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PATH CONNECTEDNESS OF SOLUTION SETS FOR PARTIALLY ORDERED SET OPTIMIZATION PROBLEMS

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Abstract. The aim of this paper is to study the path connectedness of the minimal and weak minimal solution sets for set optimization problems equipped with partial set order relations defined on the family of bounded sets by means of Minkowski difference. We propose some new notions of generalized convexity for set-valued maps and a lower level map. Further, by investigating several properties of this level map, we study path connectedness of the solution sets for set optimization problems with respect to partial order relations. Finally, we give an application of the derived results to Nash equilibrium games with vector-valued maps under uncertainty.

Keywords. Generalized convexity; Path connectedness; Partial order relations; Nash equilibrium games; Weak minimal solution.

1. INTRODUCTION

The study of optimization problems with set-valued objective functions is referred as set optimization. In the last two decades, set optimization has gained increasing interest due to numerous applications in uncertain optimization, optimal control, medical sciences, economics, differential equations, finance, and image processing. For more details, we refer to [4, 16] and the references therein.

Set optimization problems can be studied in several ways. Among them, the approach involving the comparison of values of objective maps is more popular and useful. Kuroiwa [17] initially proposed six different kinds of set comparisons which were further investigated by Jahn and Ha [15]. Since then, many authors investigated set optimization problems by using Kuroiwa's approach; see [2, 5, 7, 8, 12, 13, 15, 18, 19, 21] and the references therein. Karaman et al. [21] proposed set relations by using the Minkowski difference. In comparison to Kuroiwa's set relations, these relations are partially ordered on the family of bounded sets and hence provide a new way, namely, the Karaman's approach, to explore set optimization problems.

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The most important aspect of set optimization problems is to study topological properties, namely stability [3, 20], well-posedness [12], continuity [12, 13], connectedness [9, 23, 24], and path connectedness [14], of their solution sets. Among such properties, path connectedness has great importance, as it provides the possibility to move from one solution to another solution by means of a continuous path. Han et al. [14] studied the arcwise connectedness of the solution sets in set optimization in sense of Kuroiwa's approach. Further, Han [10, 11] investigated connectedness and arcwise connecedness of the minimal and weak minimal solutions in set optimization problems via linear and nonlinear scalarization functions. Hieu [11] studied the disconnectedness and unboundedness of the solution sets of monotone vector variational inequalities. Sharma and Lalitha [25] studied the connectedness of minimal and weak minimal solution sets for semi-infinite set optimization problems. However, the work is still in its initial stages. Khushboo and Lalitha [18] investigated the relationships among several kind of solution sets for set optimization problems in sense of Kuroiwa's approach and Karaman's approach and observed that the solution sets may be different. Thus it is important to study the path connectedness of the solution sets in sense of Karaman's approach.

Motivated by [14, 25], we propose some new notions of cone convexity and cone quasi-convexity for set-valued maps by using partial set relations, which will pave our way to achieve the main results of the paper. Further, we study the path connectedness of the solution sets for set optimization problems under partial order relations. We also apply our derived result to Nash equilibrium games with vector-valued maps under uncertainty.

The remaining part of the paper proceeds as follows. Section 2 includes some basic definitions and notions which are used in the sequel of the paper. In Section 3, we introduce some new concepts of cone quasi-convexity and illustrate these concepts with the help of several examples. Also, we introduce a lower level map and study different properties of this map which are useful to achieve our goal. In Section 4, we investigate the path connectedness of the solution sets in the sense of Karaman's approach. In Section 5, we exemplify the utility of our results by providing an application of our results to Nash equilibrium games with vector-valued maps under uncertainty. Section 6 concludes this paper.

2. Preliminaries

Let X and Y be normed vector spaces. Let Y^* be the topological dual space of Y, and let $\mathbf{0}$ stand for the zero vector of Y. We denote by $\mathscr{P}(Y)$ (respectively, $\mathscr{B}(Y)$) the family of all nonempty proper (respectively, nonempty proper bounded) subsets of Y. For $A \subseteq Y$, the topological interior and topological closure of A are denoted by $\text{int}A$ and $\text{cl}A$, respectively.

For $A, B \in \mathscr{P}(Y)$, the sets $A + B := \{a + b : a \in A, b \in B\}$ and $A \dot{-} B := \{x \in X : x + B \subseteq A\} = \bigcap_{b \in B} (A - b)$ are called Minkowski sum and Minkowski (Pontryagin) difference of A and B, respectively. For further detail on Minkowski (Pontryagin) difference, we refer to [22].

Let C be a cone in Y and C^* be defined by $C^* := \{l \in Y^* : l(c) \geq 0 \text{ for all } c \in C\}$. The convex cone $C \subseteq Y$ induces an ordering on Y as $a \preceq_C b \Leftrightarrow b - a \in C, \quad$ for all $a, b \in Y$. If $\text{int}C \neq \emptyset$, then $a \prec_C b \Leftrightarrow b - a \in \text{int}C$ for all $a, b \in Y$. Further, if C is pointed, then the ordering $\preceq_C$ is a partial order relation on Y.

We now recall the set relations proposed by Kuroiwa [17]. For $A, B \in \mathscr{P}(Y)$ and for a closed convex pointed cone $C \subseteq Y$ with $\text{int}C \neq \emptyset$, the lower set less relation $\preceq_C^l$ is defined as $A \preceq_C^l B \Leftrightarrow B \subseteq A + C$, and the strictly lower set less relation $\prec_C^l$ is defined as $A \prec_C^l B \Leftrightarrow B \subseteq A + \text{int}C$.

We now recall m-lower and strictly m-lower set order relations on $\mathscr{P}(Y)$, which were proposed by Karaman et al. [21] in set optimization. For $A,B,K \in \mathscr{P}(Y)$, the m-lower set less relation, denoted by $\preceq_K^{ml}$, is defined as $A \preceq_K^{ml} B \Leftrightarrow (A \dot{-} B) \cap (-K) \neq \emptyset$, and strictly m-lower set less relation, denoted by $\prec_K^{ml}$, is defined as $A \prec_K^{ml} B \Leftrightarrow (A \dot{-} B) \cap (-\text{int}K) \neq \emptyset$.

