J. Nonlinear Var. Anal. 7 (2023), No. 6, pp. 957-970
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.6.05

ON A BOUNDARY VALUE PROBLEM FOR HALE TYPE FRACTIONAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS WITH CAUSAL MULTIOPERATORS IN A BANACH SPACE

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Abstract. We prove the existence of mild solutions to a nonlocal boundary value problem for fractional-functional differential inclusions of the Hale type in a separable Banach space. We assume that the linear part of an inclusion is an infinitesimal generator of a bounded C_0-semigroup of linear operators, and the nonlinear part is a causal multivalued operator.

Keywords. Caputo fractional derivative; Causal multivalued operator; Condensing multivalued map; Fractional functional differential inclusion; Measure of noncompactness; Topological degree.

1. INTRODUCTION

In the last decades, the theory of fractional calculus has become one of the most popular and important areas of contemporary mathematics. Interest in this topic has increased as the result of the fact that numerous modern problems of science and technology find a fairly adequate description in terms of fractional differential equations and inclusions. Many physical, economic, biological and engineering problems, primarily related to the evolution of processes in dynamical systems, lead to the necessity of investigation of boundary value problems for fractional differential equations and inclusions (see, e.g., [1, 2, 3, 4, 5]). In recent years, the study of the whole complex of problems related to differential equations and inclusions of fractional order has been very intensively carried out by many researchers (see, e.g., [6, 7, 8, 9, 10, 11, 12]). Fractional functional differential and integro-differential inclusions of Hale type occupy an intermediate position between functional differential inclusions with delay and neutral type inclusions. The monograph of Hale [13] is devoted to the corresponding equations in the finite-dimensional case. For the infinite-dimensional case, this theory was developed by Ilolov et al. in [14, 15, 16, 17, 18].

Recently, the attention of many researchers (see the monograph [19] and references therein) has been attracted to the generalizations of differential equations and inclusions to the class

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Received January 5, 2023; Accepted April 1, 2023.

of functional equations and inclusions with causal operators. The term of a causal or Volterra operator in the sense of Tikhonov (see [20]) was used in mathematical physics to solve problems of differential equations, integro-differential equations, functional differential equations with a finite or infinite delay, integral equations of Volterra type, functional equations of a neutral type, etc. The results presented in [21, 22, 23, 24, 25, 26, 27, 28] are devoted to the study of equations and inclusions with causal operators of various types, the justification of the existence of solutions, the description of qualitative properties of solutions and a number of applications.

In this paper, we prove the existence of mild solutions to a nonlocal boundary value problem for fractional differential inclusions of the Hale type. We assume that the linear part of an inclusion is an infinitesimal generator of a bounded C_0-semigroup of linear operators, and the nonlinear part is a causal multivalued operator. The paper is organized in the following way. In the preliminaries section, we collect the basic concepts and statements from the fractional calculus, theory of multivalued maps and measures of noncompactness, and the Hale-Kato phase space. In the third section, we define the notion of a causal multivalued operator, and present examples and necessary properties. In the forth section, we introduce the resolving operator for the considered boundary value problem and study its properties. In the last section, the concluding section, by using the fixed point theory for condensing multivalued maps, we present the existence result.

2. Preliminaries

2.1. **The fractional integral and Caputo fractional derivative.** For the considering of the main problem, we need the following notions from fractional calculus (see, e.g., monographs [2, 3]).

Definition 2.1. The fractional integral of an order $q > 0$ of a function $g : [0,T] \to E$ is the function $I_0^q g$ of the following form:

$$I_0^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)\, ds,$$

where Γ is the Euler gamma function

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x} dx.$$

Definition 2.2. The Caputo fractional derivative of an order $q \geq 0$ of a function $g \in C^n([0,T];E)$ is the function ${}^C D_0^q g$ of the following form:

$${}^C D_0^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds,$$

where n and q are related by equality $n = [q] + 1$.

2.2. **Multivalued maps and measures of noncompactness.** Let X be a metric space and Y a normed space. We introduce the following notation:

$P(Y)$ denotes the collection of all non-empty subsets of Y;
$Pb(Y)$ denotes the collection of all non-empty and bounded subsets of Y;
$C(Y)$ denotes the collection of all non-empty and closed subsets of Y;
$Cv(Y)$ denotes the collection of all non-empty, closed and convex subsets of Y;
$K(Y)$ denotes the collection of all non-empty and compact subsets of Y;

$Kv(Y)$ denotes the collection of all non-empty, compact and convex subsets of Y.

Let us recall some notions (see, e.g., [29], [30]).

Definition 2.3. A multivalued map (multimap) $\mathscr{F} : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) at a point $x \in X$ if, for every open set $V \subset Y$ such that $\mathscr{F}(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $\mathscr{F}(U(x)) \subset V$.

Definition 2.4. A multivalued map $\mathscr{F} : X \to P(Y)$ is called closed if its graph $G_{\mathscr{F}} = \{(x,y) : x \in X, y \in \mathscr{F}(x)\}$ is a closed subset of $X \times Y$.

Definition 2.5. For a given $p \geq 1$, a multifunction $G : [0,T] \to K(Y)$ is called:

- L^p-integrable *if it admits an* L^p*–Bochner integrable selection, i.e., there exists a function* $g \in L^p([0,T];Y)$ *such that* $g(t) \in G(t)$ *for a.e.* $t \in [0,T]$;
- L^p-integrably bounded *if there exists a function* $\xi \in L^p([0,T])$ *such that* $\|G(t)\| \leq \xi(t)$ *for a.e.* $t \in [0,T]$.

Let $\mathscr{E}$ be a Banach space.

