J. Nonlinear Var. Anal. 8 (2024), No. 6, pp. 935-952
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.8.2024.6.06

A VARIATIONAL INEQUALITY TRADE NETWORK MODEL IN PRICES AND QUANTITIES UNDER COMMODITY LOSSES

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Abstract. Multicommodity trade enables the production, consumption, and flow of commodities across the globe from agricultural ones to precious metals. Mathematical formalisms to model, analyze, and solve such problems have advanced and are also relevant to policy and decision-making. In this paper, we construct a variational inequality trade network model in price and quantity variables, which captures possible losses on transportation routes, which can occur because of perishability of commodities, as in the case of agricultural ones, or outright thefts. The equilibrium conditions are stated and the variational inequality formulation derived. Qualitative properties of existence and uniqueness of the equilibrium supply price, commodity shipment, and demand price pattern are provided under reasonable conditions. Illustrative examples help to demonstrate the model. An algorithm that is proposed yields closed form expressions at each iteration and can also be interpreted as a discrete time adjustment process for the evolution of the economic variables. A spectrum of algorithmically solved numerical examples, with full input and output data provided, yields insights into the impacts of commodity losses, increased congestion, as well as enhanced marketing on producers as well as consumers. This new model expands the scope of spatial price equilibrium modeling under commodity losses.

Keywords. Agriculture; Equilibrium problems; Generalized networks; Multicommodity trade networks; Variational inequalities.

1. INTRODUCTION

Commodities are products that have not yet been transformed through a production process with important examples being agricultural commodities, such as fresh produce, energy in the form of oil and gas, and metals such as gold and silver. According to Hayes [1] commodities are often classified as being "hard" with such commodities including natural resources that need to be extracted or mined (precious metals, lithium, etc.), whereas "soft" commodities include agricultural products (fruits, vegetables, corn, wheat, etc.) or livestock. Commodities are the fundamental components of the global economy.

Given the importance of commodities to health in the form of agricultural products and as essential raw materials to high technology products, as well as to energy, multicommodity trade has been the subject of mathematical modeling. Of particular interest is the determination of

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Received 29 April 2024; Accepted 6 September 2024; Published online 25 October 2024.

prices on the production side in the form of supply market prices as well as on the consumption side in the form of demand market prices plus the flows of the commodities between producing regions and consuming ones. In fact, one might even argue that the development of methodological advances has been stimulated, in part, by the need to model, analyze, and solve increasingly more general commodity trade network problems. For example, spatial price equilibrium models, which have served as the foundation for the formulation, analysis, and computation of solutions to commodity trade problems, have significantly advanced since the pioneering work of Samuelson [2] and Takayama and Judge [3, 4]. In particular, both complementarity formulations (see [5, 6, 7]) and variational inequality ones ([8, 9, 10, 11, 12]) have enabled more realistic modeling of multiple commodities, asymmetric price functions, as well as multiple routes joining supply and demand markets. Prior to such contributions, assumptions needed to be made to allow for optimization reformulations of the governing equilibrium conditions. In addition, the evaluation of the impacts of a wide range of policies, from goal targets and quotas to tariffs, is now more tractable in the context of computable, more general spatial price equilibrium models (see, e.g., [13, 14, 15, 16, 17]).

The value of commodities in global trade is immense. For example, the value of the fresh food commodity market globally, which includes fresh fruits, vegetables, and meats, was estimated by Statista [18] at 3,077 billion US dollars in 2021. The number is expected to increase to more than 3,922 billion US dollars by 2027. The global precious metals market, in turn, according to Fortune Business Insights [19], is projected to grow from \$275.40 billion US dollars in 2021 to \$403.08 billion in 2028.

One of the major challenges associated with commodity trade is that of "losses," from that of perishability of fresh produce to outright theft of various commodities. According to the Food and Agricultural Organization [20], quoting [21], approximately 14% of the world's food continues to be lost after harvest and before it reaches retailers. Plus, food and beverage products in the United States in 2023 were top targets for freight theft, with an average loss of \$214,000 according to LaRocco [22]. Goncalves [23] reports that food and drink products were the most commonly stolen items in global supply chains, comprising almost 25% of the reported stolen products in Britain alone. Thefts in Europe, while food is in transit, was noted as being "commonplace." Additional information and data on thefts of high value cargo, including certain foods, such as nuts and salmon, can be found in Nagurney et al. [24].

Furthermore, in an environment of heightened geopolitical risk and negative impacts of climate change, trade is being challenged by a plethora of disasters, both slow-onset and sudden-onset ones, necessitating further research into the quantification of associated economic costs as well as accompanying tools. For collected, edited volumes on dynamics of disasters, we refer the reader to Kotsireas, Nagurney, and Pardalos [25] and Kotsireas et al. [26]. Recent research on spatial price equilibrium modeling in the context of disaster scenarios, and associated disruptions, using variational inequality theory, with a focus on international trade, quantity variables, and with no commodity losses, can be found in Passacantando and Raciti [27] and in Nagurney et al. [28].

In this paper, we construct a new trade network equilibrium model that captures commodity losses, which can be due to perishability, theft, accidents, such as in the case of maritime transportation of cargo, or even attacks, as are happening now in the Red Sea and Suez Canal areas ([29]) and Ukraine (see [30]). The model considers multiple commodities, multiple possible routes joining supply markets with demand markets, and, in equilibrium, yields the supply prices, the demand prices, the value of the commodities at the demand markets plus the commodity shipments (flows) between the supply and demand markets. The underlying functions can be nonlinear and asymmetric. The model integrates generalized networks and variational inequalities and captures commodity losses from supply markets to demand markets on different transportation routes.

Thore [31] was the first to consider the use of arc multipliers for perishability in the context of spatial price equilibrium problems, but in the case of separable functions. Nagurney and Aronson [32] constructed a dynamic spatial price equilibrium model with gains and losses using variational inequality theory. Yu and Nagurney [33] developed an oligopolistic food supply chain network equilibrium model with arc multipliers using game theory with a focus on fresh produce. More recently, Nagurney and Besik [34] proposed aspatial price network equilibrium model in which the arc multipliers are flow-dependent. The model, however, was a single commodity one and assumed a single route between each supply and demand market pair. Nagurney [35] introduced trade policies into a spatial price equilibrium model with perishability but there was only a single route option between each supply and demand market. The majority of the spatial price equilibrium models proposed to-date using variational inequality theory are in quantity variables only.