We observe from [21] that

(a) if K is a convex cone in Y and $\mathbf{0} \in K$, then $\preceq_K^{ml}$ is a pre-order relation on $\mathscr{P}(Y)$, that is, reflexive and transitive order relation on $\mathscr{P}(Y)$;
(b) if K is a pointed convex cone in Y and $\mathbf{0} \in K$, then $\preceq_K^{ml}$ is a partial order relation on $\mathscr{B}(Y)$, that is, reflexive, transitive and antisymmetric order relation on $\mathscr{B}(Y)$;
(c) if K is a pointed convex cone in Y, then the relation $\prec_K^{ml}$ may not be reflexive and hence $\prec_K^{ml}$ may not be a partial order relation on $\mathscr{B}(Y)$.

Let S be a nonempty subset of X, and let $F : S \rightrightarrows Y$ be a set-valued map such that $F(x) \neq \emptyset$ for all $x \in S$. Consider the following set optimization problem:

$$\begin{aligned} &\min F(x) \\ &\text{subject to } x \in S. \end{aligned} \tag{P}$$

From now onwards, we assume that $F(x) \in \mathscr{B}(Y)$, for all $x \in S$, $F(x) \dot{-} F(y) \neq \emptyset$ for all $x,y \in X$, and $K := C$ is a closed, convex, and pointed cone with $\text{int}C \neq \emptyset$.

We recall the solution concepts of (P) with respect to the relations $\preceq_C^l$ and $\prec_C^l$.

Definition 2.1. An element $x_0 \in S$ is called

(a) an l-minimal solution of (P) if
$$F(x) \preceq_C^l F(x_0),\ x \in S \quad \Rightarrow \quad F(x_0) \preceq_C^l F(x);$$
(b) a weakly l-minimal solution of (P) if there does not exist any $x \in S$ such that $F(x) \prec_C^l F(x_0)$.

We denote by $l-\text{Min}(F,S)$ and $l-\text{WMin}(F,S)$, the set of l-minimal solutions and the set of weakly l-minimal solutions of (P), respectively. Clearly, $l-\text{Min}(F,S) \subseteq l-\text{WMin}(F,S)$. However, the reverse inclusion may not be true (see [18]).

We recall the solution concepts of (P) with respect to the relations $\preceq_C^{ml}$ and $\prec_C^{ml}$.

Definition 2.2. An element $x_0 \in S$ is called

(a) an ml-minimal solution of (P) if there does not exist any $x \in S$ such that $F(x) \preceq_C^{ml} F(x_0)$ and $F(x) \neq F(x_0)$, that is, either $F(x) \npreceq_C^{ml} F(x_0)$ or $F(x) = F(x_0)$ for all $x \in S$;
(b) a weakly ml-minimal solution of (P) if there does not exist any $x \in S$ such that $F(x) \prec_C^{ml} F(x_0)$.

We denote by $ml-\text{Min}(F,S)$ and $ml-\text{WMin}(F,S)$, the set of ml-minimal solutions and the set of weakly ml-minimal solutions of (P), respectively. Clearly, $ml-\text{Min}(F,S) \subseteq ml-\text{WMin}(F,S)$. However, the reverse inclusion may not be true (see [18]).

Remark 2.1. If $x_0 \in ml-\text{Min}(F,S), y_0 \in S$, and $F(y_0) \preceq_C^{ml} F(x_0)$, then $y_0 \in ml-\text{Min}(F,S)$. Indeed, assume to the contrary that $y_0 \notin ml-\text{Min}(F,S)$. Then there exists $x \in S$ such that $F(x) \preceq_C^{ml} F(y_0)$ and $F(x) \neq F(y_0)$. Since $y_0 \in S$ and $F(y_0) \preceq_C^{ml} F(x_0)$, by transitivity of the relation $\preceq_C^{ml}$, we have
$$F(x) \preceq_C^{ml} F(x_0) \quad \text{and} \quad F(x) \neq F(x_0),$$

which contradict $x_0 \in ml - \text{Min}(F,S)$. Thus $y_0 \in ml - \text{Min}(F,S)$.

Theorem 2.1. [18] *If C is a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$, then*

$$l - \text{WMin}(F,S) \subseteq ml - \text{WMin}(F,S).$$

However, the converse of the above theorem may not be true (see [18]). It is also pointed out in [18] that there is no relation between $l - \text{Min}(F,S)$ and $ml - \text{Min}(F,S)$. For further detail on relations among different kinds of optimal solutions, we refer to [18].

3. Generalized Convexity for Set-valued Maps

We recall the notion of upper semi-continuity and lower semi-continuity for set-valued maps, which are needed in the sequel.

Definition 3.1. [16, Definition 3.1.1] Let X and Y be topological spaces. A set-valued map $F : X \rightrightarrows Y$ is said to be

(a) upper semi-continuous at $x_0 \in X$ if, for any neighbourhood V of $F(x_0)$, there exists a neighbourhood U of x_0 such that $F(x) \subseteq V$ for all $x \in U$;
(b) lower semi-continuous at $x_0 \in X$ if, for any $x \in F(x_0)$ and any neighbourhood V of x, there exists a neighbourhood U of x_0 such that $F(x') \cap V \neq \emptyset$ for all $x' \in U$.

We say that F is upper semi-continuous and lower semi-continuous on X if it is upper semi-continuous and lower semi-continuous at each point $x \in X$, respectively. We say that F is continuous on X if it is both upper semi-continuous and lower semi-continuous on X.

We need the following characterizations of upper semi-continuity and lower semi-continuity in terms of sequences which play an important role to establish the path connectedness of the solution set of problem (P).

Proposition 3.1. [16] *Let X and Y be real normed vector spaces and $F : X \rightrightarrows Y$ be a set-valued map.*

(a) *F is lower semi-continuous at $x_0 \in X$ if and only if, for any sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ and for any $u \in F(x_0)$, there exists $u_n \in F(x_n)$ such that $u_n \to u$.*
(b) *For any given $x_0 \in X$, if $F(x_0)$ is compact, then F is upper semi-continuous at $x_0 \in X$ if and only if, for any sequence $\{x_n\} \subseteq X$, $\{u_n\} \subseteq Y$ with $x_n \to x_0$ and $u_n \in F(x_n) \setminus F(x_0)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0 \in F(x_0)$.*

Following [13], we propose new concepts of convexity and quasi-convexity for set-valued maps by using set order relations $\preceq_C^{ml}$ and $\prec_C^{ml}$.

