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NEW BREGMAN PROJECTION ALGORITHMS FOR SOLVING THE SPLIT FEASIBILITY PROBLEM

YAXIN HAO, JING ZHAO*

College of Science, Civil Aviation University of China, Tianjin 300300, China

Abstract. Bregman distance iterative methods for solving optimization problems are important and interesting because of the numerous applications of Bregman distance techniques. In this paper, for solving a split feasibility problem, we introduce a new Bregman projection algorithm and construct two selection strategies of stepsizes. Moreover, a relaxed Bregman projection algorithm is proposed with two selection strategies of stepsizes, where the two closed and convex sets are both level sets of convex functions. Weak convergence results of the proposed algorithms are obtained under suitable assumptions. In addition, using the proposed algorithms with different Bregman distances, a numerical experiment solving signal processing problem is also given to demonstrate the effectiveness of the proposed algorithms.

Keywords. Bregman projection; Split feasibility problem; Self-adaptive stepsize; Weak convergence.

1. INTRODUCTION

The split feasibility problem (SFP) was firstly introduced by Censor and Elfving [1] for modelling some inverse problems. Since then, it has played an important role in many real-world application problems, such as signal processing, image reconstruction, machine learning, radiation therapy, and so on [3–7]. Let H_1 and H_2 be real Hilbert spaces, and let $A : H_1 \to H_2$ be a bounded linear operator. The SFP can mathematically be formulated as the problem of finding a point $\hat{x}$ with the property

$$\hat{x} \in C \text{ and } A\hat{x} \in Q, \tag{1.1}$$

where C and Q are nonempty, convex, and closed subsets of H_1 and H_2, respectively. In particular, when $Q = \{b\}$, SFP (1.1) becomes the following convex constrained linear inverse problem:

$$\hat{x} \in C \text{ and } A\hat{x} = b.$$

For solving SFP (1.1), Byrne [2] introduced the following celebrated CQ algorithm, which generates an iterative sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \lambda_n A^*(I - P_Q)A)x_n, \tag{1.2}$$

where $\lambda_n \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A, P_C and P_Q are the projections onto C and Q, respectively.

*Corresponding author.
E-mail address: zhaojing200103@163.com (J. Zhao).
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We assume that SFP (1.1) is consistent (i.e., (1.1) at least has a solution) and use Γ to denote the solution set of SFP (1.1), i.e., $\Gamma = \{\hat{x} \in C : A\hat{x} \in Q\}$. We know that Γ is a nonempty, convex, and closed set. And $\hat{x} \in \Gamma$ if and only if $\hat{x}$ is the solution to the following fixed point equation:

$$\hat{x} = P_C(I - \lambda A^*(I - P_Q)A)\hat{x},$$

where $\lambda > 0$. This implies that we can use fixed point algorithms (see, e.g., [8–12]) to solve the SFP (1.1).

It is observed that, in CQ algorithm (1.2), stepsize λ_n depends on the bounded linear operator (matrix) norm $\|A\|$ (or the largest eigenvalue of A^*A). It is not always easy in practice to calculate the operator (matrix) norm $\|A\|$. To avoid this difficulty, there have been many self-adaptive algorithms that the stepsize dose not depend on the norm of operator A. In [13], Lopez et al. improved CQ algorithm (1.2) by selecting the following stepsize:

$$\lambda_n = \frac{\rho_n h(x_n)}{\|\nabla h(x_n)\|^2},$$

where $\inf_n \rho_n(4 - \rho_n) > 0$ and $h(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$.

We note that CQ algorithm (1.2) and numerous pertinent iterative algorithms involve the calculations of the projections, P_C and P_Q, onto sets C and Q, respectively. However, in some cases, it is very difficult to calculate projections, and hence the efficiency of CQ algorithm (1.2) is seriously affected. To overcome this difficulty, in [14], for the level sets C and Q of convex functions, Yang introduced the following relaxed CQ algorithm for solving SFP (1.1):

$$x_{n+1} = P_{C_n}(x_n - \lambda_n A^*(I - P_{Q_n})Ax_n), \tag{1.3}$$

where $\lambda_n \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of A^*A. In relaxed CQ algorithm (1.3), convex and closed sets C and Q were replaced with two half-spaces C_n and Q_n, respectively. Recently, numerous authors presented various relaxed CQ algorithms for solving SFP (1.1); see, e.g., [12, 13, 15–19]. We know the SFP (1.1) has a close connection with the variational inequality problem (VIP). Let C be a nonempty, convex and closed subset of H, and let $F : C \to H$ be an operator. The VIP is to find a point $\hat{x} \in C$ such that

$$\langle F\hat{x}, z - \hat{x} \rangle \geq 0, \ \forall z \in C. \tag{1.4}$$

$\hat{x}$ solves SFP (1.1) if and only if that there is a vector $\hat{x} \in C$ such that $A\hat{x} - q = 0$ for some $q \in Q$. This motivates us to introduce the (convex) objective function:

$$h(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2.$$

Therefore, SFP (1.1) becomes the following convex minimization problem: $\min_{x \in C} h(x)$. The objective function h is differentiable and its gradient is given by $\nabla h(x) = A^*(I - P_Q)Ax$. Hence, SFP (1.1) can be converted to the following VIP: $\langle A^*(I - P_Q)A\hat{x}, z - \hat{x} \rangle \geq 0$ for all $z \in C$. It is known that the Bregman distance is a useful substitute for a distance, obtained from the various choices of functions. The applications of the Bregman distance instead of the norm gives us alternative ways for more flexibility in the selection of projections. Let the function $f : H \to \mathbb{R}$ be σ-strongly convex, Frééchet differentiable ,and bounded on bounded subsets of H. The Bregman projection with respect to f of $x \in int(domf)$ is denoted by Π_C^f. In [20], Sunthrayuth et al. proposed the following Bregman projection algorithm for solving the VIP (1.4):

$$\begin{cases} y_n = \Pi_c^f (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n F x_n), \\ x_{n+1} = (\nabla f)^{-1}(\nabla f(y_n) - \lambda_n (F y_n - F x_n)), \end{cases}$$

where $F : H \to H$ is pseudo-monotone, λ_{n+1} is chosen by

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle F x_n - F y_n, x_{n+1} - y_n\rangle}, \lambda_n\}, & \text{if } \langle F x_n - F y_n, x_{n+1} - y_n\rangle > 0, \\ \lambda_n, & \text{otherwise} \end{cases}$$

and $\mu \in (0, \sigma)$.