Definition 2.6. A sequence of functions $\{\xi_n\} \subset L^p([0,T];\mathscr{E}), p \geq 1$, is called L^p–semicompact if it is L^p–integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in $\mathscr{E}$ for a.e. $t \in [0,T]$.

Definition 2.7. Let $(\mathscr{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathscr{E}) \to \mathscr{A}$ is called the measure of noncompactness (MNC) in $\mathscr{E}$ if, for each $\Omega \in Pb(\mathscr{E})$, $\beta(\overline{\text{co}}\,\Omega) = \beta(\Omega)$, where $\overline{\text{co}}\,\Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone* if, for each $\Omega_0, \Omega_1 \in Pb(\mathscr{E})$, $\beta(\Omega_0) \leq \beta(\Omega_1)$ for $\Omega_0 \subseteq \Omega_1$;
2) *nonsingular* if, for each $a \in E$ and each $\Omega \in Pb(\mathscr{E})$, $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathscr{A}$ is a cone in a Banach space, the MNC β is called:

3) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathscr{E})$;
4) *real* if $\mathscr{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

As the example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathscr{E}\}.$$

As other examples, we consider the measures of noncompactness defined in the space of continuous functions $C([a,b];E)$ with values in a Banach space E:

(1) *the modulus of fiber noncompactness:*

$$\varphi(\Omega) = \sup_{t \in [a,b]} \chi_E(\Omega(t)),$$

where χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(2) *the fading modulus of fiber noncompactness:*

$$\gamma(\Omega) = \sup_{t \in [a,b]} e^{-Lt} \chi_E(\Omega(t)),$$

where $L > 0$ is a given number;

(3) *the modulus of equicontinuity:*

$$\operatorname{mod}_C(\Omega) = \lim_{\delta \to 0} \sup_{y \in \Omega} \max_{|t_1 - t_2| \le \delta} \|y(t_1) - y(t_2)\|.$$

These measures of noncompactness satisfy all the above properties, except for the regularity.

Definition 2.8. A multimap $\mathscr{F} : X \subseteq \mathscr{E} \to K(\mathscr{E})$ is called condensing with respect to a MNC β (or β–condensing) if, for each bounded set $\Omega \subseteq X$ which is not relatively compact, $\beta(F(\Omega)) \not\geq \beta(\Omega)$.

Let $\mathscr{D} \subset \mathscr{E}$ be a non-empty closed convex subset, V be a non-empty bounded open subset of $\mathscr{D}$, β be a monotone nonsingular MNC in $\mathscr{E}$, and $\mathscr{F} : \overline{V} \to Kv(\mathscr{D})$ be a u.s.c. β-condensing map such that $x \notin \mathscr{F}(x)$ for all $x \in \partial V$, where $\overline{V}$ and ∂V denote the closure and the boundary of the set V in the relative topology of $\mathscr{D}$.

In such a setting, the (relative) topological degree

$$\deg_{\mathscr{D}}\left(i - \mathscr{F}, \overline{V}\right)$$

of the corresponding vector field $i - \mathscr{F}$, satisfying the standard properties is defined (see, for example, [29] and [30]). In particular, the condition

$$\deg_{\mathscr{D}}\left(i - \mathscr{F}, \overline{V}\right) \neq 0$$

implies that the fixed points set $Fix\mathscr{F} = \{x : x \in \mathscr{F}(x)\}$ is a nonempty subset of V.

The application of topological degree theory leads to the following fixed point principles, which will be used in the sequel.

Theorem 2.1. *([29], Theorem 3.3.4). Let $V \subset \mathscr{D}$ be a bounded open neighborhood of a point $a \in V$ and $\mathscr{F} : \overline{V} \to Kv(\mathscr{D})$ a u.s.c. β-condensing multimap, where β is a monotone nonsingular MNC in $\mathscr{E}$, satisfying the boundary condition*

$$x - a \notin \lambda(\mathscr{F}(x) - a)$$

for all $x \in \partial V$ and $0 < \lambda \le 1$. Then $Fix\mathscr{F} \neq \emptyset$ is a non-empty compact set.

2.3. **Phase space.** We will use the axiomatic definition of the phase space $\mathscr{B}$, introduced by Hale and Kato (see [31] and [32]). The space $\mathscr{B}$ will be considered as a linear topological space of functions defined on $(-\infty, 0]$ with values in a Banach space E endowed with the seminorm $\|\cdot\|_{\mathscr{B}}$. For all function x: $(-\infty, T] \to E$, where $T > 0$, and every $t \in (-\infty, T]$, x_t describes the prehistory and is defined as

$$x_t(\theta) = x(t + \theta), \ \theta \in (-\infty, 0].$$

We will assume that $\mathscr{B}$ satisfies the following axioms:

($\mathscr{B}1$) if a function $x : (-\infty; T] \to E$ is continuous on $[0; T]$ and $x_0 \in \mathscr{B}$, then, for each $t \in [0; T]$,
- (i) $x_t \in \mathscr{B}$;
- (ii) the function $t \mapsto x_t$ is continuous;
- (iii) $\|x_t\|_{\mathscr{B}} \le K(t) \sup_{0 \le \tau \le t} \|x(\tau)\| + H(t)\|x_0\|_{\mathscr{B}}$, where the functions $K, H : [0; \infty) \to [0; \infty)$ are independent of x, K is strictly positive and continuous, and H is locally bounded.

($\mathscr{B}2$) there exists $l > 0$ such that $\|\psi(0)\|_E \le l \|\psi\|_{\mathscr{B}}$ for all $\psi \in \mathscr{B}$.

Notice that under these conditions the space C_{00} of all continuous functions with compact support mapping $(-\infty,0]$ to E is a subset of phase space $\mathscr{B}$ ([32, Proposition 1.2.1]).