Definition 3.2. Let S be a nonempty convex subset of X. A set-valued map $F : S \rightrightarrows Y$ is said to be

(a) ml-type C-convex on S if, for all $x_1, x_2 \in S$ and all $t \in [0,1]$,

$$F(tx_1 + (1-t)x_2) \preceq_C^{ml} tF(x_1) + (1-t)F(x_2);$$

(b) strictly ml-type C-convex on S if, for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ and all $t \in (0,1)$,

$$F(tx_1 + (1-t)x_2) \prec_C^{ml} tF(x_1) + (1-t)F(x_2);$$

(c) natural *ml*-type C-quasi-convex on S if, for all $x_1, x_2 \in S$ and all $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$F(tx_1 + (1-t)x_2) \preceq_C^{ml} \lambda F(x_1) + (1-\lambda)F(x_2);$$

(d) strictly natural *ml*-type C-quasi-convex on S if, for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ and all $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$F(tx_1 + (1-t)x_2) \prec_C^{ml} \lambda F(x_1) + (1-\lambda)F(x_2).$$

To show that the class of *ml*-type C-convex and natural *ml*-type C-quasi-convex set-valued maps is nonempty, we give the following examples.

Example 3.1. Let $S = [0,1] \subseteq X = \mathbb{R}$, $Y = \mathbb{R}$, and $C = \mathbb{R}_+$. Consider the set-valued map $F : S \rightrightarrows Y$ defined by $F(x) = [0, x+1], \quad$ for all $x \in S$. Then F is *ml*-type C-convex set-valued map.

Example 3.2. Let $S = [0,1] \subseteq X = \mathbb{R}$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$. Consider the set-valued map $F : S \rightrightarrows Y$ defined by $F(x) = \{(x, 1-x)\}, \quad$ for all $x \in S$. Then it can be easily seen that F is natural *ml*-type C-quasi-convex set-valued map.

If F is strictly *ml*-type C-convex on S, then it is strictly natural *ml*-type C-quasi-convex on S. The following example shows that the class of strictly natural *ml*-type C-quasi-convex maps is larger than the class of strictly *ml*-type C-convex set-valued maps.

Example 3.3. Let $Y = \mathbb{R}^2$, $X = \mathbb{R}$, $S = [0, \frac{\pi}{2}]$, and $C = \mathbb{R}^2_+$. We denote by $\mathbb{B}_Y$ the closed unit ball in Y. Define the set-valued map $F : X \rightrightarrows Y$ by $F(x) = (\sin x, x) + \mathbb{B}_Y$ for all $x \in X$. Then it is easy to see that F is strictly natural *ml*-type C-quasi-convex, but not strictly *ml*-type C-convex on S.

We now define a lower level map by using relation $\preceq_C^{ml}$; however, it was defined in [14] by using the relation $\preceq_C^{l}$.

Definition 3.3. Let S be a nonempty subset of X and $F : S \rightrightarrows Y$ be a set-valued map with nonempty values. A set-valued map $L : S \rightrightarrows S$, defined by $L(x) := \{y \in S : F(y) \preceq_C^{ml} F(x)\}$ for all $x \in S$, is called a lower level map.

Lemma 3.1. *For any $x \in S$, we have $ml - \mathrm{Min}(F, L(x)) \subseteq ml - \mathrm{Min}(F, S)$.*

Proof. If $ml - \mathrm{Min}(F, L(x)) = \emptyset$, then there is nothing to prove. Assume $ml - \mathrm{Min}(F, L(x)) \neq \emptyset$. Let $x_0 \in ml - \mathrm{Min}(F, L(x))$. We need to show that $x_0 \in ml - \mathrm{Min}(F, S)$. Assume contrary that $x_0 \notin ml - \mathrm{Min}(F, S)$. Then there exists a $y \in S$ such that $F(y) \preceq_C^{ml} F(x_0)$ and $F(y) \neq F(x_0)$. Since $x_0 \in ml - \mathrm{Min}(F, L(x))$, we have $x_0 \in L(x) \subseteq S$ and $F(x_0) \preceq_C^{ml} F(x)$ for all $x \in L(x) \subseteq S$. Thus, by transitivity of the relation $\preceq_C^{ml}$, we have $F(y) \preceq_C^{ml} F(x)$ for all $x \in S$, so $y \in L(x)$. Since $x_0 \in ml - \mathrm{Min}(F, L(x))$, it follows that $F(x_0) \preceq_C^{ml} F(y)$. Therefore, $x_0 \in ml - \mathrm{Min}(F, S)$. □

We now study several properties, namely, convexity, closedness, upper semi-continuity, and lower semi-continuity of the lower level map L.

Proposition 3.2. *Let S be a nonempty convex subset of X and $F : S \rightrightarrows Y$ be natural ml-type C-quasi-convex on S with nonempty values. Then the lower level map L is convex valued, that is, for all $x \in S$, $L(x)$ is convex.*

Proof. Let $x \in S$ be arbitrary and $y_1, y_2 \in L(x)$. Then, $F(y_1) \preceq_C^{ml} F(x)$ and $F(y_2) \preceq_C^{ml} F(x)$, that is,

$$(F(y_1) \dot{-} F(x)) \cap (-C) \neq \emptyset \text{ and } (F(y_2) \dot{-} F(x)) \cap (-C) \neq \emptyset.$$

Hence, there exists $c_1, c_2 \in C$ such that

$$-c_1 \in F(y_1) \dot{-} F(x) \text{ and } -c_2 \in F(y_2) \dot{-} F(x),$$

which means that

$$-c_1 + F(x) \subseteq F(y_1), \tag{3.1}$$

and

$$-c_2 + F(x) \subseteq F(y_2). \tag{3.2}$$

We need to show that $ty_1 + (1-t)y_2 \in L(x)$ for all $y_1, y_2 \in L(x)$ and $t \in [0,1]$. Since F is natural ml-type C-quasi-convex on S, for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$F(ty_1 + (1-t)y_2) \preceq_C^{ml} \lambda F(y_1) + (1-\lambda)F(y_2),$$

that is,

$$[F(ty_1 + (1-t)y_2) \dot{-} (\lambda F(y_1) + (1-\lambda)F(y_2))] \cap (-C) \neq \emptyset,$$

