_n = 1/(n+1)^s,\ \frac{1}{2} < r < s \leq 1.$$

Remark 3.5. As a direct application, we present a new method for solving the SFPMOS. This method is given by the following formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau_n \sum_{i=0}^{N} A_i^*(I - P_{Q_i})A_i x_n\right],$$

where $\tau_n = \rho_n / \|\sum_{i=0}^{N} A_i^*(I-P_{Q_i})A_i x_n\|$. This method is distinct from Reich, Truong, and Mai's method, and its step size selection does not depend on the value of $\max_{1\leq i\leq N}\|A_i\|$.

4. A Numerical Experiment

In this section, we explore numerical experiments that demonstrate the practical applications of the proposed algorithm to inverse problems in signal processing. Compressed sensing is a very active domain of research and applications, based on the fact that an N-sample signal x with exactly m nonzero components can be recovered from $m \ll k < N$ measurements as long as the number of measurements is smaller than the number of signal samples and at the same time much larger than the sparsity level of x. Likewise the measurements are required to be incoherent, which means that the information contained in the signal is spread out in the domain. Since $k < N$, the problem of recovering x from k measurements is ill conditioned because we encounter an underdeterminated system of linear equations. With a sparsity prior, it turns out that reconstructing x from y is possible as long as the number of nonzero elements is small enough.

To formulate compressed sensing, we begin with the equation system $y_i = A_i x + \varepsilon$ for $i = 1,2,\ldots,10$, where $x \in \mathbb{R}^N$ is the signal that we wish to recover, $y_i \in \mathbb{R}^k$ is the vector of noisy measurements, and ε represents the noise. However, the linear observation operator $A_i : \mathbb{R}^N \to \mathbb{R}^k$ is often ill-conditioned due to the loss of information in the measurement process. To overcome this challenge, we adopt a sparse representation of the signal, which assumes that the signal can be represented by a series expansion with respect to an orthonormal basis that has only a small number of large coefficients. By leveraging convex analysis, we can formulate the compressed sensing problem as a particular SFPMOS, where $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q_i = \{y_i\}$. Our proposed method can solve this problem efficiently. In this case, P_C has a closed form and is nothing but the projection onto the closed ℓ_1 ball in $\mathbb{R}^N$ (see [29]). To perform the experiment, we start with an initial signal $x_0 = 0$ and run 300 iterations. We measure the restoration accuracy using the mean squared error: $\mathrm{MSE} = (1/N)\|x^* - x\|^2$, where x^* is the estimated signal of x.

In Figure 1, we present three plots: the top one demonstrates the true signal, while the second and third plots demonstrate the signal reconstructed using Reich, Truong, and Mai's method and our method, respectively. For Reich, Truong, and Mai's method, we used a stepsize of

$\tau_n \equiv (10\max_i \|A_i\|^2)^{-1}$, while for our method, we used a stepsize of

$$\tau_n = \frac{1}{(n+1)\|\sum_{i=0}^{N} A_i^*(I-P_{Q_i})A_i x_n\|}.$$

As the figure demonstrates, our method provides a relatively accurate estimate of the signal x, while requiring the shortest CPU time compared to Reich, Truong, and Mai's method. This difference in efficiency becomes more pronounced as the dimensionality of N increases. Overall, our results suggest that our method is a promising approach for problem (1.7).