The new model proposed in this paper is in price and in quantity variables. Nagurney and Zhao [36] developed a spatial price equilibrium model but only the demand functions were functions of prices and not the supply functions, as is the case here. Plus, that model was a single commodity one; also, only a single route was assumed between supply and demand market pairs. Furthermore, the model was a pure network flow one and not a generalized network model as is the case here. Nagurney, Salarpour, and Dong [37] developed a multicommodity spatial price equilibrium model in price and quantity variables and trade policies but without commodity losses. Nagurney, Takayama, and Zhang [38], earlier, constructed a projected dynamical systems model, whose stationary points coincide with a solution to an appropriate variational inequality problem in price and quantity variables. But that model was a single commodity one and assumed a single route joining each pair of supply and demand markets. Also, in contrast to the model in this paper, there were no commodity losses captured.

This paper is organized as follows. In Section 2, the variational inequality multicommodity trade network model in prices and quantities is developed with equilibrium conditions stated, along with qualitative properties of existence and uniqueness of an equilibrium, under appropriate conditions. In addition, illustrative examples are presented to highlight the concepts. In Section 3, we recall the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney [39] and which can be interpreted as a discrete-time adjustment process for the evolution of the commodity supply and demand prices and the commodity shipments until an equilibrium is achieved. This algorithm yields closed form expressions at each time step (iteration) and is easy to implement. We also provide conditions for convergence. The algorithm is then applied in Section 4 to compute solutions to a series of numerical examples, with accompanying insights provided. Section 5 summarizes our results and presents suggestions for future research.

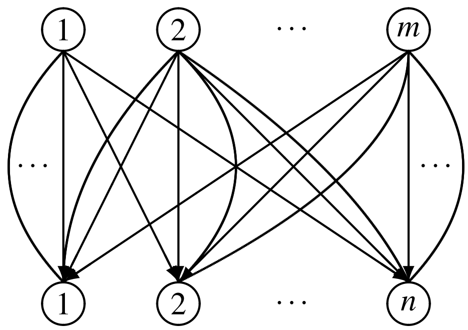

FIGURE 1. The Trade Network Topology

2. THE VARIATIONAL INEQUALITY TRADE NETWORK MODEL IN PRICES AND QUANTITIES

In this section, we present the multicommodity trade network equilibrium model in price and quantity variables that captures commodity losses. The formalism utilized is that of the theory of variational inequalities coupled with the use of arc multipliers as in generalized networks to capture losses associated with the transportation of the commodities from supply markets, where they are produced, to demand markets, where they are purchased. In the trade network model there are m supply markets and n demand markets, with the former involved in the production of K commodities. The supply markets and the demand markets are spatially separated and may be located in the same country or in different countries. If they are located in different countries, we assume that the price functions and the unit transportation cost functions are all in the same currency. A typical commodity is denoted by k, a typical supply market is denoted by i, and a typical demand market by j. We assume that there are n_{ij} routes joining each pair of supply and demand markets (i, j) with a typical route denoted by r. Hence, the number of routes joining a specific pair of markets may differ from the number associated with another pair of markets. A route r joining a pair of supply and demand markets does not need to correspond to the same route r connecting another pair of markets. There is a total of P routes in the trade network. The network representation of the model is depicted in Figure 1.

The basic model notation is given in Table 1. All vectors are column vectors.

Note that, in our new trade network model, the variables are the commodity shipments between supply and demand markets as well as the commodity supply market prices and their demand market prices. In the literature, the spatial price equilibrium models in quantity variables have dominated (see, e.g., [8, 9, 17, 28, 30, 34, 40, 41] and the references therein). It may be easier, however, to estimate supply and demand functions directly and, therefore, having a model such as the one introduced here is very relevant. Plus, it is the first time that commodity losses are included within such a model. The model also extends the theoretical framework for trade networks.

TABLE 1. Notation for the Trade Network Model in Price and Quantity Variables

Notation	Parameter Definition
α^k_{ijr}	the route multiplier associated with commodity k on route r joining i with j; $k = 1,\dots,K$; $i = 1,\dots,m$; $j = 1,\dots,n$; $r = 1,\dots,n_{ij}$. Each such multiplier lies in the range $(0,1]$ with 1 representing no loss.
Notation	**Variable Definition**
π^k_i	the supply price for commodity k; $k = 1,\dots,K$, at supply market i; $i = 1,\dots,m$. We group the supply prices into the vector $\pi \in R^{Km}_+$.
Q^k_{ijr}	the shipment of the commodity k; $k = 1,\dots,K$, from supply market i; $i = 1,\dots,m$, to demand market j; $j = 1,\dots,n$, on route r; $r = 1,\dots,n_{ij}$. We group the commodity shipments into the vector $Q \in R^{KP}_+$.
ρ^k_j	the demand price for commodity k; $k = 1,\dots,K$, at demand market j; $j = 1,\dots,n$. We group the demand prices into the vector $\rho \in R^{Kn}_+$.
Notation	**Function Definition**
$s^k_i(\pi)$	the supply function for commodity k and supply market i; $k = 1,\dots,K$; $i = 1,\dots,m$.
$c^k_{ijr}(Q)$	the unit transportation cost function associated with shipping the commodity k; $k = 1,\dots,K$, from supply market i; $i = 1,\dots,m$, to demand market j; $j = 1,\dots,n$ via route r; $r = 1,\dots,n_{ij}$.
$d^k_j(\rho)$	the demand function for commodity k at demand market j; $k = 1,\dots,K$; $j = 1,\dots,n$.

We assume that the supply, demand, and unit transportation cost functions are all continuous. We now present the trade network equilibrium conditions under commodity losses and then provide the variational inequality formulation.