Hence, there exists $c' \in C$ such that

$$-c' \in [F(ty_1 + (1-t)y_2) \dot{-} (\lambda F(y_1) + (1-\lambda)F(y_2))],$$

which means that

$$-c' + \lambda F(y_1) + (1-\lambda)F(y_2) \subseteq F(ty_1 + (1-t)y_2). \tag{3.3}$$

On the other hand, multiplying (3.1) and (3.2) by λ and $(1-\lambda)$, respectively, and then adding the resultants, we obtain

$$-(\lambda c_1 + (1-\lambda)c_2) + \lambda F(x) + (1-\lambda)F(x) \subseteq \lambda F(y_1) + (1-\lambda)F(y_2). \tag{3.4}$$

From (3.3) and (3.4), we see that

$$-c' - (\lambda c_1 + (1-\lambda)c_2) + F(x) \subseteq F(ty_1 + (1-t)y_2).$$

Set $c := c' + (\lambda c_1 + (1-\lambda)c_2) \in C$. Then, $-c \in F(ty_1 + (1-t)y_2) \dot{-} F(x)$, and clearly, $-c \in -C$. Therefore, $(F(ty_1 + (1-t)y_2) \dot{-} F(x)) \cap (-C) \neq \emptyset$. Hence, $F(ty_1 + (1-t)y_2) \preceq_C^{ml} F(x)$, and therefore, $ty_1 + (1-t)y_2 \in L(x)$. □

Proposition 3.3. *Let S be a nonempty closed subset of X, and let $F : S \rightrightarrows Y$ be upper semi-continuous on S with nonempty compact values. Then the lower level map L is closed valued, that is, for all $x \in S$, $L(x)$ is closed.*

Proof. Let $x \in S$ be arbitrary fixed and $\{y_n\}$ be a sequence in $L(x)$ such that $y_n \to y_0$. We prove that $y_0 \in L(x)$. Clearly, $y_0 \in S$ because S is closed. Since $\{y_n\} \subseteq L(x)$, we have $F(y_n) \preceq_C^{ml} F(x)$ for all $n \in \mathbb{N}$, that is, $(F(y_n) \dot{-} F(x)) \cap (-C) \neq \emptyset$. Then, there exists $c_n \in C$ such that $-c_n \in F(y_n) \dot{-} F(x)$, which means that $-c_n + F(x) \subseteq F(y_n)$, that is, $-c_n + v \in F(y_n)$ for all $v \in F(x)$ and all $n \in \mathbb{N}$. By Proposition 3.1 (b), one sees that there exist a point $v_0 \in F(y_0)$ and a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that $-c_{n_k} + v \to v_0$, which implies that $\{c_{n_k}\}$ has a convergent subsequence. Without loss of generality, we assume that $c_{n_k} \to c_0$. Clearly, $c_0 \in C$ because C is closed. Thus $-c_0 + v = v_0 \in F(y_0)$, which gives $-c_0 + F(x) \subseteq F(y_0)$, and hence $F(y_0) \preceq_C^{ml} F(x)$ for any $x \in S$. Thus, $y_0 \in L(x)$. □

Proposition 3.4. *Let S be a nonempty compact subset of X and $F : S \rightrightarrows Y$ be continuous on S with nonempty compact values. Then the lower level map L is upper semi-continuous on S.*

Proof. Assume to the contrary that the lower level map L is not upper semi-continuous at $x_0 \in S$. Then there exists a neighbourhood V of $L(x_0)$ in S, a sequence $\{x_n\}$ with $x_n \to x_0$ and $y_n \in L(x_n)$ such that $y_n \notin V$ for all $n \in \mathbb{N}$. Since S is compact and $y_n \in S$, it follows that $y_n \to y_0$ for some $y_0 \in S$. We claim that $y_0 \in L(x_0)$. Since $y_n \in L(x_n)$, it follows that $F(y_n) \preceq_C^{ml} F(x_n)$, that is, $(F(y_n) \dot{-} F(x_n)) \cap (-C) \neq \emptyset$. Hence, there exists $c_n \in C$ such that $-c_n \in (F(y_n) \dot{-} F(x_n))$, which means that

$$-c_n + F(x_n) \subseteq F(y_n), \qquad \text{for all } n \in \mathbb{N}, \tag{3.5}$$

that is, $-c_n + v_n \in F(y_n)$ for all $v_n \in F(x_n)$ and all $n \in \mathbb{N}$. Since F is upper semi-continuous with compact values on S, the sequences $\{v_n\}$ and $\{v_n - c_n\}$ have convergent subsequences, which implies that $\{c_n\}$ has a convergent subsequence. Without loss of generality, we assume that $c_{n_k} \to c_0$. Clearly, $c_0 \in C$ because C is closed. In fact, for any $z_0 \in F(x_0)$, by Proposition 3.1 (a), one sees that there exists $z_n \in F(x_n)$ such that $z_n \to z_0$. It follows from (3.5) that there exists $w_n \in F(y_n)$ such that

$$-c_n + z_n = w_n. \tag{3.6}$$

Since F is upper semi-continuous with compact values, by Proposition 3.1 (b), one sees that there exist a point $w_0 \in F(y_0)$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to w_0$. It follows from (3.6) that $-c_0 + z_0 = w_0 \in F(y_0)$, which gives $-c_0 + F(x_0) \subseteq F(y_0)$, and hence $F(y_0) \preceq_C^{ml} F(x_0)$ for any $x \in S$. Thus $y_0 \in L(x_0)$. Hence, $y_0 \in L(x_0) \subseteq V$. Since $y_n \notin V$ and V is open, it follows that $y_0 \notin V$, which is a contradiction. □

Proposition 3.5. *Let S be a nonempty, convex and compact subset of X and $F : S \rightrightarrows Y$ be continuous and strictly natural ml-type C-quasi-convex on S with nonempty compact values. Then the lower level map L is lower semi-continuous on S.*

Proof. Suppose contrary that L is not lower semi-continuous at $x_0 \in S$. Then there exist a point $y_0 \in L(x_0)$, a neighbourhood W_0 of $y_0 \in S$ and a sequence $\{x_n\}$ with $x_n \to x_0$ such that

$$(y_0 + W_0) \cap L(x_n) = \emptyset, \qquad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