In addition, we will assume that the following condition is satisfied:

($\mathscr{BC}$1) if a uniformly bounded sequence $\{\psi_n\}_{n=1}^{+\infty} \subset C_{00}$ converges to a function ψ compactly (i.e. uniformly on each compact subset $(-\infty,0]$), then $\psi \in \mathscr{B}$ and $\lim\limits_{n\to+\infty} \|\psi_n - \psi\|_{\mathscr{B}} = 0$.

The condition ($\mathscr{BC}$1) implies that the Banach space of bounded continuous functions $BC = BC((-\infty,0];E)$ is continuously embedded into $\mathscr{B}$. More precisely, the following assertion is true.

Theorem 2.2. *(*[32, Proposition 7.1.1]*).*

- (i) *$BC \subset \overline{C_{00}}$, where $\overline{C_{00}}$ denote the closure of C_{00} in $\mathscr{B}$;*
- (ii) *if a uniformly bounded sequence $\{\psi_n\}$ in BC converges to a function ψ compactly on $(-\infty,0]$, then $\psi \in \mathscr{B}$ and $\lim\limits_{n\to+\infty} \|\psi_n - \psi\|_{\mathscr{B}} = 0$;*
- (iii) *there exists $L > 0$ such that $\|\psi\|_{\mathscr{B}} \le L\|\psi\|_{BC}$ for all $\psi \in BC$.*

Finally, we assume that the following condition is satisfied:

($\mathscr{BC}$2) if $\psi \in BC$ and $\|\psi\|_{BC} \neq 0$, then $\|\psi\|_{\mathscr{B}} \neq 0$.

This assumption implies that BC, endowed with $\|\cdot\|_{\mathscr{B}}$, is a normed space. We denote it by $\mathscr{BC}$.

We may consider the following examples of phase spaces satisfying all the above properties:

(1) for $\gamma > 0$ let $\mathscr{B} = C_\gamma$ be the space of continuous functions $\varphi : (-\infty;0] \to E$ for which there exists $\lim\limits_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta)$ and

$$\|\varphi\|_{\mathscr{B}} = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}\|\varphi(\theta)\|.$$

(2) (Spaces of "fading memory") Let $\mathscr{B} = C_\rho$ be the space of functions $\varphi : (-\infty;0] \to E$ such that

   (a) φ is continuous on $[-r;0], r > 0$;

   (b) φ is Lebesgue measurable on $(-\infty;r)$ and there exists a nonnegative Lebesgue integrable function $\rho : (-\infty;-r) \to \mathbb{R}^+$ such that $\rho\varphi$ is Lebesgue integrable on $(-\infty;r)$. Moreover, there exists a locally bounded function $P : (-\infty;0] \to \mathbb{R}^+$ such that, for all $\xi \le 0$, $\rho(\xi+\theta) \le P(\xi)\rho(\theta)$ a.e. $\theta \in (-\infty;-r)$.

   Then

$$\|\varphi\|_{\mathscr{B}} = \sup_{-r\le\theta\le 0} \|\varphi(\theta)\| + \int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|d\theta.$$

A simple example of such space is given by taking $\rho(\theta) = e^{\mu\theta}$, $\mu \in \mathbb{R}$.

3. Causal Multioperators

Let E be a separable Banach space. By $L^p([0,T];E)$, $1 \le p \le \infty$, we denote the Banach space of all Bochner p-summable functions $f : [0,T] \to E$ with the usual norm. For each subset $\mathscr{N} \subset L^p([0,T];E)$ and $\tau \in (0,T)$ we define restriction $\mathscr{N}$ on $[0,\tau]$ as

$$\mathscr{N}\mid_{[0,\tau]} = \{f\mid_{[0,\tau]} : f \in \mathscr{N}\}.$$

We denote by $\mathscr{C}((-\infty;T];E)$ the normed space of bounded continuous functions $x:(-\infty;T]\to E$ such that $x\mid_{(-\infty;0]}=x_0\in\mathscr{BC}$, which is endowed with the norm

$$\|x\|_{\mathscr{C}}=\|x_0\|_{\mathscr{BC}}+\|x\mid_{[0;T]}\|_C.$$

Definition 3.1. A multivalued map $\mathscr{Q}:\mathscr{C}((-\infty,T];E)\multimap L^p([0,T];E)$ is said to be a causal multioperator if, for each $\tau\in(0,T)$ and for every $u(\cdot),v(\cdot)\in\mathscr{C}((-\infty,T];E)$, the condition $u\mid_{(-\infty,\tau]}=v\mid_{(-\infty,\tau]}$ implies that $\mathscr{Q}(u)\mid_{[0,\tau]}=\mathscr{Q}(v)\mid_{[0,\tau]}$.

Let us give examples of causal multioperators.

Example 3.1. We assume that the multimap $F:[0,T]\times\mathscr{BC}\times E\to Kv(E)$ satisfies the following conditions:

- $(F1)$ for each $(\psi,x)\in\mathscr{BC}\times E$, $F(\cdot,\psi,x):[0,T]\to Kv(E)$ admits a measurable selection;
- $(F2)$ for a.e. $t\in[0,T]$, $F(t,\cdot,\cdot):\mathscr{BC}\times E\to Kv(E)$ is u.s.c.;
- $(F3)$ there exists a function $\alpha\in L^p_+[0,T]$, $1\le p\le\infty$, such that

$$\|F(t,\psi,x)\|_E:=\sup\{\|z\|_E:z\in F(t,\psi,x)\}\le\alpha(t)(1+\|\psi\|_{\mathscr{BC}}+\|x\|_E)$$

  for a.e. $t\in[0,T]$ and $(\psi,x)\in\mathscr{BC}\times E$.