Motivated and inspired by the results mentioned above, we, in this paper, introduce new Bregman projection algorithms for solving SFP (1.1) in real Hilbert spaces. The paper is organized as follows. In Section 2, we present definitions and notions that are need for the rest of the paper. In Section 3 and Section 4, we introduce a new Bregman projection algorithm and construct two selection strategies of stepsizes. We also obtain weak convergence results under mild conditions. In Section 5, we modify the relaxed CQ algorithm (1.3) by employing the Bregman projection and obtain weak convergence theorems for the proposed algorithms. Finally, a numerical experiment is given to illustrate the effectiveness of our proposed algorithms in Section 6, the last section.

2. Preliminaries

From now on, we denote the inner product by $\langle \cdot, \cdot \rangle$ and the norm by $\|\cdot\|$. Let H be a real Hilbert space, and let C be a nonempty, convex, and closed subset of H. One uses $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively, and use $\omega_w(x_n)$ to denote the weak limit set of $\{x_n\}$.

Recall that the projection from H on to C, denoted P_C, is defined in such a way that, for each $x \in H$, $P_C x$ is the unique point in C with $P_C x = \arg\min\{\|x - y\| : y \in C\}$. The following is a useful characterization of the projection: given $x \in H$ and $z \in C$, $z = P_C x$ if and only if, for all $y \in C$, $\langle x - z, y - z\rangle \leq 0$.

Let $T : H \to H$ be an operator. Recall that T is said to be

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in H$;

(ii) *firmly nonexpansive* if $2T - I$ is nonexpansive or, equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2$$

for all $x,\ y \in H$;

(iii) demiclosed at the origin if, for any sequence $\{x_n\}$, which converges weakly to x, $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

It is well known that both P_C and $I - P_C$ are firmly nonexpansive.

Recall that the Bregman bifunction $D_f : dom f \times int(dom f) \to [0, \infty)$ corresponding to the convex and differentiable function f with its gradient ∇f is defined by $D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y\rangle$. The Bregman projection with respect to f of $x \in int(dom f)$ is denoted by Π_C^f and $\Pi_C^f(x) = \arg\min\{D_f(y, x) : y \in C\}$. In addition, $\Pi_C^f(x)$ has the following property [21]: for each $x \in H$, $z = \Pi_C^f(x)$ if and only if, for all $y \in C$, $\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0$.

Recall that a convex and differentiable function f is said to be σ-strongly convex if there exists a constant $\sigma > 0$ such that

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\sigma}{2}\|x - y\|^2,$$

for any $x \in dom f$ and $y \in int(dom f)$. If the function f is σ-strongly convex, we find from the definition of the Bregman distance the following inequality:

$$D_f(x,y) \geq \frac{\sigma}{2}\|x - y\|^2. \tag{2.1}$$

For any two sequences $\{x_n\}$ and $\{y_n\}$ in H, one has $\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Longrightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0$.

Finally, we also need the following lemmas.

Lemma 2.1. [22] *Let $f\colon H \longrightarrow \mathbb{R}$ be a strongly convex and differentiable function. And its gradient ∇f is sequentially weak-to-weak continuous. Suppose that $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, then $\liminf_{n\to\infty} D_f(x, x_n) < \liminf_{n\to\infty} D_f(y, x_n)$, for all $y \in H$ with $y \neq x$.*

Lemma 2.2. [23,24] *Let E be a uniformly convex Banach space. Let K be a nonempty, convex, and closed subset of E, and let $T : K \to K$ be a nonexpansive operator. Then $I - T$ is demiclosed at origin.*

3. The Bregman Projection Algorithm

We assume that the following conditions hold.

Condition 3.1. The function $f : H \to \mathbb{R}$ is σ-strongly convex and differentiable function with its gradient ∇f being sequentially weak-to-weak continuous.

Condition 3.2. (1) Γ denotes the solution set of the SFP (1.1), and Γ is nonempty. (2) $A : H_1 \to H_2$ be a bounded linear operator with $A \neq \mathbf{0}$.

Algorithm 3.1. (Bregman Projection Algorithm for Solving SFP (1.1))

Let $x_1 \in H_1$ be arbitrary. For $n \geq 1$, compute

$$x_{n+1} = \Pi_C^f (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A^*(I - P_Q)Ax_n),$$

where $\lambda_n > 0$.

Theorem 3.1. *Assume that Conditions 3.1-3.2 hold. If $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2\sigma}{\|A\|^2}$, then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a solution of the SFP (1.1).*

Proof. First, we show that $\{x_n\}$ is bounded. Let $p \in \Gamma$ and $z_n = (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A^*(I - P_Q)Ax_n)$. Then $x_{n+1} = \Pi_C^f(z_n)$. It follows that

$$\langle \nabla f(x_{n+1}) - \nabla f(z_n), p - x_{n+1} \rangle \geq 0. \tag{3.1}$$

By the definition of the Bregman distance, we have

$$\begin{aligned} D_f(p, x_{n+1}) &= f(p) - f(x_{n+1}) - \langle \nabla f(x_{n+1}), p - x_{n+1} \rangle \\ &= f(p) - f(x_{n+1}) - \langle \nabla f(x_{n+1}) - \nabla f(z_n) + \nabla f(z_n), p - x_{n+1} \rangle \\ &= f(p) - f(x_{n+1}) - \langle \nabla f(x_{n+1}) - \nabla f(z_n), p - x_{n+1} \rangle - \langle \nabla f(z_n), p - x_{n+1} \rangle. \end{aligned}$$