Clearly, $x_0 \in L(x_0)$. Assume that $y_0 = x_0 \in L(x_0)$. Then $x_n \in L(x_n)$ and $x_n \to x_0 = y_0$, which contradicts (3.7). Thus $x_0 \neq y_0$. Since F is strictly natural *ml*-type C-quasi-convex on S for any $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$F(ty_0 + (1-t)x_0) \prec_C^{ml} \lambda F(y_0) + (1-\lambda)F(x_0),$$

that is,

$$[F(ty_0 + (1-t)x_0) \dot{-} (\lambda F(y_0) + (1-\lambda)F(x_0))] \cap (-\text{int}C) \neq \emptyset.$$

Hence, there exists $c_1 \in \text{int}C$ such that

$$-c_1 \in [F(ty_0 + (1-t)x_0) \dot{-} (\lambda F(y_0) + (1-\lambda)F(x_0))],$$

which means that

$$-c_1 + \lambda F(y_0) + (1-\lambda)F(x_0) \subseteq F(ty_0 + (1-t)x_0). \tag{3.8}$$

It follows from $y_0 \in L(x_0)$ that $F(y_0) \preceq_C^{ml} F(x_0)$, that is, $(F(y_0) \dot{-} F(x_0)) \cap (-C) \neq \emptyset$. Hence, there exists $c_2 \in C$ such that $-c_2 \in F(y_0) \dot{-} F(x_0)$, which means that

$$-c_2 + F(x_0) \subseteq F(y_0). \tag{3.9}$$

Multiplying (3.9) by λ and adding $(1-\lambda)F(x_0)$ both sides, we have

$$-\lambda c_2 + \lambda F(x_0) + (1-\lambda)F(x_0) \subseteq \lambda F(y_0) + (1-\lambda)F(x_0).$$

Using (3.8), we arrive at

$$-\lambda c_2 - c_1 + F(x_0) \subseteq -c_1 + \lambda F(y_0) + (1-\lambda)F(x_0) \subseteq F(ty_0 + (1-t)x_0).$$

Since C is a closed, convex and pointed cone with $\text{int}C \neq \emptyset$, $C + \text{int}C = \text{int}C$, and hence, $\bar{c} = c_1 + \lambda c_2 \in \text{int}C$. Therefore,

$$-\bar{c} + F(x_0) \subseteq F(ty_0 + (1-t)x_0). \tag{3.10}$$

Let $y(t) := ty_0 + (1-t)x_0$. Then there exists a $t_0 \in (0,1)$ such that $y(t_0) \in y_0 + W_0$. Thus there exists $N_0 \in \mathbb{N}$ such that

$$-\bar{c} + F(x_n) \subseteq F(y(t_0)), \quad \text{when } n \geq N_0. \tag{3.11}$$

In fact, if not, without loss of generality, we may assume that $-\bar{c} + F(x_n) \not\subseteq F(y(t_0))$ for any $n \in \mathbb{N}$. Thus there exists $u_n \in F(x_n)$ such that

$$-\bar{c} + u_n \notin F(y(t_0)). \tag{3.12}$$

It follows from Proposition 3.1 (b) that there exist a point $u_0 \in F(x_0)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$. Thus, (3.10) implies that there exists $v_0 \in F(y(t_0))$ such that $u_0 - v_0 \in \text{int}C$. which implies $u_{n_k} - v_0 \in \text{int}C$ for all k and hence $u_n - v_0 \in \text{int}C$ for all n large enough, which contradicts (3.12). This shows that (3.11) holds. Now it follows from (3.11) that $y(t_0) \in L(x_n)$. By $y(t_0) \in y_0 + W_0$, we have $y(t_0) \in (y_0 + W_0) \cap L(x_n)$, which contradicts (3.7). Therefore, L is lower semi-continuous on S. ☐

Proposition 3.6. *Assume that S is a convex subset of X and $F : S \rightrightarrows Y$ is a strictly natural ml-type C-quasi-convex map on S with nonempty values. Then, $ml - \text{Min}(F,S) = ml - \text{WMin}(F,S)$.*

Proof. It suffices to prove that $ml - \text{WMin}(F,S) \subseteq ml - \text{Min}(F,S)$. Let $x_0 \in ml - \text{WMin}(F,S)$. If there exists $\bar{x} \in S$ such that $F(\bar{x}) \preceq_C^{ml} F(x_0)$, then $(F(\bar{x}) \dot{-} F(x_0)) \cap (-C) \neq \emptyset$. Therefore, there exists $c_1 \in C$ such that $-c_1 \in (F(\bar{x}) \dot{-} F(x_0))$, which means that

$$-c_1 + F(x_0) \subseteq F(\bar{x}). \tag{3.13}$$

Suppose that $\bar{x} \neq x_0$. Since F is strictly natural ml-type C-quasi-convex on S, for any $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$F(tx_0 + (1-t)\bar{x}) \prec_C^{ml} \lambda F(x_0) + (1-\lambda)F(\bar{x}).$$

It follows that

$$[F(tx_0 + (1-t)\bar{x}) \dot{-} (\lambda F(x_0) + (1-\lambda)F(\bar{x}))] \cap (-\text{int}C) \neq \emptyset.$$

Hence there exists $c_2 \in \text{int}C$ such that

$$-c_2 \in [F(tx_0 + (1-t)\bar{x}) \dot{-} (\lambda F(x_0) + (1-\lambda)F(\bar{x}))],$$

which means that

$$-c_2+\lambda F(x_0)+(1-\lambda)F(\bar{x})\subseteq F(tx_0+(1-t)\bar{x}). \tag{3.14}$$

Multiplying (3.13) by $(1-\lambda)$ and adding $\lambda F(x_0)$ both the sides, we obtain

$$-(1-\lambda)c_1+(1-\lambda)F(x_0)+\lambda F(x_0)\subseteq \lambda F(x_0)+(1-\lambda)F(\bar{x}).$$

It follows from (3.14) that

$$-(1-\lambda)c_1-c_2+F(x_0)\subseteq -c_2+\lambda F(x_0)+(1-\lambda)F(\bar{x})\subseteq F(tx_0+(1-t)\bar{x}).$$