From $(F1)$-$(F3)$ and $(\mathscr{B}1)$, it follows that the multimap $\mathscr{P}_F:\mathscr{C}((-\infty;T];E)\to P(L^p([0,T];E))$ as

$$\mathscr{P}_F(x)=\{f\in L^p([0,T];E):f(t)\in F(t,x_t,x(t))\text{ a.e. }t\in[0,T]\}$$

is well defined (see, e.g., [29] and [30]). It is clear that the multioperator $\mathscr{P}_F$ is causal.

Example 3.2. Let $F:[0,T]\times\mathscr{BC}\to Kv(E)$ be a multimap satisfying conditions $(F1)-(F3)$ from Example 3.1. Suppose that $\{K(t,s):0\le s\le t\le T\}$ is a continuous (with respect to the corresponding norm) family of bounded linear operators in E and $m\in L^1([0,T];E)$ is a given function. Consider the Volterra integral multioperator $\mathscr{V}:\mathscr{C}((-\infty,T];E)\multimap L^1([0,T];E)$ defined as

$$\mathscr{V}(u)(t)=m(t)+\int_0^t K(t,s)F(s,u_s)ds,$$

i.e.,

$$\mathscr{V}(u)=\{y\in L^1([0,T];E):y(t)=m(t)+\int_0^t K(t,s)f(s)ds:f\in\mathscr{P}_F(u)\}.$$

It is also clear that the multioperator $\mathscr{V}$ is causal.

We will assume that the causal multioperator $\mathscr{Q}:\mathscr{C}((-\infty,T];E)\multimap C(L^p([0,T];E))$ satisfies the following conditions:

- $(\mathscr{Q}1)$ $\mathscr{Q}$ is weakly closed in the following sense: conditions $\{u_n\}_{n=1}^\infty\subset\mathscr{C}((-\infty,T];E)$, $\{f_n\}_{n=1}^\infty\subset L^p([0,T];E)$, $1\le p\le\infty$, $f_n\in\mathscr{Q}(u_n)$, $n\ge1$, $u_n\to u_0$, $f_n\xrightarrow{L^1}f_0$ implies $f_0\in\mathscr{Q}(u_0)$;
- $(\mathscr{Q}2)$ there exists a function $\alpha\in L^\infty_+([0,T])$ such that

$$\|\mathscr{Q}(u)(t)\|_E\le\alpha(t)(1+\|u\|_{\mathscr{C}}),\quad\text{for a.e. }t\in[0,T],$$

  for all $u\in\mathscr{C}((-\infty,T];E)$;
- $(\mathscr{Q}3)$ there exists a function $\omega:[0,T]\times\mathbb{R}_+\to\mathbb{R}_+$ such that
  - $(\omega1)$ for all $x\in\mathbb{R}_+:\omega(\cdot,x)\in L^p_+([0,T])$, $1\le p\le\infty$;

(ω2) for a.e. $t \in [0,T]$ a function $\omega(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing, and quasihomogeneous in the sense that $\omega(t,\lambda x) \leq \lambda \omega(t,x)$ for all $x \in \mathbb{R}_+$ and $\lambda \geq 0$;

(ω3) for each bounded set $\Delta \subset \mathscr{C}((-\infty,T];E)$ we have

$$\chi(\mathscr{Q}(\Delta)(t)) \leq \omega\left(t, \sup_{s\in[0,t]} \varphi(\Delta_s)\right) \text{ for a.e. } t \in [0,T],$$

where the set $\Delta_s = \{y_s : y \in \Delta\} \subset \mathscr{BC}$ and φ is the modulus of fiber noncompactness in $\mathscr{BC}$.

Notice that condition (ω2) means that $\omega(t,0) = 0$ for a.e. $t \in [0,T]$ and as an example of such a function we can consider $\omega(t,x) = k(t)\cdot x$, where $k(\cdot) \in L^p_+([0,T])$.

Consider a linear operator $\mathscr{S} : L^p([0,T];E) \to C([0,T];E)$, which is causal in the sense that for every $\tau \in (0,T]$ and $f,g \in L^p([0,T];E)$ condition $f(t) = g(t)$ for a.e. $t \in [0,\tau]$ implies $(\mathscr{S}f)(t) = (\mathscr{S}g)(t)$ for all $t \in [0,\tau]$. Following [29], we impose the following conditions on operator $\mathscr{S}$:

($\mathscr{S}$1) for $1 \leq p < \infty$, there exist $D \geq 0$ such that

$$\|\mathscr{S}f(t) - \mathscr{S}g(t)\|^p_E \leq D\int_0^t \|f(s)-g(s)\|^p_E ds$$

for all $f,g \in L^p([0,T];E)$ and $0 \leq t \leq T$;

if $p = \infty$, then there exist $D_1 \geq 0$ such that

$$\|\mathscr{S}f(t) - \mathscr{S}g(t)\|_E \leq D_1\int_0^t \|f(s)-g(s)\|_E ds$$

for all $f,g \in L^\infty([0,T];E)$ and $0 \leq t \leq T$.

($\mathscr{S}$2) for an arbitrary compact set $K \subset E$ and a sequence $\{f_n\}_{n=1}^\infty \subset L^p([0,T];E)$, $1 \leq p \leq \infty$, such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for almost all $t \in [0,T]$, the weak convergence $f_n \overset{L^1}{\rightharpoonup} f_0$ implies $\mathscr{S}f_n \to \mathscr{S}f_0$ in $C([0,T];E)$.

Also we suppose that $\mathscr{S}$ satisfies the relation:

($\mathscr{S}$3) $(\mathscr{S}f)(0) = 0$ for each function $f \in L^p([0,T];E)$.