Definition 2.1. (The International Trade Network Equilibrium Conditions Under Commodity Losses) A multicommodity supply price, shipment, and demand price pattern $(\pi^*, Q^*, \rho^*) \in \mathscr{K}$, where

$$\mathscr{K} \equiv \{(\pi, Q, \rho) | (\pi, Q, \rho) \in R^{Km+KP+Kn}_+\}$$

is a trade network equilibrium under commodity losses, if the following conditions hold: for all commodities k; $k = 1,\dots,K$, and for all supply markets i; $i = 1,\dots,m$:

$$s^k_i(\pi^*) \begin{cases} = \sum_{j=1}^n \sum_{r=1}^{n_{ij}} Q^{k*}_{ijr}, & \text{if } \pi^{k*}_i > 0, \\ \geq \sum_{j=1}^n \sum_{r=1}^{n_{ij}} Q^{k*}_{ijr}, & \text{if } \pi^{k*}_i = 0; \end{cases} \tag{1}$$

for all commodities k; $k = 1,\dots,K$; for all supply and demand market pairs: (i,j); $i = 1,\dots,m$; $j = 1,\dots,n$, and for all routes r; $r = 1,\dots,n_{ij}$:

$$\pi^{k*}_i + c^k_{ijr}(Q^*) \begin{cases} = \alpha^k_{ijr}\rho^{k*}_j, & \text{if } Q^{k*}_{ijr} > 0, \\ \geq \alpha^k_{ijr}\rho^{k*}_j, & \text{if } Q^{k*}_{ijr} = 0; \end{cases} \tag{2}$$

and for all commodities k; $k = 1, \ldots, K$, and for all demand markets j; $j = 1, \ldots, n$:

$$d_j^k(\rho^*) \begin{cases} = \sum_{i=1}^m \sum_{r=1}^{n_{ij}} \alpha_{ijr}^k Q_{ijr}^{k*}, & \text{if } \rho_j^{k*} > 0, \\ \leq \sum_{i=1}^m \sum_{r=1}^{n_{ij}} \alpha_{ijr}^k Q_{ijr}^{k*}, & \text{if } \rho_j^{k*} = 0. \end{cases} \tag{3}$$

The above trade network equilibrium conditions extend the spatial price equilibrium conditions in the case of quantity variables only and losses captured through arc multipliers. In particular, in the special case of a single commodity and no losses, and only a single route between each pair of supply and demand markets, the above conditions collapse to those articulated in Nagurney [42] for the single commodity spatial price equilibrium model in price and quantity variables, which is a pure network flow model. The above equilibrium conditions broaden the scope of problems that can be modeled, and, as we shall also demonstrate, solved, providing additional tools for decision-making and policy-making in the important realm of commodity trade.

We now interpret the above trade network equilibrium conditions. Equilibrium conditions (1) state that, for each commodity and each supply market, the supply at the equilibrium is equal to the sum of the shipments on the routes from the supply market of the commodity to all the demand markets if the supply price of the commodity at the supply market is positive. If on the other hand, the price is equal to zero, then there can be an excess supply. Equilibrium conditions (2), in turn, state that, for each pair of supply and demand markets, if there is a positive shipment of a commodity between them on a route then the supply price of the commodity plus the unit transportation cost is equal to the value of the commodity at the demand market. Note that the value is equal to the multiplier on the route for the commodity times the equilibrium price at the demand market for the commodity (see also Thore [31]). If there is a zero commodity shipment on that route of that commodity then the supply price plus the unit transportation cost on the route for the commodity can exceed the value of the commodity at the demand market. Finally, equilibrium conditions (3) state that, in equilibrium, the actual shipments of a commodity from all the supply markets to a demand market (which reflect the quantities that arrive at the demand markets, after the losses) must be equal to the demand at the demand market for the commodity if the demand price is positive. On the other hand, if there is an excess amount of shipments of the commodity to a demand market, then the price of the commodity at the demand market will be equal to zero.

We now provide the variational inequality formulation of the above equilibrium conditions.

Theorem 2.1. *(Variational Inequality Formulation of the Trade Network Equilibrium Conditions Under Commodity Losses) A multicommodity supply price, commodity shipment, and demand price pattern $(\pi^*, Q^*, \rho^*) \in \mathcal{K}$ is a trade network equilibrium under commodity losses, according to Definition 2.1, if and only if it satisfies the variational inequality:*

$$\sum_{k=1}^K \sum_{i=1}^m \left[s_i^k(\pi^*) - \sum_{j=1}^n \sum_{r=1}^{n_{ij}} Q_{ijr}^{k*} \right] \times (\pi_i^k - \pi_i^{k*})$$
$$+ \sum_{k=1}^K \sum_{i=1}^m \sum_{j=1}^n \sum_{r=1}^{n_{ij}} \left[\pi_i^{k*} + c_{ijr}^k(Q^*) - \alpha_{ijr}^k \rho_j^{k*} \right] \times (Q_{ijr}^k - Q_{ijr}^{k*})$$
$$+ \sum_{k=1}^K \sum_{j=1}^n \left[\sum_{i=1}^m \sum_{r=1}^{n_{ij}} \alpha_{ijr}^k Q_{ijr}^{k*} - d_j^k(\rho^*) \right] \times (\rho_j^k - \rho_j^{k*}) \geq 0, \quad \forall (\pi, Q, \rho) \in \mathcal{K}. \tag{4}$$

Proof. First, necessity is established; i.e., if $(\pi^*, Q^*, \rho^*) \in \mathscr{K}$ satisfies equilibrium conditions (1) through (3), then it also satisfies variational inequality (4). From equilibrium conditions (1), for a fixed k, i, and equilibrium pattern, we know that

$$\left[s_i^k(\pi^*) - \sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q_{ijr}^{k*}\right] \times (\pi_i^k - \pi_i^{k*}) \geq 0, \quad \forall \pi_i^k \geq 0. \tag{5}$$

Summing (5) over all indices k, i yields

$$\sum_{k=1}^{K}\sum_{i=1}^{m}\left[s_i^k(\pi^*) - \sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q_{ijr}^{k*}\right] \times (\pi_i^k - \pi_i^{k*}) \geq 0, \quad \forall \pi \geq R_+^{Km}. \tag{6}$$

From equilibrium conditions (2), in turn, for an equilibrium commodity shipment and price pattern, and for fixed k, i, j, r, we have that

$$\left[\pi_i^{k*} + c_{ijr}^k(Q^*) - \alpha_{ijr}^k \rho_j^{k*}\right] \times (Q_{ijr}^k - Q_{ijr}^{k*}) \geq 0, \quad \forall Q_{ijr}^k \geq 0. \tag{7}$$

Indeed, since, if $Q_{ijr}^{k*} > 0$, then the left-hand side in (7) is zero, so (7) holds. Observe $Q_{ijr}^k \geq Q_{ijr}^{k*}$. If $Q_{ijr}^{k*} = 0$, then the left-hand side expression in (7) is nonnegative, and (7) holds. Because (7) is true for any k, i, j, r, its summation over all these indices results in

$$\sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{n_{ij}}\left[\pi_i^{k*} + c_{ijr}^k(Q^*) - \alpha_{ijr}^k \rho_j^{k*}\right] \times (Q_{ijr}^k - Q_{ijr}^{k*}) \geq 0, \quad \forall Q \in R_+^{KP}. \tag{8}$$