Since C is a closed, convex, and pointed cone with $\mathrm{int}C\neq\emptyset$, $C+\mathrm{int}C=\mathrm{int}C$, and hence, $\bar{c}=(1-\lambda)c_1+c_2\in\mathrm{int}C$. Therefore,

$$-\bar{c}+F(x_0)\subseteq F(tx_0+(1-t)\bar{x}). \tag{3.15}$$

In fact, if not, then there exists $\beta\in F(x_0)$ such that

$$-\bar{c}+\beta\notin F(tx_0+(1-t)\bar{x}). \tag{3.16}$$

By (3.13), there exist $\eta\in F(\bar{x})$ such that $-c_1+\beta=\eta$. It follows from (3.14) that

$$-c_2+\lambda\beta+(1-\lambda)\eta\in F(tx_0+(1-t)\bar{x}) \tag{3.17}$$

Note that $-c_1+\beta=\eta$. Putting value of η in (3.17), we have

$$-c_2-(1-\lambda)c_1+\beta\in F(tx_0+(1-t)\bar{x}),$$

which contradicts (3.16). Thus (3.15) holds, so $F(tx_0+(1-t)\bar{x})\prec_C^{ml} F(x_0)$. By Definition 2.2 (b) and nonemptyness of the set of weakly m-lower minimal solutions, we can see that $x_0\notin ml-\mathrm{WMin}(F,S)$, which contradicts the fact that $x_0\in ml-\mathrm{WMin}(F,S)$. Thus $\bar{x}=x_0$. Now, it is clear that $F(x_0)\preceq_C^{ml} F(\bar{x})$, so $x_0\in ml-\mathrm{Min}(F,S)$. □

Lemma 3.2. *Let $F: X\rightrightarrows Y$ be a set-valued map with nonempty values. For $l\in C^*\setminus\{\mathbf{0}\}$ and $x\in X$, let $\Phi(x):=\inf_{z\in F(x)} l(z)$. Then the following statements hold:*

(a) *If F is continuous on X with nonempty compact values, then Φ is continuous on X.*
(b) *For all $x,y\in X$, if $F(y)$ is compact and $F(x)\prec_C^{ml} F(y)$, then $\Phi(x)<\Phi(y)$.*
(c) *If F is strictly natural ml-type C-quasi-convex on X with nonempty compact values, then Φ is strictly natural quasi-convex on X, that is, for any $x_1,x_2\in X$ with $x_1\neq x_2$ and for any $t\in(0,1)$, there exists $\lambda\in[0,1]$ such that*

$$\Phi(tx_1+(1-t)x_2)<\lambda\Phi(x_1)+(1-\lambda)\Phi(x_2).$$

Proof. (a) From [1, Proposition 19 and 21 of Chapter 3], it can be obtained immediately.

(b) Since l is continuous and $F(y)$ is compact, there exists $y_0\in F(y)$ such that

$$\Phi(y)=\inf_{z\in F(y)} l(z)=l(y_0).$$

Since $F(x)\prec_C^{ml} F(y)$, we have $(F(x)\dot{-}F(y))\cap(-\mathrm{int}C)\neq\emptyset$, that is, there exists $c\in\mathrm{int}C$ such that $-c\in F(x)\dot{-}F(y)$ which means that $-c+F(y)\subseteq F(x)$. Hence, there exists $\bar{y}\in F(x)$ such that $-c+y_0=\bar{y}$, that is, $y_0=\bar{y}+c$. Therefore,

$$\Phi(y)=l(y_0)=l(\bar{y}+c)=l(\bar{y})+l(c)>l(\bar{y})\geq \inf_{z\in F(x)} l(z)=\Phi(x).$$

(c) Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since $F(x_1)$ and $F(x_2)$ are compact, then there exists $y_i \in F(x_i), i = 1,2$, such that

$$\Phi(x_i) = \inf_{z \in F(x_i)} l(z) = l(y_i), \quad i = 1,2. \tag{3.18}$$

Since F is strictly natural ml-type C-quasi-convex on X, for any $t \in (0,1)$, then there exists $\lambda \in [0,1]$ such that

$$F(tx_1 + (1-t)x_2) \prec_C^{ml} \lambda F(x_1) + (1-\lambda)F(x_2),$$

that is,

$$[F(tx_1 + (1-t)x_2) \dot{-} (\lambda F(x_1) + (1-\lambda)F(x_2))] \cap (-\text{int}C) \neq \emptyset.$$

Then there exists $c \in \text{int}C$ such that

$$-c \in [F(tx_1 + (1-t)x_2) \dot{-} (\lambda F(x_1) + (1-\lambda)F(x_2))],$$

which means that

$$-c + \lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(tx_1 + (1-t)x_2). \tag{3.19}$$

Hence there exists $y_t \in F(tx_1 + (1-t)x_2)$ such that $-c + \lambda y_1 + (1-\lambda)y_2 = y_t$, that is, $\lambda y_1 + (1-\lambda)y_2 = y_t + c$. Then

$$\begin{aligned} l(\lambda y_1 + (1-\lambda)y_2) &= \lambda l(y_1) + (1-\lambda)l(y_2) \\ &= l(y_t + c) > l(y_t) \\ &\geq \inf_{z \in F(tx_1 + (1-t)x_2)} l(z) \\ &= \Phi(tx_1 + (1-t)x_2). \end{aligned}$$

which together with (3.18) yields

$$\Phi(tx_1 + (1-t)x_2) < \lambda \Phi(x_1) + (1-\lambda)\Phi(x_2).$$

□

4. Path Connectedness of Solution Sets of Set Optimization Problems

In this section, we establish the path connectedness of $ml - \text{Min}(F,S)$ and $ml - \text{WMin}(F,S)$. Throughout this section, we assume that $ml - \text{Min}(F,S) \neq \emptyset$.

Recall that a nonempty subset A of a topological space X is said to be path connected if, for every two points $x, y \in A$, there exists a continuous function $\psi : [0,1] \to A$ such that $\psi(0) = x$ and $\psi(1) = y$.