From (3.1), we obtain

$$\begin{aligned} D_f(p,x_{n+1}) \leq & f(p) - f(x_{n+1}) - \langle \nabla f(z_n), p - x_{n+1} \rangle \\ = & f(p) - f(x_{n+1}) - \langle \nabla f(x_n) - \lambda_n A^*(I - P_Q)Ax_n, p - x_{n+1} \rangle \\ = & f(p) - f(x_{n+1}) - \langle \nabla f(x_n), p - x_{n+1} \rangle + \lambda_n \langle A^*(I - P_Q)Ax_n, p - x_{n+1} \rangle \\ = & f(p) - f(x_{n+1}) - f(x_n) + f(x_n) - \langle \nabla f(x_n), p - x_n \rangle - \langle \nabla f(x_n), x_n - x_{n+1} \rangle \\ & + \lambda_n \langle A^*(I - P_Q)Ax_n, p - x_{n+1} \rangle \\ = & D_f(p,x_n) - D_f(x_{n+1},x_n) + \lambda_n \langle A^*(I - P_Q)Ax_n, p - x_n \rangle \\ & + \lambda_n \langle A^*(I - P_Q)Ax_n, x_n - x_{n+1} \rangle. \end{aligned} \tag{3.2}$$

In view of $Ap \in Q$, we have $\langle Ax_n - P_Q Ax_n, Ap - P_Q Ax_n \rangle \leq 0$, which implies that

$$\begin{aligned} \lambda_n \langle A^*(I - P_Q)Ax_n, p - x_n \rangle = & \lambda_n \langle (I - P_Q)Ax_n, Ap - P_Q Ax_n \rangle + \lambda_n \langle (I - P_Q)Ax_n, P_Q Ax_n - Ax_n \rangle \\ \leq & -\lambda_n \|(I - P_Q)Ax_n\|^2. \end{aligned} \tag{3.3}$$

For all $\mu > 0$, we have

$$\begin{aligned} \lambda_n \langle A^*(I - P_Q)Ax_n, x_n - x_{n+1} \rangle \leq & \lambda_n \|A^*(I - P_Q)Ax_n\| \cdot \|x_n - x_{n+1}\| \\ \leq & \frac{\mu \lambda_n}{2} \|A^*(I - P_Q)Ax_n\|^2 + \frac{\lambda_n}{2\mu} \|x_n - x_{n+1}\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$\begin{aligned} D_f(p,x_{n+1}) \leq & D_f(p,x_n) - D_f(x_{n+1},x_n) - \lambda_n \|(I - P_Q)Ax_n\|^2 \\ & + \frac{\mu \lambda_n}{2} \|A^*(I - P_Q)Ax_n\|^2 + \frac{\lambda_n}{2\mu} \|x_n - x_{n+1}\|^2. \end{aligned}$$

Using (2.1), we have

$$\begin{aligned} D_f(p,x_{n+1}) \leq & D_f(p,x_n) - D_f(x_{n+1},x_n) - \lambda_n (1 - \frac{\mu}{2} \|A\|^2) \|(I - P_Q)Ax_n\|^2 + \frac{\lambda_n}{2\mu} \frac{2}{\sigma} D_f(x_{n+1},x_n) \\ = & D_f(p,x_n) - (1 - \frac{\lambda_n}{\mu \sigma}) D_f(x_{n+1},x_n) - \lambda_n (1 - \frac{\mu}{2} \|A\|^2) \|(I - P_Q)Ax_n\|^2. \end{aligned} \tag{3.5}$$

Take $\mu > 0$ with $\frac{1}{\sigma} \limsup_{n\to\infty} \lambda_n < \mu < \frac{2}{\|A\|^2}$. Since $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2\sigma}{\|A\|^2}$, we see that

$$\liminf_{n\to\infty} \lambda_n (1 - \frac{\mu}{2} \|A\|^2) > 0 \tag{3.6}$$

and

$$\liminf_{n\to\infty} (1 - \frac{\lambda_n}{\mu \sigma}) > 0. \tag{3.7}$$

From (3.6) and (3.7), we obtain $D_f(p,x_{n+1}) \leq D_f(p,x_n)$, which shows that $\lim_{n\to\infty} D_f(p,x_n)$ exists and hence $\{D_f(p,x_n)\}$ is bounded. In view of (2.1), we have that $\{x_n\}$ is bounded. From (3.5), we have

$$(1 - \frac{\lambda_n}{\mu \sigma}) D_f(x_{n+1},x_n) + \lambda_n (1 - \frac{\mu}{2} \|A\|^2) \|(I - P_Q)Ax_n\|^2 \leq D_f(p,x_n) - D_f(p,x_{n+1}).$$

Since $\lim_{n\to\infty} D_f(p,x_n)$ exists, we have from (3.6) and (3.7) that

$$\lim_{n\to\infty} D_f(x_{n+1},x_n) = \lim_{n\to\infty} \|(I-P_Q)Ax_n\| = 0 \tag{3.8}$$

and hence $\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0$.

Next we show $\omega_w(x_n) \subseteq \Gamma$. By the boundedness of $\{x_n\}$, we have $\omega_w(x_n) \neq \emptyset$. Taking $\hat{x} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$. Since $x_{n_k} \rightharpoonup \hat{x}$, then $Ax_{n_k} \rightharpoonup A\hat{x}$ as $k \to \infty$. By Lemma 2.2 and (3.8), we can obtain $(I-P_Q)A\hat{x} = 0$, so $A\hat{x} \in Q$. Hence, we have $\omega_w(x_n) \subseteq \Gamma$.