Plus, from equilibrium conditions (3), we have that, for fixed k, j, and equilibrium pattern,

$$\left[\sum_{i=1}^{m}\sum_{r=1}^{n_{ij}} \alpha_{ijr}^k Q_{ijr}^{k*} - d_j^k(\rho^*)\right] \times (\rho_j^k - \rho_j^{k*}) \geq 0, \quad \forall \rho_j^k \geq 0. \tag{9}$$

Summing (9) over all indices k, j, we have

$$\sum_{k=1}^{K}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{r=1}^{n_{ij}} \alpha_{ijr}^k Q_{ijr}^{k*} - d_j^k(\rho^*)\right] \times (\rho_j^k - \rho_j^{k*}) \geq 0, \quad \forall \rho \in R_+^{Kn}. \tag{10}$$

Adding (6), (8), and (10) yields variational inequality (4). Necessity is done.

We now establish sufficiency.

Setting $Q_{ijr}^k = Q_{ijr}^{k*}$ for all k, i, j, r; $\rho_j^k = \rho_j^{k*}$ for all k, j; and $\pi_i^k = \pi_i^{k*}$ for all k, i except for $k = \tilde{k}$, $i = \tilde{i}$, and substituting the resultants into (4), one reduces the variational inequality (4) to

$$\left[s_{\tilde{i}}^{\tilde{k}}(\pi^*) - \sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q_{\tilde{i}jr}^{\tilde{k}*}\right] \times (\pi_{\tilde{i}}^{\tilde{k}} - \pi_{\tilde{i}}^{\tilde{k}*}) \geq 0, \quad \forall \pi_{\tilde{i}}^{\tilde{k}} \geq 0. \tag{11}$$

This indicates equilibrium conditions (1) holds.

Setting now $\pi_i^k = \pi_i^{k*}$ for all k, i; $\rho_j^k = \rho_j^{k*}$ for all k, j; and $Q_{ijr}^k = Q_{ijr}^{k*}$ for all k, i, j, r except for $k = \tilde{k}$, $i = \tilde{i}$, $j = \tilde{j}$, and $r = \tilde{r}$, and substituting the resultants into (4), one reduces the variational inequality (4) to

$$\left[\pi_{\tilde{i}}^{\tilde{k}*} + c_{\tilde{i}\tilde{j}\tilde{r}}^{\tilde{k}}(Q^*) - \alpha_{\tilde{i}\tilde{j}\tilde{r}}^{\tilde{k}} \rho_{\tilde{j}}^{\tilde{k}*}\right] \times (Q_{\tilde{i}\tilde{j}\tilde{r}}^{\tilde{k}} - Q_{\tilde{i}\tilde{j}\tilde{r}}^{\tilde{k}*}) \geq 0, \quad \forall Q_{\tilde{i}\tilde{j}\tilde{r}}^{\tilde{k}} \geq 0, \tag{12}$$

from which it follows that equilibrium conditions (2) hold.

Finally, setting $Q^k_{ijr} = Q^{k*}_{ijr}$ for all k,i,j,r; $\pi^k_i = \pi^{k*}_i$ for all k,i; and $\rho^k_j = \rho^{k*}_j$ for all k,j except for $k = \tilde{k}$, $j = \tilde{j}$, and substituting the resultants into (4), we have

$$\left[d^{\tilde{k}}_{\tilde{j}}(\rho^*) - \sum_{i=1}^{m}\sum_{r=1}^{n_{ij}} \alpha^{\tilde{k}}_{i\tilde{j}r} Q^{\tilde{k}*}_{i\tilde{j}r}\right] \times (\rho^{\tilde{k}}_{\tilde{j}} - \rho^{\tilde{k}*}_{\tilde{j}}) \geq 0, \quad \forall \rho^{\tilde{k}}_{\tilde{j}} \geq 0. \tag{13}$$

Hence, equilibrium conditions (3) must hold. Sufficiency is now established. □

Remark 2.1. We note that, in the case of a policy intervention, such as that of a unit tariff, τ^k_{ij} associated with specific commodities k and supply and demand market pairs (i,j), one can adapt equilibrium condition (2) by adding the term τ^k_{ij} to the left-hand side of the expression with recognition that only certain commodities and market pairs may be subject to tariffs. We now put variational inequality (4) is into standard form (cf. [42]), VI$(F,\mathscr{K})$, where one seeks to determine a vector $X^* \in \mathscr{K} \subset R^N$ such that

$$\langle F(X^*), X - X^*\rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{14}$$

with F being a given continuous function from $\mathscr{K}$ to R^N, where $\mathscr{K}$ is a given closed, convex set, and $\langle\cdot,\cdot\rangle$ denotes the inner product in N-dimensional Euclidean space. We define $X \equiv (\pi, Q, \rho)$ and $N \equiv Km + KP + Kn$. The feasible set $\mathscr{K}$ remains as before. Here we have $N = Km + KP + Kn$. $F(X)$ is the vector function mapping $\mathscr{K}$ into $R^{Km+KP+Kn}_{+}$ and is defined by the vector

$$F(X) \equiv (S(X), T(X), D(X)), \tag{15}$$

where $S(X)$ has components S_{ki} such that

$$S_{ki} = s^k_i(\pi) - \sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q^k_{ijr}, \quad k = 1,\ldots,K; i = 1,\ldots,m, \tag{16a}$$

$T(X)$ has components T_{kijr} such that

$$T_{kijr} = \pi^k_i + c^k_{ijr}(Q) - \alpha^k_{ijr}\rho^k_j, \quad k = 1,\ldots,K; i = 1,\ldots,m; j = 1,\ldots,n; r = 1,\ldots,n_{ij}, \tag{16b}$$

and D has components D_{kj}, where

$$D_{kj} = \sum_{i=1}^{m}\sum_{r=1}^{n_{ij}} \alpha^k_{ijr} Q^k_{ijr} - d^k_j(\rho), \quad k = 1,\ldots,K; j = 1,\ldots,n. \tag{16c}$$

Variational inequality (4) can, clearly, be put into standard form (14).