Notice that condition ($\mathscr{S}$1) implies that $\mathscr{S}$ satisfies the Lipschitz condition:

($\mathscr{S}$1′) $\|\mathscr{S}f - \mathscr{S}g\|_C \leq D\|f-g\|_{L^1}$.

Consider the following important examples.

(i) Let a closed linear operator $A : D(A) \subset E \to E$ be the infinitesimal generator of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$. The operator $\mathscr{L} : L^1([0,T];E) \to C([0,T];E)$ defined as

$$\mathscr{L}f(t) = \int_0^t e^{A(t-s)} f(s)ds$$

is a special case of the causal operator $\mathscr{S}$.

Taking $A = 0$, we obtain, in a particular, the usual integral operator $\mathscr{L}_I : L^1([0,T];E) \to C([0,T];E)$,

$$\mathscr{L}_I f(t) = \int_0^t f(s)ds.$$

(ii) Let $A : D(A) \to E$ be a closed linear operator E generating a C_0-semigroup $\{U(t)\}_{t\geq 0}$. The operator $G : L^p([0,T];E) \to C([0,T];E)$, $p > 1/q$, defined as

$$Gf(t) = \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds, \quad 0 < q < 1, \tag{3.1}$$

where

$$\mathscr{T}(t) = q\int_0^\infty \theta\xi_q(\theta)U(t^q\theta)d\theta, \tag{3.2}$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}),$$

and

$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in \mathbb{R}^+,$$

is a special case of the causal operator $\mathscr{S}$.

Lemma 3.1. *(*[29, Lemma 4.2.1], [33, Lemma 3.4]*). The operators $\mathscr{L}$ and G satisfy conditions* $(\mathscr{S}1)-(\mathscr{S}3)$.

4. The Resolving Multioperator and Its Properties

We consider in a separable Banach spaces E a system governed by a Hale type fractional functional-differential inclusion of the following form:

$$^C D_0^q[y(t) - k(t,y_t)] \in Ay(t) + \mathscr{Q}(y)(t), \ t \in [0,T], \tag{4.1}$$

$$y(\tau) + g(\widetilde{y})(\tau) = \upsilon(\tau), \ \tau \in (-\infty, 0], \tag{4.2}$$

where $^C D_0^q$ is the Caputo fractional derivative of an order $0 < q < 1, A : D(A) \subset E \to E$ is a closed linear (not necessariliy bounded) operator, $k : [0,T] \times \mathscr{BC} \to E$, $\upsilon \in \mathscr{BC}$ are given functions, $g : C([0,T];E) \to \mathscr{BC}$ is a nonlinear map, and $\widetilde{y} = y\mid_{[0;T]}$. Denote by $C = C([0,T];\mathbb{R})$ and $CE = C([0,T];E)$. We will suppose that the multioperator $\mathscr{Q} : \mathscr{C}((-\infty,T];E) \multimap C(L^p([0,T];E))$ in problem (4.1)-(4.2) is causal, satisfies conditions $(\mathscr{Q}1)-(\mathscr{Q}3)$, and the following assumptions hold true.

(A) The operator $A : D(A) \subset E \to E$ is an infinitesimal generator of a bounded C_0-semigroup $\{U(t)\}_{t\geq 0}$ of linear operators in E. Denote

$$M = \sup\{\|U(t)\|; t \geq 0\}.$$

$(K1)$ The function $k : [0,T] \times \mathscr{BC} \to E$ is completely continuous, and for every bounded set $\mathscr{N} \subset \mathscr{C}((-\infty,T];E)$ the set of functions $\{t \to k(t,y_t) : y \in \mathscr{N}\}$ is equicontinuous in the space CE.

$(K2)$ There exists a continuous function $b_1 : [0,T] \to \mathbb{R}_+, b_1(0) = 0$, and a constant $b_2 > 0$ such that

$$\|k(t,\phi)\|_E \leq b_1(t)\|\phi\|_{\mathscr{BC}} + b_2, \ t \in [0,T], \ \phi \in \mathscr{BC}.$$

We impose the following conditions on the map g :

$(g1)$ $g : CE \to \mathscr{BC}$ is completely continuous;

$(g2)$ there exist constants $N_1, N_2 \geq 0$ such that

$$\|g(z)\|_{\mathscr{BC}} \leq N_1\|z\|_{CE} + N_2 \ \textit{and} \ \|g(z)(0)\|_E \leq N_2,$$

for each $z \in CE$;

$(g3)$ g is an affine operator in the following sense: for every $z_1, z_2 \in CE$ and each $0 \le \lambda \le 1$,
$$g(\lambda z_1 + (1-\lambda)z_2) = \lambda g(z_1) + (1-\lambda)g(z_2).$$

As an example of g, the following function can be considered:
$$g(z)(\tau) = \sum_{i=1}^{k} c_i(\tau) z(t_i),$$
where $z \in CE$, $c_i : (-\infty, 0] \to \mathbb{R}, i = 1, \ldots .k$, are given bounded continuous linearly independent functions such that $c_i(0) = 0$ and $0 \le t_1 < \ldots < t_k \le T$. For a function $y \in CE$ such that $y(0) = \upsilon(0) - g(y)(0)$, we define the function $y[\upsilon] \in \mathscr{C}((-\infty, T]; E)$ as
$$y[\upsilon](t) = \begin{cases} \upsilon(t) - g(y)(t), & -\infty < t \le 0, \\ y(t), & 0 \le t \le T. \end{cases}$$
We denote by $\mathscr{D}$ the closed convex subset of CE, which consists of all functions y, satisfying the condition $y(0) = \upsilon(0) - g(y)(0)$.