Finally, we show the uniqueness of the weak cluster points of $\{x_n\}$. Indeed, let $x^{'}$ be other weak cluster point of $\{x_n\}$. Then $x^{'} \in \Gamma$ and there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $x_{m_j} \rightharpoonup x^{'}$ as $j \to \infty$. Since $\lim_{n\to\infty} D_f(u,x_n)$ exists for any $u \in \Gamma$, it follows from Lemma 2.1 that

$$\begin{aligned}\lim_{n\to\infty} D_f(\hat{x},x_n) &= \lim_{k\to\infty} D_f(\hat{x},x_{n_k}) = \lim_{k\to\infty} \inf D_f(\hat{x},x_{n_k}) \\ &< \lim_{k\to\infty} \inf D_f(x^{'},x_{n_k}) = \lim_{j\to\infty} D_f(x^{'},x_{m_j}) \\ &= \lim_{n\to\infty} D_f(x^{'},x_n).\end{aligned}$$

In a similar way as above, we have $\lim_{k\to\infty} D_f(x^{'},x_n) < \lim_{n\to\infty} D_f(\hat{x},x_n)$. This is a contradiction. Hence $\hat{x} = x^{'}$ and we conclude that $\{x_n\}$ converges weakly to a point in Γ. This completes the proof. □

Remark 3.1. If $f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f(x) = x$, $\Pi_C^f = P_C$, and $\sigma = 1$. In this case, Algorithm 3.1 reduces to Byrne's *CQ* algorithm (1.2). Moreover, we can select a different Bregman distance which is more flexible than the squared Euclidean distance.

4. Self-Adaptive Stepsize

As we see from the previous section, the selection of λ_n requires the norm of A (or the largest eigenvalue of A^*A). To avoid computing the norm of the bounded linear operator A, in this section, we choose self-adaptive stepsizes to modify the Bregman projection algorithm.

Algorithm 4.1. (Bregman Projection Algorithm with Self-Adaptive Stepsizes)

Let $x_1 \in C$ be arbitrary. For $n \geq 1$, if $Ax_n = P_Q Ax_n$, then stop and x_n is a solution to SFP (1.1). *Otherwise, compute*

$$x_{n+1} = \Pi_C^f (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A^*(I-P_Q)Ax_n),$$

where λ_n is chosen by

$$\lambda_n = \min\{ \tfrac{\rho\sigma\|(I-P_Q)Ax_n\|^2}{\|A^*(I-P_Q)Ax_n\|^2}, \lambda_{n-1}\} \tag{4.1}$$

with $0 < \rho < 2$.

Remark 4.1. (i) In Algorithm 4.1, stepsize λ_n is chosen by a self-adaptive way. We give a way of selecting the stepsize such that the implementation of the algorithm does not need any prior information about the norm of the bounded linear operator. (ii) We see that if Algorithm 4.1 terminates in a finite step of iterations, then x_n is a solution to the SFP (1.1). In the rest of this paper, we assume that Algorithm 4.1 does not terminate in any finite iterations, and hence generates an infinite sequence $\{x_n\}$.

From the following lemma, we see that λ_n is well-defined.

Lemma 4.1. *λ_n defined by (4.1) is well-defined.*

Proof. Fix $x \in \Gamma$, i.e., $x \in C$ and $Ax \in Q$. Since $I - P_Q$ is firmly nonexpansive, we have

$$\begin{aligned}\|A^*(I-P_Q)Ax_n\| \cdot \|x_n - x\| \geq & \langle A^*(I-P_Q)Ax_n, x_n - x\rangle \\ = & \langle (I-P_Q)Ax_n, Ax_n - Ax\rangle \\ \geq & \|(I-P_Q)Ax_n\|^2.\end{aligned}$$

Consequently, when $\|(I-P_Q)Ax_n\| \neq 0$, we have $\|A^*(I-P_Q)Ax_n\| > 0$. This guarantees that λ_n is well-defined. □

Theorem 4.1. *Assume that Conditions 3.1-3.2 hold, then the sequence $\{x_n\}$ generated by Algorithm 4.1 converges weakly to a solution of the SFP (1.1).*

Proof. First, we prove that $\{x_n\}$ is bounded. For all $\mu > 0$, we can deduce that

$$\begin{aligned}D_f(p, x_{n+1}) \leq & D_f(p, x_n) - D_f(x_{n+1}, x_n) - \lambda_n\|(I-P_Q)Ax_n\|^2 \\ & + \frac{\mu\lambda_n}{2}\|A^*(I-P_Q)Ax_n\|^2 + \frac{\lambda_n}{2\mu}\|x_n - x_{n+1}\|^2.\end{aligned}$$

Using (2.1), we see that

$$\begin{aligned}& D_f(p, x_{n+1}) \\ & \leq D_f(p, x_n) - (1 - \frac{\lambda_n}{\mu\sigma})D_f(x_{n+1}, x_n) - \lambda_n\|(I-P_Q)Ax_n\|^2(1 - \frac{\mu\|A^*(I-P_Q)Ax_n\|^2}{2\|(I-P_Q)Ax_n\|^2}).\end{aligned} \tag{4.2}$$

Since $\frac{2\|(I-P_Q)Ax_n\|^2}{\|A^*(I-P_Q)Ax_n\|^2} \geq \frac{2}{\|A\|^2} > 0$, then $\inf_n \frac{2\|(I-P_Q)Ax_n\|^2}{\|A^*(I-P_Q)Ax_n\|^2} \geq \frac{2}{\|A\|^2} > 0$. By the definition of λ_n and $0 < \rho < 2$, we have

$$\frac{\lambda_n}{\sigma} \leq \inf_{k \leq n} \frac{\rho\|(I-P_Q)Ax_k\|^2}{\|A^*(I-P_Q)Ax_k\|^2} < \inf_{k \leq n} \frac{2\|(I-P_Q)Ax_k\|^2}{\|A^*(I-P_Q)Ax_k\|^2}.$$