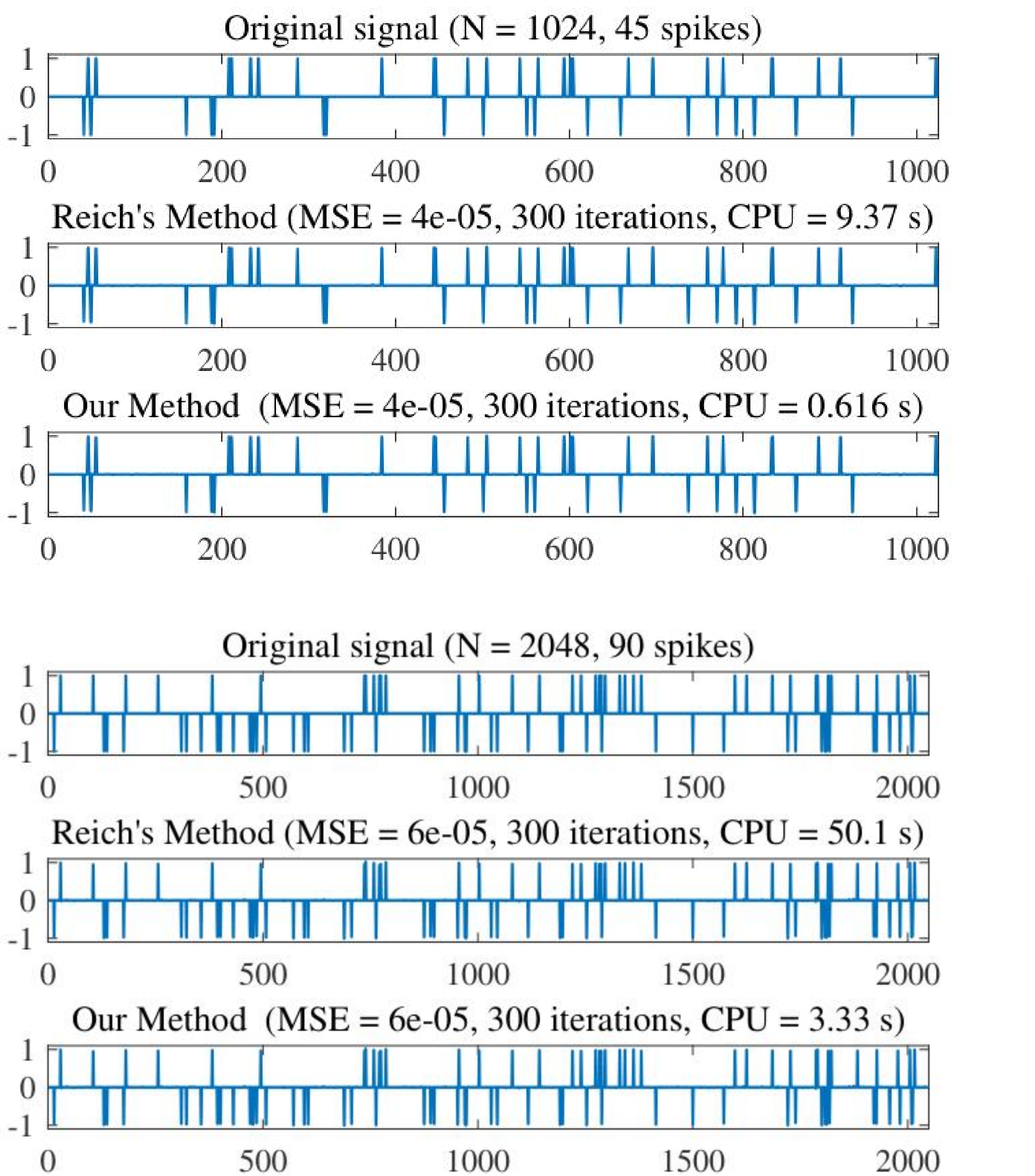

FIGURE 1. Numerical Results on Different Choices of N

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REFERENCES

[1] Y. Censor T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algo. 8 (1994), 221-239.
[2] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2003), 103-120.
[3] R. Li, Local gap density for clustering high-dimensional data with varying densities, Knowledge-based Sys. 184 (2019), 104905.
[4] Y. Censor, S. Petra, Christoph Schnörr, Superiorization vs. accelerated convex optimization: The superiorized / regularized least-squares case, J. Appl. Numer. Optim. 2 (2020), 15-62.
[5] T. Huang, M. Li, X. Qin, W. Zhu, A CNN-based policy for optimizing continuous action control by learning state sequences, Neurocomputing, 468 (2022), 286-295.
[6] T. Humphries, M. Loreto, B. Halter, W. O'Keeffe, L. Ramirez, Comparison of regularized and superiorized methods for tomographic image reconstruction, J. Appl. Numer. Optim. 2 (2020), 77-99.
[7] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51 (2006), Article ID 2353.
[8] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, Inverse Probl. 18 (2002), 441-453.
[9] H.-K. Xu, A variable Krasnoselskii–Mann algorithm and the multiple-set split feasibility problem, Inverse Problems, 22 (2006), 2021-2034.
[10] H.-K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Probl. 26 (2010), Article ID 105018.
[11] H.-K. Xu, A. Cegielski, The landweber operator approach to the split equality problem, SIAM J. Optim. 31 (2021), 626-652.
[12] X. Qin, J.C. Yao, Projection splitting algorithms for nonself operators, J. Nonlinear Convex Anal. 18 (2017), 925–935.
[13] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018), 157-165.
[14] A. Moudafi, The split common fixed-point problem for demicontractive mappings, Inverse Probl. 26 (2010), 055007.
[15] H. Cui, L. Ceng, Iterative solutions of the split common fixed point problem for strictly pseudo-contractive mappings, J. Fixed Point Theory Appl. 20, (2018), 1-12.
[16] G. López, V. Márquez, F. Wang, H.-K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, Inverse Probl. 28 (2012), Article ID 085004.
[17] X. Qin, L. Wang, J.C. Yao, Inertial splitting method for maximal monotone mappings, J. Nonlinear Convex Anal. 21 (2020), 2325-2333.
[18] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009), 587-600.
[19] S. Reich, M. Truong, T. Mai. The split feasibility problem with multiple output sets in Hilbert spaces, Optim. Lett., 14 (2020), 2335-2353.
[20] J. Kim, T. Tuyen, M. Ha, Two projection methods for solving the split common fixed point problem with multiple output sets in Hilbert spaces, Numer. Funct. Anal. Optim. 42 (2021), 973-988.
[21] G. Marino, H.-K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 336-346.

[22] S. Maruster, C. Popirlan, On the Mann-type iteration and the convex feasibility problem, J. Comput. Appl. Math. 212 (2008), 390-396.

[23] H. Bauschke, Patrick L Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, Berlin, 2011.

[24] H.K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66 (2002), 240-256.

[25] H. Yu, F. Wang, Alternating iterative algorithms for the split equality problem without prior knowledge of operator norms, J. Inequal. Appl. 2020 (2020), Article ID 214.

[26] Q. Yang, On variable-step relaxed projection algorithm for variational inequalities, J. Math. Anal. Appl. 302 (2005), 166-179.

[27] F. Wang, Polyak's gradient method for split feasibility problem constrained by level sets, Numer. Algor. 77 (2018), 925-938.

[28] F. Wang, W. Cao, Z. Xu, Convergence of multi-block bregman ADMM for nonconvex composite problems, Sci. Chin. Info. Sci. 61 (2018), Article ID 122101.

[29] Y. Singer, J.C. Duchi, S. Shalevshwartz, Efficient projections onto the ℓ_1 -ball for learning in high dimensions, International Conference on Machine Learning, pp. 272–279, 2014.