Definition 4.1. A function $y \in \mathscr{C}((-\infty, T]; E)$ is called a mild solution to problem (4.1)–(4.2) if it satisfies conditions:

(1) $y(s) = \upsilon(s) - g(\widetilde{y})(s)$, for $s \in (-\infty, 0]$, where $\widetilde{y} = y\mid_{[0;T]}$;
(2) the function $\widetilde{y} \in \mathscr{D}$, and y on $[0, T]$ satisfies the relation
$$y(t) = k(t, y_t) + \mathscr{G}(t)\big(\upsilon(0) - g(\widetilde{y})(0) - k(0, \upsilon - g(\widetilde{y}))\big) + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) f(s) ds,$$
where
$$\mathscr{G}(t) = \int_0^\infty \xi_q(\theta) U(t^q \theta) d\theta,$$
and the operator–function $\mathscr{T}$ is defined by (3.2), $f \in \mathscr{Q}(y[\upsilon])$.

Lemma 4.1. *(see [33]) The operator functions $\mathscr{G}$ and $\mathscr{T}$ possess the following properties:*

1) *For each $t \in [0, T]$, $\mathscr{G}(t)$ and $\mathscr{T}(t)$ are linear bounded operators. More precisely, for each $x \in E$, we have*
$$\|\mathscr{G}(t)x\|_E \le M \|x\|_E, \|\mathscr{T}(t)x\|_E \le \frac{qM}{\Gamma(1+q)} \|x\|_E.$$
2) *the operator functions $\mathscr{G}(\cdot)$ and $\mathscr{T}(\cdot)$ are strongly continuous, i.e., the functions $t \in [0, T] \to \mathscr{G}(t)x$ and $t \in [0, T] \to \mathscr{T}(t)x$ are continuous for each $x \in E$.*

Consider the multioperator $\Gamma : \mathscr{D} \multimap \mathscr{D}$ given by
$$\Gamma(y) = \{z \in \mathscr{D} : z(t) = k(t, y_t) + \mathscr{G}(t)\big(\upsilon(0) - g(\widetilde{y})(0) - k(0, \upsilon - g(\widetilde{y}))\big) + Gf(t)\},$$
where $f \in \mathscr{Q}(y[\upsilon])$ and the operator G is defined by formula (3.1). It is clear that if the function y is a fixed point of the multioperator Γ, then $y[\upsilon]$ is a solution to problem (4.1)-(4.2). Thus our goal is to prove the existence of a fixed point of the multioperator Γ.

We can formulate a modification of the Theorem 5.1.1 from [29] in the following form.

Lemma 4.2. *For every L^∞-semicompact sequence $\{f_n\}_{n=1}^\infty \subset L^\infty([0, T]; E)$, $\{Gf_n\}_{n=1}^\infty$ is relatively compact in CE and. Moreover, the weak convergence $f_n \overset{L^1}{\rightharpoonup} f_0$ implies $Gf_n \to Gf_0$ in CE.*

Lemma 4.3. *(see [21]) Let the multioperator $\mathcal{Q}$ satisfy conditions $(\mathcal{Q}1)$–$(\mathcal{Q}3)$ and the operator $\mathcal{S}$ satisfy $(\mathcal{S}1')$, $(\mathcal{S}2)$. Then the composition $\mathcal{S}\circ\mathcal{Q} : \mathscr{C}((-\infty,T];E) \multimap CE$ is a u.s.c. multimap with compact values.*

From the last lemma, we obtain that multioperator Γ is u.s.c. Let us find conditions under which the multioperator Γ is condensing with respect to an appropriate MNC. For this, we need the following assertions.

Lemma 4.4. *(see [21, Lemma 3.5]) Let a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^{\infty}([0,T];E)$ be L^{∞}-integrally bounded and there exists a function $\psi \in L^{\infty}_{+}([0,T])$ such that*

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq \psi(t) \text{ for a.e. } t \in [0,T].$$

Then

$$\chi\left(\{Gf_n(t)\}_{n=1}^{\infty}\right) \leq 2D_1 \int_0^t \psi(s)ds,$$

where D_1 is the constant from condition $(\mathcal{S}1)$.

Consider the measure of noncompactness ν in the space CE with values in the cone $\mathbb{R}^2_+$. On a bounded subset of $\Omega \subset CE$, we define the values of ν the following way:

$$\nu(\Omega) = \left(\gamma(\Omega), \quad \operatorname{mod}_{CE}(\Omega)\right),$$

where mod_{CE} is the modulus of equicontinuity, and γ is the fading modulus of fiber noncompactness

$$\gamma(\Omega) = \sup_{t\in[0,T]} e^{-Lt}\chi(\Omega(t)).$$

We assume that the constant $L > 0$ is chosen so that

$$\beta = \sup_{t\in[0,T]} \left(2D_1 \int_0^t e^{Ls}\omega(s,1)\,ds\right) < 1, \tag{4.3}$$

where the constant D_1 is from condition $(\mathcal{S}1)$, and ω is a function from condition $(\mathcal{Q}3)$. It is easy to see that the MNC ν is monotone, nonsingular, and algebraically semi-additive. It follows from the Arzela–Ascoli theorem that it is also regular.