Since $\{\lambda_n\}$ is non-increasing and $\lambda_n \geq \frac{\rho\sigma}{\|A\|^2}$, we have that $\lim_{n\to\infty}\lambda_n$ exists, so

$$\frac{1}{\sigma}\lim_{n\to\infty}\lambda_n < \liminf_{n\to\infty}\frac{2\|(I-P_Q)Ax_n\|^2}{\|A^*(I-P_Q)Ax_n\|^2}.$$

Take μ with $\frac{1}{\sigma}\lim_{n\to\infty}\lambda_n < \mu < \liminf_{n\to\infty}\frac{2\|(I-P_Q)Ax_n\|^2}{\|A^*(I-P_Q)Ax_n\|^2}$. Then,

$$\liminf_{n\to\infty}(1 - \frac{\lambda_n}{\mu\sigma}) > 0 \tag{4.3}$$

and

$$\liminf_{n\to\infty}(1 - \frac{\mu\|A^*(I-P_Q)Ax_n\|^2}{2\|(I-P_Q)Ax_n\|^2}) > 0. \tag{4.4}$$

So $D_f(p, x_{n+1}) \leq D_f(p, x_n)$, which yields that $\lim_{n\to\infty}D_f(p, x_n)$ exists and hence $\{D_f(p, x_n)\}$ is bounded. By using (2.1), we have that $\{x_n\}$ is bounded. It follows from (4.2)-(4.4) that

$$\lim_{n\to\infty}D_f(x_{n+1}, x_n) = \lim_{n\to\infty}\|(I-P_Q)Ax_n\| = 0.$$

Similar to the proof of Theorem 3.1, we can obtain $\omega_w(x_n) \subseteq \Gamma$. And the sequence $\{x_n\}$ generated by Algorithm 4.1 converges weakly to a solution of the SFP (1.1). □

5. Relaxed Bregman Projection Algorithms

In this section, we propose relaxed Bregman projection algorithms for solving SFP (1.1). Let $C = \{u \in H_1 : c(u) \leq 0\}$, $Q = \{v \in H_2 : q(v) \leq 0\}$, where $c : H_1 \to R$ and $q : H_2 \to R$ are convex and lower semi-continuous functions. We assume that c and q are subdifferentiable on H_1 and H_2, respectively. For all $u \in C$ and $v \in Q$, the subdifferentials are

$$\partial c(u) = \{z \in H_1 : c(x) \geq c(u) + \langle z, x-u\rangle, \ \ x \in H_1\} \neq \emptyset$$

and

$$\partial q(v) = \{w \in H_2 : q(y) \geq q(v) + \langle w, y-v\rangle, \ \ y \in H_2\} \neq \emptyset.$$

We also assume that ∂c and ∂q are bounded on bounded sets.

Algorithm 5.1. (Relaxed Bregman Projection Algorithm for Solving SFP (1.1))
Let $x_1 \in H_1$ be arbitrary. For $n \geq 1$, set

$$C_n = \{u \in H_1 : c(x_n) + \langle \xi_n, u - x_n\rangle \leq 0\} \tag{5.1}$$

and

$$Q_n = \{v \in H_2 : q(Ax_n) + \langle \eta_n, v - Ax_n\rangle \leq 0\}, \tag{5.2}$$

where $\xi_n \in \partial c(x_n)$ and $\eta_n \in \partial q(Ax_n)$. Compute

$$x_{n+1} = \Pi^f_{C_n}(\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A^*(I - P_{Q_n})Ax_n),$$

where $\lambda_n > 0$.

Remark 5.1. Obviously, C_n and Q_n are half-spaces. From the subdifferentiable inequality, It is easy to verify that $C \subseteq C_n$ and $Q \subseteq Q_n$ for every $n \geq 1$.

Theorem 5.1. *Assume that Conditions 3.1-3.2 hold. If $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2\sigma}{\|A\|^2}$, then the sequence $\{x_n\}$ generated by Algorithm 5.1 converges weakly to a solution of the SFP (1.1).*

Proof. First, we prove that $\{x_n\}$ is bounded. Let $p \in \Gamma \subseteq C_n$ and $z_n = (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A^*(I - P_{Q_n})Ax_n)$. Then $x_{n+1} = \Pi^f_{C_n}(z_n)$. It follows that $\langle \nabla f(x_{n+1}) - \nabla f(z_n), p - x_{n+1}\rangle \geq 0$. Using Q_n to substitute Q, we obtain

$$\begin{aligned} D_f(p, x_{n+1}) \leq D_f(p, x_n) - D_f(x_{n+1}, x_n) + \lambda_n \langle A^*(I - P_{Q_n})Ax_n, p - x_n\rangle \\ + \lambda_n \langle A^*(I - P_{Q_n})Ax_n, x_n - x_{n+1}\rangle. \end{aligned} \tag{5.3}$$

In view of $Ap \in Q_n$, we have $\langle (I - P_{Q_n})Ax_n, Ap - P_{Q_n}Ax_n\rangle \leq 0$. Similar to (3.3) and (3.4), we see that

$$\lambda_n \langle A^*(I - P_{Q_n})Ax_n, p - x_n\rangle \leq -\lambda_n \|(I - P_{Q_n})Ax_n\|^2 \tag{5.4}$$

and

$$\lambda_n \langle A^*(I - P_{Q_n})Ax_n, x_n - x_{n+1}\rangle \leq \frac{\mu\lambda_n}{2}\|A^*(I - P_{Q_n})Ax_n\|^2 + \frac{\lambda_n}{2\mu}\|x_n - x_{n+1}\|^2, \tag{5.5}$$

where $\mu > 0$. Substituting (5.4) and (5.5) into (5.3), we have

$$D_f(p,x_{n+1}) \le D_f(p,x_n) - D_f(x_{n+1},x_n) - \lambda_n\|(I-P_{Q_n})Ax_n\|^2 + \frac{\mu\lambda_n}{2}\|A^*(I-P_{Q_n})Ax_n\|^2 + \frac{\lambda_n}{2\mu}\|x_n - x_{n+1}\|^2.$$