2.1 Qualitative properties. We now investigate qualitative properties of existence and uniqueness of the solution to variational inequality (14) with $F(X)$ being defined as in (15). Specifically, we construct

$$\langle F(X^1) - F(X^2), X^1 - X^2\rangle$$
$$= \sum_{k=1}^{K}\sum_{i=1}^{m}\left[(s^k_i(\pi^1) - \sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q^{k1}_{ijr}) - (s^k_i(\pi^2) - \sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q^{k2}_{ijr})\right] \times (\pi^{k1}_i - \pi^{k2}_i)$$
$$+ \sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{n_{ij}}\left[(\pi^{k1}_i + c^k_{ijr}(Q^1) - \alpha^k_{ijr}\rho^{k1}_j) - (\pi^{k2}_i + c^k_{ijr}(Q^2) - \alpha^k_{ijr}\rho^{k2}_j)\right] \times (Q^{k1}_{ijr} - Q^{k2}_{ijr})$$
$$+ \sum_{k=1}^{K}\sum_{j=1}^{n}\left[(\sum_{i=1}^{m}\sum_{r=1}^{n_{ij}} \alpha^k_{ijr} Q^{k1}_{ijr} - d^k_j(\rho^1)) - (\sum_{i=1}^{m}\sum_{r=1}^{n_{ij}} \alpha^k_{ijr} Q^{k2}_{ijr} - d^k_j(\rho^2))\right] \times (\rho^{k1}_j - \rho^{k2}_j)$$

$$= \sum_{k=1}^{K}\sum_{i=1}^{m}\left[s_i^k(\pi^1) - s_i^k(\pi^2)\right] \times (\pi_i^{k1} - \pi_i^{k2})$$

$$+ \sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{n_{ij}}\left[c_{ijr}^k(Q^1) - c_{ijr}^k(Q^2)\right] \times (Q_{ijr}^{k1} - Q_{ijr}^{k2})$$

$$+ \sum_{k=1}^{K}\sum_{j=1}^{n}\left[-d_j^k(\rho^1) + d_j^k(\rho^2)\right] \times (\rho_j^{k1} - \rho_j^{k2}) \tag{17}$$

$$= \langle s(\pi^1) - s(\pi^2), \pi^1 - \pi^2\rangle + \langle c(Q^1) - c(Q^2), Q^1 - Q^2\rangle - \langle d(\rho^1) - d(\rho^2), \rho^1 - \rho^2\rangle. \tag{18}$$

We can conclude from (17) and (18) that necessary and sufficient conditions for $F(X)$ to be coercive (monotone, strictly monotone, strongly monotone) is that $s(\pi)$, $c(Q)$, and -$d(\rho)$ are coercive (monotone, strictly monotone, strongly monotone) in π, Q, and ρ, respectively.

Thus it follows from the classical theory of variational inequalities (cf. [42, 43]) that if the supply functions, the unit transportation cost functions and minus the demand price functions (in vector form), are strongly monotone, then a solution to variational inequality (14); equivalently, variational inequality (4) exists and is unique. Of course, one could also impose conditions on the prices and the commodity shipments to ensure that they are bounded, in which the feasible set $\mathscr{K}$ will be compact and the existence of a solution would be guaranteed since we have assumed continuity of the price and unit transportation cost functions. Then, strict monotonicity of $F(X)$ would guarantee uniqueness of X^*.

2.2 Illustrative examples. Illustrative examples are now presented to highlight some of the concepts. The trade network consists of a single supply market and two demand markets as depicted in Figure 2.

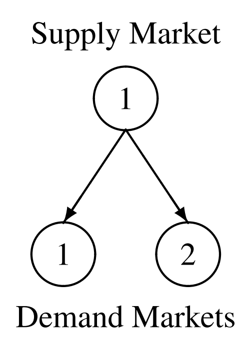

FIGURE 2. Trade Network Topology for the Illustrative Examples

The data are as follows. There is a single commodity. Thus we suppress the superscript k. The supply function is $s_1(\pi_1) = \pi_1 - 10$. The demand functions are

$$d_1(\rho_1) = -\rho_1 + 62, \quad d_2(\rho_2) = -\rho_2 + 84$$

and the unit transportation cost functions are

$$c_{111}(Q_{111}) = Q_{111} + 2, \quad c_{121}(Q_{121}) = Q_{121} + 4.$$

We first consider the situation of no losses and, therefore, $\alpha_{111} = \alpha_{121} = 1$. Using the equilibrium conditions (1) through (3), we obtain the following trade network equilibrium pattern

$$\pi_1^* = 40.0000, \quad Q_{111}^* = 10.0000, \quad Q_{121}^* = 20.0000, \quad \rho_1^* = 52.0000, \quad \rho_2^* = 64.0000,$$

with $s_1(\pi_1^*) = 30.0000$, $c_{111}(Q_{111}^*) = 12.0000$, $c_{121}(Q_{121}^*) = 24.0000$, $d_1(\rho_1^*) = 10.0000$, and $d_2(\rho_2^*) = 20.0000$. The equilibrium conditions (1) through (3) hold precisely.

We now assume that there is no loss on the route joining the supply market 1 with demand market 2, but there is a loss on the route joining the supply market with demand market 1 of 20%, so the "throughput" on that route is 80%. Thus $\alpha_{111} = .8$ and $\alpha_{121} = 1$. The rest of the data remain as in the above example.

The new equilibrium solution is

$$\pi_1^* = 37.4566, \quad Q_{111}^* = 6.1849, \quad Q_{121}^* = 21.2716, \quad \rho_1^* = 57.0520, \quad \rho_2^* = 62.7283.$$

Now the incurred function values at the equilibrium are $s_1(\pi_1^*) = 27.4565$, $c_{111}(Q^{111*}) = 8.1849$, $c_{121}(Q_{121}^*) = 25.2716$, $d_1(\rho_1^*) = 4.9479$, and $d_2(\rho_2^*) = 21.2716$. The commodity shipment Q_{111}^* to the first demand market decreases from 10.0000 to 6.1849, whereas the commodity shipment Q_{121}^* now increases from 20.0000 to 21.2716. The demand price at the first demand market is now higher than before, at about 10%, whereas that at the second demand market is lower. The overall supply of the commodity decreases by more than 10% and the supply market price is also lower. Hence, if the commodity were an agricultural one, the producers, which are the farmers, would suffer economically and also the consumers at the first demand market would suffer since they incur a higher demand price and a lower volume of the commodity than when there are no commodity losses on the route. Consumers at the second demand market benefit, signifying also a competitive element to the trade network economy.

We now conduct sensitivity analysis. As the multiplier α_{111} decreases, the volume of the commodity shipment to the first demand market also decreases. From $\alpha_{111} = .5698$, until the lower boundary of 0.0000, which is not defined, the equilibrium commodity shipment Q_{111}^* is equal to 0.0000 since each α multiplier lies in $(0,1]$.