Theorem 4.1. *Let a causal multioperator $\mathcal{Q} : \mathscr{C}((-\infty,T];E) \multimap L^{\infty}([0,T];E)$ satisfy conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$. Then, under conditions (A), $(K1)-(K2)$, $(g1)-(g3)$, the multioperator Γ is ν-condensing.*

Proof. By Lemma 4.1 and condition $(K1)$, it is sufficient to prove the assertion of the theorem for the multioperator $G\circ\mathcal{Q}$. Let $\Omega \subset \mathscr{D}$ be a bounded set such that

$$\nu\left(G\circ\mathcal{Q}(\Omega[\upsilon])\right) \geq \nu(\Omega), \tag{4.4}$$

where $\Omega[\upsilon] = \{y[\upsilon] : y \in \Omega\}$. Let us prove that the set Ω is relatively compact. Inequality (4.4) means that

$$\gamma(G\circ\mathcal{Q}(\Omega[\upsilon])) \geq \gamma(\Omega). \tag{4.5}$$

Applying condition $(\mathscr{Q}3)$ and using the properties of function ω, we obtain for a.e. $t \in [0,T]$

$$\begin{aligned}\chi\left(\{f(t) : f \in \mathscr{Q}(\Omega[\upsilon])\}\right) &\le \omega\left(t, \sup_{s\in[0,t]} \varphi\left(\{y[\upsilon]_s : y \in \Omega\}\right)\right) \\ &= \omega\left(t, \varphi\left(\{y|_{[0,t]} : y \in \Omega\}\right)\right) \\ &= \omega\left(t, e^{Lt}e^{-Lt}\varphi\left(\{y|_{[0,t]} : y \in \Omega\}\right)\right) \\ &\le \omega\left(t, e^{Lt}\gamma\left(\{y|_{[0,t]} : y \in \Omega\}\right)\right) \\ &\le \omega\left(t, e^{Lt}\gamma(\Omega)\right) \\ &\le \omega\left(t, e^{Lt}\right)\cdot\gamma(\Omega).\end{aligned}$$

By Lemma 4.4 and $(\omega 2)$, we have, for each $t \in [0,T]$,

$$\begin{aligned}\chi\left(\{Gf(t) : f \in \mathscr{Q}(\Omega[\upsilon])\}\right) &\le 2D_1 \int_0^t \omega\left(s, e^{Ls}\right) ds \cdot \gamma(\Omega) \\ &\le 2D_1 \int_0^t e^{Ls}\omega(s,1)\, ds \cdot \gamma(\Omega).\end{aligned}$$

Inequality (4.5) and the last estimate imply that

$$\gamma(\Omega) \le \sup_{t\in[0,T]} \left(2D_1 \int_0^t e^{Ls}\omega(s,1)\, ds\right)\gamma(\Omega) = \beta\cdot\gamma(\Omega).$$

It follows from (4.3) that $\gamma(\Omega) = 0$. Thus $\varphi\left(\Omega[\upsilon]_t\right) = 0$ for each $t \in [0,T]$.

Now we demonstrate that Ω is equicontinuous. We take sequences $\{y_n\}_{n=1}^{\infty} \subset \Omega$, $n \ge 1$ and $\{f_n\}_{n=1}^{\infty}$, $f_n \in \mathscr{Q}(y_n[\upsilon])$. From conditions $(\mathscr{Q}2)$ and $(\mathscr{Q}3)$, it follows that $\{f_n\}_{n=1}^{\infty}$ is L^{∞}-semicompact. By Lemma 4.2, we have tha t $\{Gf_n\}_{n=1}^{\infty}$ is relatively compact. Hence

$$\operatorname{mod}_{CE}(\{Gf_n\}_{n=1}^{\infty}) = 0.$$

Thus $\nu\left(\{G\circ\mathscr{Q}(\Omega[\upsilon])\}\right) = (0,0)$, but it follows from inequality (4.4) that $\nu(\Omega) = (0,0)$, and the last equality yields that Ω is relatively compact. □

5. The Existence Result

To prove the main theorem of this paper, we need the following Gronwall-Bellman Lemma (see [34]).

Lemma 5.1. *Let $v(t)$ and $f(t)$ be nonnegative continuous functions on the segment $[a,b]$, moreover $v(t) \le c + \int_a^t f(s)v(s)ds$, $t \in [a,b]$, where c is a positive constant. Then, for each $t \in [a,b]$, the inequality $v(t) \le ce^{\int_a^t f(s)ds}$, holds.*

Theorem 5.1. *Let a causal multioperator $\mathscr{Q} : \mathscr{C}((-\infty,T];E) \to Cv(L^{\infty}([0,T];E))$ satisfy conditions $(\mathscr{Q}1)$-$(\mathscr{Q}3)$. Then, under conditions (A), $(K1)-(K2)$, $(g1)-(g3)$ and*

$$L_1 := \|b_1\|_C\,(K + HN_1) + D_1\|\alpha\|_{L^{\infty}}THN_1 < 1$$

the set Σ_{υ} of all solutions to problem (4.1)-(4.2) is non-empty and compact set.

Proof. Let us prove that the set of all solutions $y \in \mathscr{D}$ to a one-parameter inclusion

$$y \in \lambda \Gamma(y), \quad \lambda \in [0,1], \tag{5.1}$$

is a priori bounded. If $y \in \mathscr{D}$ satisfies inclusion (5.1) for some $\lambda \in [0,1]$, then, for each $t \in [0,T]$, we obtained by using assumptions $(\mathscr{B}2), (\mathscr{S}1), (K2)$, and $(g2)$ that

$$\begin{aligned}\|y(t)\|_E &\le \lambda \|k(t,y_t)\|_E + \lambda \|\mathscr{G}(t)\big(\upsilon(0) - g(\widetilde{y})(0) - k(0, \upsilon - g(\widetilde{y}))\big)\|_E \\ &\quad + \lambda \Big\| \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) f(s) ds \Big\|_E \\ &\le \|b_1\|_C \|y_t\|_{\mathscr{BC}} + b_2 + M\|\upsilon(0) - g(\widetilde{y})(0) - k(0, \upsilon - g(\widetilde{y}))\|_E + D_1 \int_0^t \|f(s)\|_E ds \\ &\le \|b_1\|_C \|y_t\|_{\mathscr{BC}} + b_2 + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big) + D_1 \int_0^t \|f(s)\|_E ds,\end{aligned}$$