Using (2.1), it follows from that

$$D_f(p,x_{n+1}) \le D_f(p,x_n) - (1-\frac{\lambda_n}{\mu\sigma})D_f(x_{n+1},x_n) - \lambda_n(1-\frac{\mu}{2})\|A\|^2\|(I-P_{Q_n})Ax_n\|^2. \tag{5.6}$$

Take $\mu > 0$ with $\frac{1}{\sigma}\limsup_{n\to\infty}\lambda_n < \mu < \frac{2}{\|A\|^2}$. Using the same arguments as in Theorem 3.1, we find that $\lim_{n\to\infty} D_f(p,x_n)$ exists and $\{D_f(p,x_n)\}$ is bounded. We also have that $\{x_n\}$ is bounded. From (5.6), we have

$$(1-\frac{\lambda_n}{\mu\sigma})D_f(x_{n+1},x_n) + \lambda_n(1-\frac{\mu}{2}\|A\|^2)\|(I-P_{Q_n})Ax_n\|^2 \le D_f(p,x_n) - D_f(p,x_{n+1}).$$

Since $\liminf_{n\to\infty}\lambda_n(1-\frac{\mu}{2}\|A\|^2) > 0$ and $\liminf_{n\to\infty}(1-\frac{\lambda_n}{\mu\sigma}) > 0$, we have

$$\lim_{n\to\infty} D_f(x_{n+1},x_n) = \lim_{n\to\infty}\|(I-P_{Q_n})Ax_n\| = 0$$

and hence $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Next we show $\omega_w(x_n) \subseteq \Gamma$. Since ∂c is bounded on bounded sets, one sees that there exists a constant $M_1 > 0$ such that $\|\xi_n\| \le M_1$ for all $n \in N$. It follows that

$$c(x_n) \le -\langle \xi_n, x_{n+1}-x_n\rangle \le M_1\|x_{n+1}-x_n\|.$$

Hence $\limsup_{n\to\infty} c(x_n) \le 0$. By the boundedness of $\{x_n\}$, we have $\omega_w(x_n) \neq \emptyset$. Fixing $\hat{x} \in \omega_w(x_n)$, one sees that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \hat{x}$ as $j \to \infty$. From the weak lower semicontinuity of c, we have $c(\hat{x}) \le \liminf_{j\to\infty} c(x_{n_j}) \le 0$. Therefore, $\hat{x} \in C$. Since $\eta_n \in \partial q(Ax_n)$, then there exists a constant $M_2 > 0$ such that $\|\eta_n\| \le M_2$ for all $n \in N$. It follows from $P_{Q_n}Ax_n \in Q_n$ that

$$q(Ax_n) + \langle \eta_n, P_{Q_n}Ax_n - Ax_n\rangle \le 0,$$

which implies that

$$q(Ax_n) \le \langle \eta_n, Ax_n - P_{Q_n}Ax_n\rangle \le M_2\|Ax_n - P_{Q_n}Ax_n\|.$$

It follows from $\lim_{n\to\infty}\|(I-P_{Q_n})Ax_n\| = 0$, the weak lower semicontinuity of q, and the fact that $Ax_{n_j} \rightharpoonup A\hat{x}$ that $q(A\hat{x}) \le \liminf_{j\to\infty} q(Ax_{n_j}) \le 0$, that is, $A\hat{x} \in Q$. Thus $\hat{x} \in \Gamma$ and $\omega_w(x_n) \subseteq \Gamma$.

Finally, as proved in Theorem 3.1, we can deduce that $\{x_n\}$ converges weakly to a point in Γ. This completes the proof. □

Remark 5.2. If $f(x) = \frac{1}{2}\|x\|^2$, then Algorithm 5.1 is reduced to the relaxed CQ algorithm (1.3).

Algorithm 5.2. (Bregman Projection Algorithm with a Self-Adaptive Stepsize)

Let $x_1 \in H_1$ be arbitrary. For $n \ge 1$, C_n and Q_n are defined by (5.1) *and* (5.2) *respectively. Compute*

$$x_{n+1} = \Pi^f_{C_n}(\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A^*(I-P_{Q_n})Ax_n),$$

where λ_n is chosen by

$$\lambda_n = \begin{cases} \min\{\frac{\rho\sigma\|(I-P_{Q_n})Ax_n)\|^2}{\|A^*(I-P_{Q_n})Ax_n)\|^2}, \lambda_{n-1}\} & (I-P_{Q_n})Ax_n \neq 0, \\ \theta, & (I-P_{Q_n})Ax_n = 0 \end{cases} \tag{5.7}$$

with θ enough small and $0<\rho<2$.

From Lemma 4.1, we can obtain the following Lemma immediately.