Proposition 4.1. *Assume that $F : S \rightrightarrows Y$ is strictly natural ml-type C-quasi-convex on S with nonempty values. If $x_0 \in ml - \text{Min}(F,S)$, then $L(x_0) = \{x_0\}$.*

Proof. Let $x_0 \in ml - \text{Min}(F,S)$. By the reflexivity of the relation $\preceq_C^{ml}$, we have $x_0 \in L(x_0)$. Suppose that there exists $\bar{x} \in L(x_0)$ such that $\bar{x} \neq x_0$. Since F is strictly natural ml-type C-quasi-convex on S, for all $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$F(tx_0 + (1-t)\bar{x}) \prec_C^{ml} \lambda F(x_0) + (1-\lambda)F(\bar{x}),$$

that is,

$$[F(tx_0 + (1-t)\bar{x}) \dot{-} (\lambda F(x_0) + (1-\lambda)F(\bar{x}))] \cap (-\text{int}C) \neq \emptyset.$$

Hence, there exists $c_1 \in \text{int}C$ such that

$$-c_1 \in [F(tx_0+(1-t)\bar{x}) \dot{-} (\lambda F(x_0)+(1-\lambda)F(\bar{x}))],$$

which means that

$$-c_1+\lambda F(x_0)+(1-\lambda)F(\bar{x}) \subseteq F(tx_0+(1-t)\bar{x}). \tag{4.1}$$

Since $\bar{x} \in L(x_0)$, we have $F(\bar{x}) \preceq_C^{ml} F(x_0)$, that is, $(F(\bar{x}) \dot{-} F(x_0)) \cap (-C) \neq \emptyset$. Hence, there exists $c_2 \in C$ such that $-c_2 \in F(\bar{x}) \dot{-} F(x_0)$, which means that

$$-c_2+F(x_0) \subseteq F(\bar{x}). \tag{4.2}$$

Multiplying (4.2) by $(1-\lambda)$ and adding $\lambda F(x_0)$ both the sides, we have

$$-(1-\lambda)c_2+(1-\lambda)F(x_0)+\lambda F(x_0) \subseteq \lambda F(x_0)+(1-\lambda)F(\bar{x}).$$

It follows from (4.1) that

$$-(1-\lambda)c_2-c_1+F(x_0) \subseteq -c_1+\lambda F(x_0)+(1-\lambda)F(\bar{x}) \subseteq F(tx_0+(1-t)\bar{x}).$$

Since C is a closed convex pointed cone with $\text{int}C \neq \emptyset$, $C+\text{int}C = \text{int}C$, and hence, $\bar{c} = (1-\lambda)c_2+c_1 \in \text{int}C$. Therefore,

$$-\bar{c}+F(x_0) \subseteq F(tx_0+(1-t)\bar{x}).$$

Thus $F(tx_1+(1-t)x_2) \prec_C^{ml} F(x_0)$, which contradicts $x_0 \in ml-\text{WMin}(F,S) = ml-\text{Min}(F,S)$. This completes the proof. □

Theorem 4.1. *Assume that S is a nonempty, convex, and compact subset of X. If $F: S \rightrightarrows Y$ is continuous and strictly natural ml-type C-quasi-convex on S with nonempty compact values, then $ml-\text{Min}(F,S)$ is path connected. Moreover, $ml-\text{WMin}(F,S)$ is path connected.*

Proof. Define a set-valued map $\mathscr{G}: S \rightrightarrows S$ by

$$\mathscr{G}(x) = \{y \in L(x) : \Phi(y) = \inf\{\Phi(z) : z \in L(x)\}\},$$

where Φ is the same as in Lemma 3.2. We first show that $\mathscr{G}(x)$ is nonempty for all $x \in S$. It follows from Proposition 3.3 that $L(x)$ is a closed subset of a compact set S, and hence compact. By using the reflexivity of the relation $\preceq_C^{ml}$, we have $x \in L(x)$ and so $L(x)$ is nonempty. By Lemma 3.2 (a), we see that Φ is continuous. Therefore, $\mathscr{G}(x)$ is nonempty.

Next, we prove that, for any $x \in S$,

$$\mathscr{G}(x) \subseteq ml-\text{WMin}(F,L(x)) = ml-\text{Min}(F,L(x)) \subseteq ml-\text{Min}(F,S). \tag{4.3}$$

For any $x \in S$, we assume that $y_0 \in \mathscr{G}(x)$ but $y_0 \notin ml-\text{WMin}(F,L(x))$. Then it follows from Definition 2.2 (b) that there exists $\bar{y} \in L(x)$ such that $F(\bar{y}) \prec_C^{ml} F(y_0)$. By Lemma 3.2 (b), we know that $\Phi(\bar{y}) < \Phi(y_0)$, which contradicts $y_0 \in \mathscr{G}(x)$. Thus, $\mathscr{G}(x) \subseteq ml-\text{WMin}(F,L(x))$. The conclusion follows from Lemma 3.1 and Proposition 3.6.

We now show that, for any $x \in S$, $\mathscr{G}(x)$ is singleton. Assume contrary that there exist $y_1, y_2 \in \mathscr{G}(x)$ such that $y_1 \neq y_2$. Then, $y_1, y_2 \in L(x)$ and

$$\Phi(y_1) = \Phi(y_2) = \inf\{\Phi(z) : z \in L(x)\}. \tag{4.4}$$

From Proposition 3.2, $L(x)$ is convex for any $x \in S$, so $ty_1+(1-t)y_2 \in L(x)$ for all $t \in [0,1]$. It follows from Lemma 3.2 (c) that Φ is strictly natural quasi-convex on S. Thus, for any $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that $\Phi(ty_1+(1-t)y_2) < \lambda\Phi(y_1)+(1-\lambda)\Phi(y_2)$. It follows from (4.4) that $\Phi(ty_1+(1-t)y_2) < \inf\{\Phi(z) : z \in L(x)\}$. Since $ty_1+(1-t)y_2 \in L(x)$, one has $\Phi(ty_1+(1-t)y_2) \geq \inf\{\Phi(z) : z \in L(x)\}$, which is a contradiction.

We show that $\mathscr{G} : S \to S$ is continuous on S. It follows from Proposition 3.4 and 3.5 that L is continuous on S. Moreover, $L(x)$ is nonempty and compact for any $x \in S$. By Lemma 3.2 (a), we see that Φ is continuous on S. Let $\psi(x) = -\Phi(x)$ for all $x \in S$. Then,

$$\mathscr{G}(x) = \{y \in L(x) : \psi(y) = \sup\{\psi(z) : z \in L(x)\}\}.$$

By Proposition 23 of Chapter 3 in [1], we know that $\mathscr{G}$ is upper semi-continuous on S. Also, for any $x \in S$, $\mathscr{G}(x)$ is singleton. Hence, $\mathscr{G}$ is continuous on S.