where $f \in \mathscr{Q}(y[\upsilon])$. By condition $(\mathscr{Q}2)$, $\|f(s)\|_E \le \alpha(s)(1 + \|y[\upsilon]\|_{\mathscr{C}})$. Then

$$\begin{aligned}\|y(t)\|_E &\le \|b_1\|_C \|y_t\|_{\mathscr{BC}} + b_2 + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big) + D_1 \int_0^t \alpha(s)(1 + \|y[\upsilon]\|_{\mathscr{C}}) ds \\ &\le \|b_1\|_C \|y_t\|_{\mathscr{BC}} + b_2 + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big) \\ &\quad + D_1 \|\alpha\|_{L^\infty} \int_0^t \left(1 + \|y[\upsilon]_s\|_{\mathscr{BC}} + \sup_{s\in[0,t]} \|y(s)\|_E\right) ds.\end{aligned}$$

From property $\mathscr{B}1$ (iii), it follows that

$$\|y[\upsilon]_s\|_{\mathscr{BC}} + \sup_{s\in[0,t]} \|y(s)\|_E \le H\|\upsilon - g(\widetilde{y})\|_{\mathscr{BC}} + (K+1) \sup_{s\in[0,t]} \|y(s)\|_E,$$

where $H(t) \le H, K(t) \le K, t \in [0,T]$. Then,

$$\begin{aligned}\|y(t)\|_E &\le \|b_1\|_C (K \sup_{\tau\in[0,t]} \|y(\tau)\|_E + H\|\upsilon - g(\widetilde{y})\|_{\mathscr{BC}}) + b_2 + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big) \\ &\quad + D_1 \|\alpha\|_{L^\infty} \int_0^t \left(1 + H\|\upsilon - g(\widetilde{y})\|_{\mathscr{BC}} + (K+1) \sup_{s\in[0,t]} \|y(s)\|_E\right) ds \\ &\le \|b_1\|_C K \sup_{t\in[0,T]} \|y(t)\|_E + H\|b_1\|_C (\|\upsilon\|_{\mathscr{BC}} + N_1 \|y\|_{CE} + N_2) + b_2 \\ &\quad + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big) + D_1 \|\alpha\|_{L^\infty} T (1 + H(\|\upsilon\|_{\mathscr{BC}} + N_1 \|y\|_{CE} + N_2)) \\ &\quad + D_1 \|\alpha\|_{L^\infty} \int_0^t (K+1) \sup_{s\in[0,t]} \|y(s)\|_E ds \\ &= (\|b_1\|_C (K + HN_1) + D_1 \|\alpha\|_{L^\infty} THN_1) \|y\|_{CE} + H\|b_1\|_C (\|\upsilon\|_{\mathscr{BC}} + N_2) + b_2 \\ &\quad + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big) \\ &\quad + D_1 \|\alpha\|_{L^\infty} T (1 + H (\|\upsilon\|_{\mathscr{BC}} + N_2)) + D_1 \|\alpha\|_{L^\infty} (K+1) \int_0^t \sup_{s\in[0,t]} \|y(s)\|_E ds \\ &= L_1 \|y\|_{CE} + L_2 + L_3 \int_0^t \sup_{s\in[0,t]} \|y(s)\|_E ds,\end{aligned}$$

where

$$L_1 = \|b_1\|_C (K + HN_1) + D_1 \|\alpha\|_{L^\infty} THN_1,$$

$$L_2 = H\|b_1\|_C\left(\|\upsilon\|_{\mathscr{BC}} + N_2\right) + b_2 + M\big(l\|\upsilon\|_{\mathscr{BC}} + N_2 + b_2\big)$$
$$+ D_1\|\alpha\|_{L^\infty} T\left(1 + H\left(\|\upsilon\|_{\mathscr{BC}} + N_2\right)\right),$$

and

$$L_3 = D_1\|\alpha\|_{L^\infty}(K+1).$$

The last expression is a non-decreasing function of t, so we have

$$\sup_{t\in[0,T]} \|y(t)\|_E \leq L_2(1-L_1)^{-1} + L_3(1-L_1)^{-1}\int_0^t \sup_{s\in[0,t]} \|y(s)\|_E ds.$$

This means that $v(s) = \sup_{s\in[0,t]} \|y(s)\|_E$ satisfies the estimate

$$v(t) \leq L_2(1-L_1)^{-1} + L_3(1-L_1)^{-1}\int_0^t v(s)ds.$$

Applying Lemma 5.1, we obtain the required a priori boundedness:

$$v(t) = \|y\|_{CE} \leq L_2(1-L_1)^{-1}e^{L_3(1-L_1)^{-1}T}.$$

Now, if we take

$$R > L_2(1-L_1)^{-1}e^{L_3(1-L_1)^{-1}T},$$

then we can guarantee that the set $V \subset \mathscr{D}$, given as $V = \{y \in \mathscr{D} : \|y\|_{CE} < R\}$, contains all solutions of inclusion (5.1). Thus the multioperator Γ satisfies on ∂V the condition of Theorem 2.1 with $a = 0$. Hence, the set of its fixed points is non-empty and compact. □

Acknowledgments
The work of the first author was supported by the State contract of the Russian Ministry of Education as part of the state task (Contract FZGF-2020-0009). The work of the second author was supported by the RFBR (Project Number 19-31-60011). The work of the third author was supported by the RFBR (Project Number 20-51-15003). The work of the fourth author was supported by the National Science and Technology Council, Taiwan, under Grant No. 112-2115-M-037-001

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