Lemma 5.1. *λ_n defined by (5.7) is well-defined.*

Theorem 5.2. *Assume that Conditions 3.1-3.2 hold, then the sequence $\{x_n\}$ generated by Algorithm 5.2 converges weakly to a solution of the SFP (1.1).*

Proof. Following Theorem 5.1, we can deduce that

$$\begin{aligned} D_f(p,x_{n+1}) \le D_f(p,x_n) - D_f(x_{n+1},x_n) - \lambda_n\|(I-P_{Q_n})Ax_n\|^2 \\ + \frac{\mu\lambda_n}{2}\|A^*(I-P_{Q_n})Ax_n\|^2 + \frac{\lambda_n}{2\mu}\|x_n - x_{n+1}\|^2, \end{aligned}$$

for all $\mu>0$. If $(I-P_{Q_n})Ax_n = 0$, then $\lambda_n = \theta$ and

$$D_f(p,x_{n+1}) \le D_f(p,x_n) - (1-\frac{\theta}{\mu\sigma})D_f(x_{n+1},x_n). \tag{5.8}$$

If $(I-P_{Q_n})Ax_n \neq 0$, similar to (4.2), we obtain

$$\begin{aligned} & D_f(p,x_{n+1}) \\ & \le D_f(p,x_n) - (1-\frac{\lambda_n}{\mu\sigma})D_f(x_{n+1},x_n) - \lambda_n\|(I-P_{Q_n})Ax_n\|^2(1-\frac{\mu\|A^*(I-P_{Q_n})Ax_n\|^2}{2\|(I-P_{Q_n})Ax_n\|^2}). \end{aligned} \tag{5.9}$$

Take μ with $\frac{1}{\sigma}\lim_{n\to\infty}\lambda_n < \mu < \liminf_{n\to\infty}\frac{2\|(I-P_{Q_n})Ax_n\|^2}{\|A^*(I-P_{Q_n})Ax_n\|^2}$. Then,

$$\liminf_{n\to\infty}(1-\frac{\lambda_n}{\mu\sigma}) > 0 \tag{5.10}$$

and

$$\liminf_{n\to\infty}(1-\frac{\mu\|A^*(I-P_{Q_n})Ax_n\|^2}{2\|(I-P_{Q_n})Ax_n\|^2}) > 0. \tag{5.11}$$

Taking θ with $0<\theta<\mu\sigma$, we obtain $1-\frac{\theta}{\mu\sigma}>0$. It follows from (5.8)-(5.11) that

$$D_f(p,x_{n+1}) \le D_f(p,x_n),$$

which yields that $\lim_{n\to\infty}D_f(p,x_n)$ exists and hence $\{D_f(p,x_n)\}$ is bounded. By using (2.1), we have that $\{x_n\}$ is bounded. Moreover,

$$\lim_{n\to\infty}D_f(x_{n+1},x_n) = \lim_{n\to\infty}\|(I-P_{Q_n})Ax_n\| = 0.$$

Following the proof of Theorem 5.1, we can obtain $\omega_w(x_n)\subseteq\Gamma$. It is not hard to see that the sequence $\{x_n\}$ generated by Algorithm 5.2 converges weakly to a solution of SFP (1.1). This completes the proof. □

6. Numerical Experiment

In this section, we demonstrate the performance of the proposed Algorithm 3.1 and Algorithm 4.1 for solving the SFP (1.1). All the codes are written in MATLAB and are performed on a personal Lenovo computer with Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz 1.80GHz and RAM 8.00GB. For notational simplicity, we denote the vector with all elements 1 by e_1 in what follows. In the numerical results listed in the following tables, 'Iter.' denotes the number of iteration, and 'Time' denotes the time of iteration. We provide a numerical experiment to illustrate the numerical results of Algorithm 3.1 and Algorithm 4.1 using two Bregman distances. The following list are functions with their Bregman distances:

(i) Define the function $f^{KL}(x) = \sum_{i=1}^{m} x_i \ln x_i$ with domain $\mathrm{dom} f^{KL} = \{x = (x_1, x_2, ..., x_m)^T \in \mathbb{R}^m : x_i > 0, i = 1, 2, ...m\}$ and range $\mathrm{ran} f^{KL} = (-\infty, +\infty)$. Then

$$\nabla f^{KL}(x) = (1 + ln(x_1), 1 + ln(x_2), ..., 1 + ln(x_m))^T$$

and

$$(\nabla f^{KL})^{-1}(x) = (exp(x_1 - 1), exp(x_2 - 1), ..., exp(x_m - 1))^T.$$

Hence, we have the Kullback-Leibler distance given by

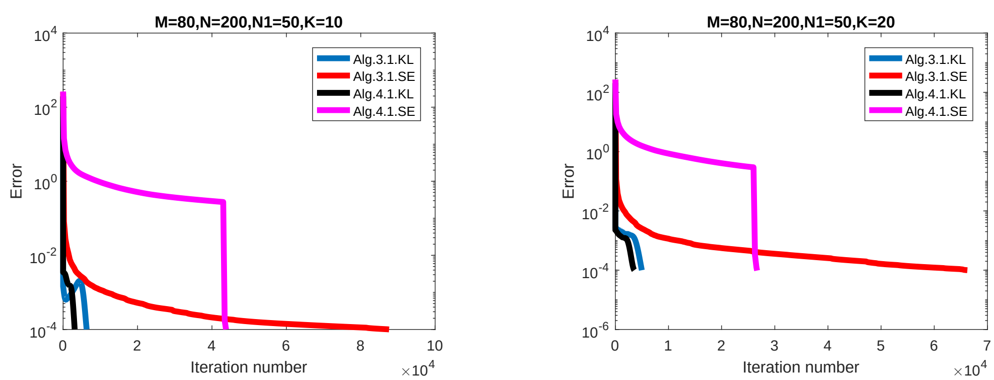
$$D_f^{KL}(x, y) = \sum_{i=1}^{m} (x_i ln(\frac{x_i}{y_i}) + y_i - x_i).$$

(ii) Define the function $f^{SE}(x) = \frac{1}{2}\|x\|^2$ with domain $\mathrm{dom} f^{SE} = H$ and range $\mathrm{ran} f^{SE} = [0, +\infty)$. Then $\nabla f^{SE}(x) = x$ and $(\nabla f^{SE})^{-1}(x) = x$. Hence, we have the squared Euclidean distance given by $D_f^{SE}(x, y) = \frac{1}{2}\|x - y\|^2$. It is clear that f^{KL} and f^{SE} satisfy strong convexity with $\sigma = 1$ (see [25]).