We now prove that $ml-\mathrm{Min}(F,S)$ and $ml-\mathrm{WMin}(F,S)$ are path connected. For any $x \in ml-\mathrm{Min}(F,S)$, it follows from Proposition 4.1 that $L(x) = \{x\}$, so $\mathscr{G}(x) = \{x\}$. For any $x_1, x_2 \in ml-\mathrm{Min}(F,S)$, let

$$h(t) = \mathscr{G}(tx_1+(1-t)x_2), \quad t \in [0,1].$$

It follows from (4.3) that $h(t) \in ml-\mathrm{Min}(F,S)$ for any $t \in [0,1]$. Since $\mathscr{G}$ is continuous on S, we get $h(\cdot)$ is continuous on $[0,1]$. Moreover, we can see that $h(0) = \mathscr{G}(x_2) = \{x_2\}$ and $h(1) = \mathscr{G}(x_1) = \{x_1\}$. Therefore, $ml-\mathrm{Min}(F,S)$ is path connected. It follows from Proposition 3.6 that $ml-\mathrm{Min}(F,S) = ml-\mathrm{WMin}(F,S)$. Thus $ml-\mathrm{WMin}(F,S)$ is also path connected. □

5. Application to Robust Nash Equilibrium under Uncertainty

In this section, we present an application to robust Nash equilibrium with vector-valued games under uncertainty. In particular, the aim is to derive the path connectedness for solution sets for a game with vector-valued maps under uncertainty.

Let $\mathbb{G}$ be a game with vector-valued objective map under uncertainty defined as

$$\mathbb{G} := (\Lambda, \{Z_\alpha\}, \{g_\alpha\}, U)_{\alpha\in\Lambda},$$

where

(a) $\Lambda := \{1,2,\cdots,n\}$ denotes the set of n number of players;
(b) for each $\alpha \in \Lambda$, the strategies of the αth player are denoted by $Z_\alpha \subseteq X$.
(c) set $S := \Pi_{\alpha\in\Lambda} Z_\alpha$;
(d) for the set of all uncertainties U and each $\alpha \in \Lambda$, the loss function for αth player is denoted by $g_\alpha : S \times U \to Y$.

Set $S_{-\alpha} := \Pi_{\beta\in\Lambda\setminus\{\alpha\}} Z_\beta$. For each $\alpha \in \Lambda$, we define

$$z_{-\alpha} := \{z_1, \cdots, z_{\alpha-1}, z_{\alpha+1}, \cdots, z_n\} \in S_{-\alpha}, \quad \forall z = (z_1, \cdots, z_n) \in S.$$

If, for each $\alpha \in \Lambda$, $w_\alpha \in Z_\alpha$, then we define

$$(w_\alpha, z_{-\alpha}) := \{z_1, \cdots, z_{\alpha-1}, z_\alpha, z_{\alpha+1}, \cdots, z_n\} \in S.$$

We define the image of g_α under U for all $x \in S$ by $g_\alpha(x,U) := \{g_\alpha(x,u) : u \in U\}$.

Motivated by [6], we define the following optimal solutions of Nash equilibria games with vector-valued maps under uncertainty using m-lower set order relations.

Definition 5.1. An element $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) \in S$ is called

(a) a Nash equilibrium for $\mathbb{G}$ if and only if, for any $\alpha \in \Lambda$, there does not exist $z_\alpha \in Z_\alpha$ such that $g_\alpha(z_\alpha, \bar{z}_{-\alpha}, U) \preceq_C^{ml} g_\alpha(\bar{z}, U)$ and $g_\alpha(z_\alpha, \bar{z}_{-\alpha}, U) \neq g_\alpha(\bar{z}, U)$;

(b) a weakly Nash equilibrium for $\mathbb{G}$ if and only if, for any $\alpha \in \Lambda$, there does not exist any $x_\alpha \in Z_\alpha$ such that $g_\alpha(z_\alpha, \bar{z}_{-\alpha}, U) \prec_C^{ml} g_\alpha(\bar{x}, U)$.

We denote by $ml - \mathrm{RMin}(F,S)$ and $ml - \mathrm{RWMin}(F,S)$, the set of Nash equilibrium and weakly Nash equilibrium robust solutions for $\mathbb{G}$, respectively. We can observe that $\mathrm{RMin}(F,S) \subseteq \mathrm{RWMin}(F,S)$. The game G is said to be path connected (respectively, weakly path connected) if the solution set $\mathrm{RMin}(F,S)$ (respectively, $\mathrm{RWMin}(F,S)$) is path connected.

An element $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) \in S$ is a (weakly) Nash equilibrium for $\mathbb{G}$ iff $\bar{z}_\alpha \in Z_\alpha$ is a (weakly) minimal solution of the set optimization problem

$$\begin{aligned} &\mathrm{Min}\ g_\alpha(z_\alpha, \bar{z}_{-\alpha}, U) \\ &\text{subject to } z_\alpha \in Z_\alpha, \end{aligned} \tag{P_α}$$

for each $\alpha \in \Lambda$.

We now derive the path connectedness of game $\mathbb{G}$.

Theorem 5.1. *Assume that*

(a) *for each $\alpha \in \Lambda$, Z_α is nonempty, convex, and compact subset of X;*
(b) *for all $x \in S$, $g_\alpha(z,U)$ is compact;*
(c) *$x \mapsto g_\alpha(x,U)$ is continuous on S;*
(d) *$x \mapsto g_\alpha(x,U)$ is strictly natural ml-type C-quasi convex set-valued map on S.*

Then the game $\mathbb{G}$ is path connected. Moreover, the game $\mathbb{G}$ is weakly path connected.

6. CONCLUSIONS

In this paper, we investigated the path connectedness of minimal and weak minimal solution sets for partially ordered set optimization problems. Our approach relies on the properties of the level set and linear functionals. We employed our results to Nash equilibrium games for vector-valued maps under uncertainty. It is of interest to obtain connnectedness or the path connectedness of solution sets under weaker assumptions.

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