3. THE ALGORITHM

For the solution of additional numerical examples, we utilize the Euler Method, which was induced by the general iterative scheme of Dupuis and Nagurney [39]. Specifically, the statement at iteration $\tau+1$ of this algorithm (see also [44]) is

$$X^{\tau+1} = P_{\mathscr{K}}(X^{\tau} - a_{\tau}F(X^{\tau})), \tag{19}$$

where $P_{\mathscr{K}}$ is the projection on the feasible set $\mathscr{K}$ and F is the function that enters the variational inequality problem (15).

As demonstrated in [39] and [44], for convergence, the sequence $\{a_{\tau}\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, and $a_{\tau} \to 0$, as $\tau \to \infty$.

Explicit Formulae for the Euler Method Applied to the Trade Network Model

The algorithm results in the following closed form expression for the commodity supply prices at iteration $\tau+1$, for $k = 1, \ldots K; i = 1, \ldots, m$:

$$\pi_i^{k,\tau+1} = \max\{0, a_{\tau}(\sum_{j=1}^{n}\sum_{r=1}^{n_{ij}} Q_{ijr}^{k,\tau} - s_i^k(\pi^{\tau})) + \pi_i^{k,\tau}\}. \tag{20}$$

It yields the following explicit formula for the commodity flows at iteration $\tau+1$: for $k = 1,\dots,K$; $i=1,\dots,m$; $j=1,\dots,n$:

$$Q_{ijr}^{k,\tau+1} = \max\{0, a_\tau(\alpha_{ijr}^k \rho_j^{k,\tau} - c_{ijr}^k(Q^\tau) - \pi_i^{k,\tau}) + Q_{ijr}^{k,\tau}\} \tag{21}$$

and the following closed form expression at iteration $\tau+1$ for the commodity demand prices for $k=1,\dots,K$; $j=1,\dots,n$:

$$\rho_j^{k,\tau+1} = \max\{0, a_\tau(d_j^k(\rho^\tau) - \sum_{i=1}^{m}\sum_{r=1}^{P} \alpha_{ijr}^k Q_{ijr}^{k,\tau}) + \rho_j^{k,\tau}\}. \tag{22}$$

Expressions (20)-(22) have an interpretation as discrete-time adjustment processes. For example, expression (20) provides a discrete time evolution of the supply market prices, given the preceding iteration's values for the supply market prices and the commodity shipments out of the supply markets. Expression (21), in turn, provides the evolution in discrete time of the commodity shipments at an iteration, given the preceding commodity shipments and the supply and demand market prices. According to (21) the commodity shipment will increase if the value of the commodity at the demand market (which includes the demand price) exceeds the supply price and the unit transportation cost associated with the cognizant pair of supply and demand markets. Also, expression (22) provides a discrete time evolution of the demand market prices. Note that these expressions guarantee that the equilibrium values of the variables never become negative, which would violate feasibility.

The convergence result is now stated. The proof follows from [44, Theorem 6.12].

Theorem 3.1. *(Convergence) If the $F(X)$ in the trade network model (cf. (15)) is strongly monotone and F is Lipschitz continuous, that is,*

$$\|F(X^1) - F(X^2)\| \le L\|X^1 - X^2\|, \quad \forall X^1, X^2 \in \mathscr{K}, \tag{23}$$

where L is a positive number known as the Lipschitz constant, then there exists a unique equilibrium $(\pi^*, Q^*, \rho^*) \in \mathscr{K}$ and any sequence generated by the algorithm as in (19), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$, converges to (π^*, Q^*, ρ^*).

4. Algorithmically Computed Solutions to Numerical Examples

In this section, we report results for numerical examples that are solved via the Euler Method. The algorithm was implemented in FORTRAN and a Linux system at the University of Massachusetts Amherst used for the computations. The algorithm was initialized with all prices and all commodity flows equal to 0.0000. The $\{a_\tau\}$ series was set to: $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\}$. The algorithm was deemed to have converged if the absolute value of each variable at two successive iterations differed by no more than 10^{-5}.

Example Set 1: 2 Supply Markets and 2 Demand Markets

The first set of algorithmically solved numerical examples consists of problems with the trade network topology given in Figure 3. There are two supply markets and two demand markets. There is a single commodity so we suppress the superscript k. There is a single route joining each pair of supply and demand markets.

Example 1 serves as the baseline. In this example, it is assumed that there are no losses. Hence, $\alpha_{111} = \alpha_{121} = \alpha_{211} = \alpha_{221} = 1$. We consider the commodity to be an agricultural one.

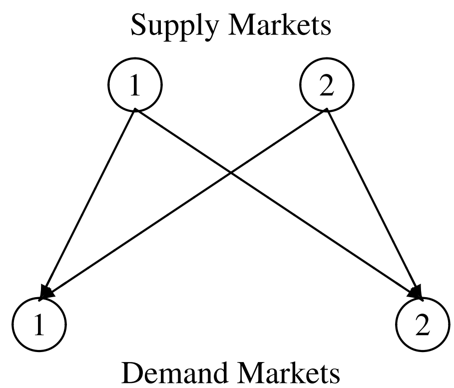

FIGURE 3. The Trade Network Topology for Numerical Examples 1 Through 4

The supply functions are $s_1(\pi_1) = .01\pi_1 + 10$ and $s_2(\pi_2) = .05\pi_2 + 12$. The unit transportation cost functions are

$$c_{111}(Q_{111}) = Q_{111} + 4,\ c_{121}(Q_{121}) = 2Q_{121} + 10,\ c_{211}(Q_{211}) = Q_{211} + 8, c_{221}(Q_{221}) = Q_{221} + 6.$$

The demand price functions are $d_1(\rho_1) = -\rho_1 + 400$ and $d_2(\rho_2) = -\rho_2 + 380$.

Example 2 has the same data as Example 1 except that we assume that there are losses associated with transport from the first supply market to the first demand market with $\alpha_{111} = .8$.

Example 3 has the same data as Example 2 but now there are also losses from the second supply market to the first demand market with $\alpha_{211} = .8$.