Example 6.1. Consider the equation system $y = Ax + \eta$, where $x \in R^N$ is the data to be recovered, y is the vector of noisy observations, and η represents the noise, sampling $A = (a_{ij})_{M \times N}$ is a matrix, $M < N$ and $a_{ij} \in [0, 1]$. The task is to recover the sparse signal x from the data y. We are interested in finding solution $x^* \in \{\hat{x} \in C | A\hat{x} \in Q\}$, where $C = \{x = (x_i) \mid x_i \leq 0, 1 \leq i \leq N_1; x_j \geq 0, N_1 < j \leq N\}$, N_1 is positive integer, $N_1 < N$, and $Q = [L, U] = \{y = (y_i) \mid L_i \leq y_i \leq U_i, 1 \leq i \leq M\}$. We know that Q is a box delimited by L and U, where $L = (L_i)$ and $U = (U_i) \in R^M$. To ensure the existence of the solution of the considered problem, K-sparse vector x^* is generated randomly in C. Take $y(t) = Ax^*$, $L = y - 0.1e_1$, and $U = y + 0.1e_1$. We use Algorithm 3.1 and Algorithm 4.1 to solve the above SFP. The metric projection P_Q can be computed by formula: $P_Q(y) = \max\{L, \min\{y, U\}\}$. In this experiment, we perform the numerical tests of Algorithm 3.1 and Algorithm 4.1 with different dimensions $(M, N, N_1) = (50, 100, 25), (80, 200, 50)$. The matrix A is generated randomly in $[0, 1]$. The sparse ratio is 0.05 and 0.1, respectively. We use the Kullback-Leibler distance and the squared Euclidean distance in Algorithm 3.1 and Algorithm 4.1. In the following, 'Alg.3.1.KL' and 'Alg.3.1.SE' denote Algorithm 3.1 with $f(x) = f^{KL}(x)$ and $f(x) = f^{SE}(x)$, respectively. 'Alg.4.1.KL' and 'Alg.4.1.SE' denote Algorithm 4.1 with $f(x) = f^{KL}(x)$ and $f(x) = f^{SE}(x)$, respectively. In all methods, we choose initial point $x_1 = (x_{ij})_{N \times 1}$, where $x_{ij} \in [0, 27]$ is generated randomly. In Algorithm 3.1, we take $\lambda_n = \frac{1.95}{\|A\|^2}$. And in Algorithm 4.1, we take $\rho = 1.95$. In the implementation, we use $error = \|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion. The numerical results for the performance of Algorithm 3.1 and Algorithm 4.1 with different Bregman distances are demonstrated in Table 1, Figure 1, and Figure 2.

TABLE 1. Numerical Results of Algorithm 3.1 and Algorithm 4.1

	$M=50, N=100, N_1=25, K=5$		$M=50, N=100, N_1=25, K=10$	
	Iter.	Time(s)	Iter.	Time(s)
$Alg.3.1.KL$	10629	0.7188	5862	0.4844
$Alg.3.1.SE$	77558	0.8906	31104	0.5156
$Alg.4.1.KL$	381	0.0313	826	0.0313
$Alg.4.1.SE$	17081	0.1875	5504	0.0625
	$M=80, N=200, N_1=50, K=10$		$M=80, N=200, N_1=50, K=20$	
	Iter.	Time(s)	Iter.	Time(s)
$Alg.3.1.KL$	2003	0.9219	5809	2.2500
$Alg.3.1.SE$	19814	1.3594	40376	2.4063
$Alg.4.1.KL$	1961	0.7500	2604	1.1406
$Alg.4.1.SE$	17824	1.2656	17562	1.2500



(a) (b)

FIGURE 1. Comparison of the iteration number with different Bregman distances of Algorithm 3.1 and Algorithm 4.1 ($M=80, N=200, N_1=50$).

Table 1 reports iterative numbers and times of Algorithm 3.1 and Algorithm 4.1 with different Bregman distances and different dimensions. Further, in Figure 1 and Figure 2, we compare

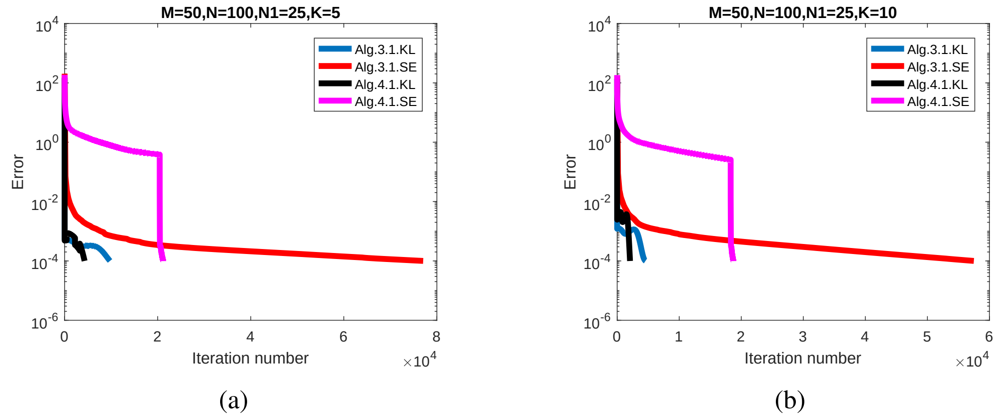


FIGURE 2. Comparison of the iteration number with different Bregman distances of Algorithm 3.1 and Algorithm 4.1 ($M = 50, N = 100, N_1 = 25$).

Algorithm 3.1 and Algorithm 4.1 with different Bregman distances and different sparse cases. Figure 1 and Figure 2 present Error value versus the iteration numbers. From Table 1, Figure 1, and Figure 2, it can be seen that the Kullback-Leibler distance have computational advantage than the squared Euclidean distance for solving Example 6.1 . We can also see that Algorithm 4.1 with the self-adaptive stepsize is more effective than Algorithm 3.1 for solving the above SFP with different dimensions and different Bregman distances.

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