Finally, Example 4 considers the most disrupted scenario with all four route multipliers equal to .8. The computed equilibrium prices and commodity shipments are reported in Table 2. It is interesting to see that, in Example 2, the commodity shipment on the route with losses, Q^*_{111}, drops to 0.0000, whereas, in Example 1, the value was positive. Furthermore, the supply price at the first supply market, π^*_1, decreases, whereas that at the second supply market, π^*_2, increases, albeit slightly. The demand price at the first demand market, ρ^*_1, increases but decreases at the second demand market (ρ^*_2). The volume of commodity shipments to demand market 1 decrease, but increase to demand market 2.

Also, in Example 3, with both routes to demand market 1 being subject to losses, the commodity shipment from supply market 1 to demand market 1 is no longer equal to 0.0000, as it was in Example 2, but the commodity shipment from supply market 2 to demand market 1, Q^*_{211}, drops to almost 0.0000. The supply market prices decrease at both supply markets. The demand market price at the first demand market, ρ^*_1, increases (as compared to the price in Example 2) but decreases at the second demand market (ρ^*_2).

In Example 4, with all the routes being subject to losses, the supplies of the agricultural commodity at the supply markets are at their lowest values, as are the supply market prices, reflecting the economic costs to producers of losses in the trade network. The demand price at the second demand market, ρ^*_2, is the highest of all the examples, and that for the first demand market, ρ^*_1, is at its next to the highest value.

This set of examples reveals the importance of ensuring that commodity losses on routes are minimized; otherwise, both producers and consumers may suffer.

Equilibrium Commodity Flows	**Ex. 1**	**Ex. 2**	**Ex. 3**	**Ex. 4**
Q^*_{111}	12.7146	0.0000	2.5852	12.5047
Q^*_{121}	0.8288	13.2338	10.5290	0.3548
Q^*_{211}	16.2005	22.4840	0.5016	14.8870
Q^*_{221}	13.1434	6.8672	26.9741	11.0918
Equilibrium Supply Prices	**Ex. 1**	**Ex. 2**	**Ex. 3**	**Ex. 4**
π^*_1	354.3705	323.4315	311.4394	285.9648
π^*_2	346.8849	347.0325	309.5230	279.5825
Incurred Transportation Costs	**Ex. 1**	**Ex. 2**	**Ex. 3**	**Ex. 4**
$c_{111}(Q^*)$	16.7146	4.0000	6.5852	16.5047
$c_{121}(Q^*)$	11.6576	36.4677	31.0579	10.7097
$c_{211}(Q^*)$	24.2005	30.4840	8.5016	22.8870
$c_{221}(Q^*)$	19.1434	12.8672	32.9742	17.0918
Equilibrium Demand Prices	**Ex. 1**	**Ex. 2**	**Ex. 3**	**Ex. 4**
ρ^*_1	371.0854	377.5165	397.5310	378.0870
ρ^*_2	366.0283	359.8995	342.4973	370.8431

TABLE 2. Equilibrium Commodity Shipments and Supply Prices, Incurred Transportation Costs, and Demand Prices for Examples 1, 2, 3, and 4

Example Set 2: 3 Supply Markets and 2 Demand Markets

The second set of algorithmically solved numerical examples consists of three supply markets and two demand markets, as depicted in Figure 4.

The data for Example 5 are identical to the data in Example 4 but with the following data for supply market 3 added. The supply function at supply market 3 is $s_3(\pi_3) = .01\pi_3 + 14$. The unit transportation cost functions from the third supply market to the demand markets are:

$$c_{311}(Q_{311}) = 2Q_{311} + 5, \quad c_{321}(Q_{321}) = 2Q_{321} + 6.$$

The additional route multipliers are $\alpha_{311} = \alpha_{321} = 1.00$. The computed equilibrium solution is reported in Table 3.

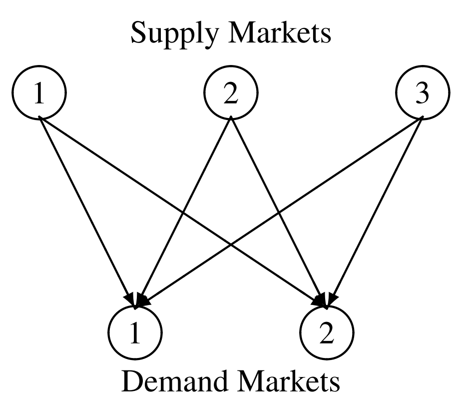

FIGURE 4. The Trade Network Topology for Numerical Examples 5 Through 9

Example 6 has the same data as that in Example 5 except we now explore the impacts of heightened congestion on the routes from supply market 3 to the demand markets. In particular, we use Bureau of Public Road (BPR) type cost functions (cf. National Research Council [45]) which are useful in capturing congestion on transportation networks and not necessarily just road networks. Hence, we add a term to the previous unit transportation cost functions emanating from supply market 3, to the fourth power, as follows:

$$c_{311}(Q_{311}) = .2Q_{311}^4 + 2Q_{311} + 5, \quad c_{321}(Q_{321}) = .2Q_{321}^4 + 2Q_{321} + 6.$$

The computed equilibrium solution is reported in Table 3.

In Example 6, with greater sensitivity to congestion, the commodity shipments from supply market 3, Q_{311}^* and Q_{321}^*, decrease, and, interestingly, the supply market price π_3^* drops to 0.0000. This is an example of the second case in equilibrium conditions (1) holding in that $s_3(\pi_3^*) = 14.0000$ is greater than $Q_{311}^* + Q_{321}^* = 12.9326$. The demand prices increase at both demand markets and the supply prices increase at supply markets 1 and 2. The transportation costs on routes from supply market 3 increase dramatically, by more than tenfold.

Equilibrium Commodity Flows	**Ex. 5**	**Ex. 6**	**Ex. 7**	**Ex. 8**	**Ex. 9**
Q_{111}^*	12.0316	12.4669	13.8392	14.7120	15.4152
Q_{121}^*	0.7577	0.3408	1.1386	1.6453	2.4536
Q_{211}^*	14.0807	14.7581	14.6297	16.4622	18.092
Q_{221}^*	11.5644	10.9719	11.0670	13.0397	15.5834
Q_{311}^*	10.3760	6.4852	6.4807	6.8850	6.8523
Q_{321}^*	7.0534	6.4475	6.4442	6.8552	6.8310
Equilibrium Supply Prices	**Ex. 5**	**Ex. 6**	**Ex. 7**	**Ex. 8**	**Ex. 9**
π_1^*	278.9560	280.917	278.7572	355.8288	440.2265
π_2^*	272.9071	274.6315	273.9674	350.0803	433.5494
π_3^*	342.9823	0.0000	0.0000	0.0000	0.0000
Incurred Transportation Costs	**Ex. 5**	**Ex. 6**	**Ex. 7**	**Ex. 8**	**Ex. 9**
$c_{111}(Q^*)$	16.0316	16.4669	17.8392	18.7120	19.4152
$c_{121}(Q^*)$	11.5154	10.6815	12.2773	13.2905	14.9073
$c_{211}(Q^*)$	22.0807	22.7581	22.6297	24.4622	26.0922
$c_{221}(Q^*)$	17.5644	16.9719	17.0670	19.0397	21.5834
$c_{311}(Q^*)$	25.7521	371.7363	370.7457	468.1780	459.6423
$c_{321}(Q^*)$	20.1068	364.5039	363.7932	461.3988	455.1340
Equilibrium Demand Prices	**Ex. 5**	**Ex. 6**	**Ex. 7**	**Ex. 8**	**Ex. 9**
ρ_1^*	368.7346	371.7365	370.7459	468.1777	459.6423
ρ_2^*	363.0893	364.5042	363.7940	461.3988	455.1341

TABLE 3. Equilibrium Commodity Shipments and Supply Prices, Incurred Transportation Costs, and Demand Prices for Examples 5, 6, 7, 8, and 9

Example 7 has the same data as Example 6 but now the supply function at supply market 1 has an additional term such that $s_1(\pi) = .01\pi_1 + .008\pi_2 + 10$. The computed equilibrium solution is also given in Table 3.

In Example 7, we see the impacts of increased competition on the supply side, which results in lower demand market prices at both demand markets. The supply market price π_1^* remains at 0.0000.

The last example in this set, Example 8, has the same data as that in Example 7, but now we consider that some marketing has taken place and the demand at both demand markets has increased, as reflected in the higher intercept terms, as follows:

$$d_1(\rho_1) = -\rho_1 + 500, \quad d_2(\rho_2) = -\rho_2 + 480.$$

Results are, again, reported in Table 3.

In Example 8, with consumers having a greater demand for the agricultural commodity, all the commodity shipments increase, as do the supply market prices, except for π_3^*, which remains at 0.0000. The demand market prices also increase, and more substantially than the supply market prices.

In Example 9, in a sense, we come "full circle." Example 9 has the identical data to that in Example 8 except that all the route multipliers are equal to 1.0000; hence, there are no commodity losses. Please refer to Table 3 for the equilibrium solutions. In Example 9, all the commodity shipments are higher than the corresponding ones in Example 8, except the ones from supply market 3, where the price is equal to 0.0000 and these shipments are reduced minimally. The supply market prices for the commodity are also higher and significantly so (except at supply market 3 where the supply price remains at 0.0000) whereas the demand market prices are lower. Hence, both farmers, the producers, at supply markets 1 and 2, as well as the consumers at both demand markets benefit from no commodity losses in the trade network.

We now proceed to conduct sensitivity analysis on the intercept of the supply function for supply market 3, which recall is: $s_3(\pi_3) = .01\pi_3 + 14$. Specifically, we retain the data as in Example 9 but vary the fixed term from 2 through 14. The results for the equilibrium price at supply market 3 are depicted in Figure 5.

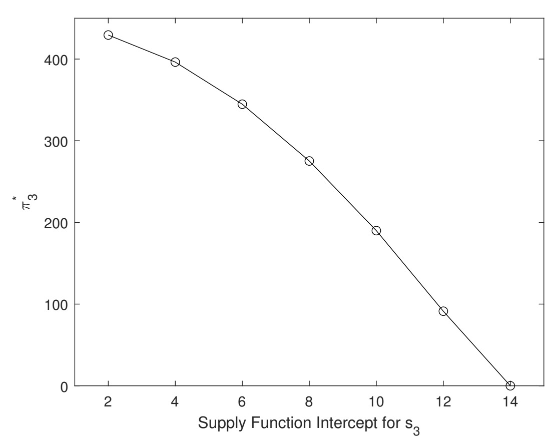

FIGURE 5. Sensitivity Analysis for the Supply Function s_3 intercept based on Example 9

In addition, we note that the equilibrium supply price at supply market 1 ranges from 443.73 for the intercept at 2 to 440.23 for the intercept of 14. Also, the equilibrium supply price at supply market 2 ranges from 437.03 to 433.55 in that same range of supply price intercept values. The high transportation costs of shipping the commodity from supply market 3 to the demand markets come to dominate as the intercept increases and, in order for transactions to take place in the form of commodity shipments, the equilibrium supply price at supply market 3 decreases until it reaches its lower bound of 0.0000.

The numerical results reinforce the importance of reducing commodity losses in transportation as we as the negative impacts of congestion.

5. Summary and Suggestions for Future Research

In this paper, we expanded the theoretical and modeling framework for trade networks under commodity losses by constructing a variational inequality model integrated with generalized networks. Commodities that were subject to losses include agricultural commodities, due to perishability, as well as high value minerals, due to theft, among others. The trade network model, unlike previous spatial price equilibrium models with losses, is in price and quantity variables. We presented the equilibrium conditions and provided conditions for existence and uniqueness of the equilibrium supply price, commodity shipment, and demand price pattern. The proposed algorithm provides a time-discretization of what may be interpreted as a continuous time evolution of the underlying economic variables. Conditions for convergence were provided. Both illustrative examples as well as algorithmically computed ones, with full input and output data reported, reveal the various scenarios that can be explored from changes in the route multipliers associated with the losses to an increase in the number of supply markets, greater competition, as well as greater sensitivity to congestion on transportation routes. In addition, the impacts of enhanced marketing on the demand side is also demonstrated. Importantly, the inclusion of the presence of commodity losses from supply markets to demand markets, as in transportation, quantitatively reveals the negative impacts on producers as well as on consumers in terms of prices and volumes of commodity shipments.

Future possible directions for research include the construction of insurance policies associated with such and, possibly, other losses in commodity trade networks, as well as the integration of alternative functional forms for transportation that explicitly include capacities since many regions on the globe are now subject to reduced capacity, including maritime route capacity due to climate change and also wars and associated attacks. Additional research integrating generalized networks and price and quantity variable spatial price network models on more complex transportation network topologies is also warranted.

Acknowledgments
The authors acknowledge the outstanding scholarship and research contributions of Professor Panos M. Pardalos as well as his role in scientific leadership, community building, education, networking, and the forging of sustained friendships across the globe. We are honored to have been invited to contribute to this special issue and we thank the Editors.

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