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CONSERVATIVE SOLUTIONS TO THE FIRST INITIAL-BOUNDARY VALUE PROBLEM FOR A NONLINEAR VARIATIONAL WAVE EQUATION

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Abstract. This paper concerns with the first initial-boundary value problem for a one-dimensional nonlinear variational wave equation which is the Euler-Lagrange equation of a variational principle. We introduce the energy-dependent coordinates into the finite interval to transform the original problem into a boundary value problem of a semilinear system. The global existence of weak solutions for the new boundary value problem is first established by deriving the a priori estimates and then expressed in terms of the original variables.

Keywords. Conservative solutions; Initial-boundary value problem; Variational wave equation; Weak solutions.

1. INTRODUCTION

We are concerned with the global existence of energy-conservative weak solutions to the first initial-boundary value problem for a one-dimensional nonlinear variational wave equation

$$(\alpha^2 u_t + \beta u_x)_t + (\beta u_t - \gamma^2 u_x)_x = \alpha\alpha_u u_t^2 + \beta_u u_t u_x - \gamma\gamma_u u_x^2, \tag{1.1}$$

with initial conditions

$$u(0,x) = u_0(x) \in H^1([a,b]), u_t(0,x) = u_1(x) \in L^2([a,b]), \tag{1.2}$$

and the homogeneous first boundary conditions

$$u(t,a) = u(t,b) = 0. \tag{1.3}$$

We here assume that the compatibility conditions $u_0(a) = u_0(b) = u_1(a) = u_1(b) = 0$ are satisfied. In Eq. (1.1), $\alpha(x,u), \beta(x,u)$, and $\gamma(x,u)$ are smooth functions satisfying

$$\begin{aligned} &0 < \alpha_1 \le \alpha(z) \le \alpha_2, \ \ |\beta(z)| \le \beta_2, \ \ 0 < \gamma_1 \le \gamma(z) \le \gamma_2, \\ &\sup_z\{|\nabla\alpha(z)|, |\nabla\beta(z)|, |\nabla\gamma(z)|\} < \infty, \quad \forall\, z \in [a,b]\times\mathbb{R}, \end{aligned} \tag{1.4}$$

for positive constants $\alpha_1, \alpha_2, \beta_2, \gamma_1$, and γ_2 such that Eq. (1.1) is strictly hyperbolic with two eigenvalues

$$\pm\lambda_\pm := \frac{\sqrt{\beta^2 + \alpha^2\gamma^2} \pm \beta}{\alpha^2} > 0.$$

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Eq. (1.1) is derived from a variational principle whose action is a quadratic function of the derivatives of the field ([1, 23, 24])

$$\delta \int A^{ij}_{\mu\nu}(\mathbf{x},u)\frac{\partial u^{\mu}}{\partial x_i}\frac{\partial u^{\nu}}{\partial x_j}\mathrm{d}\mathbf{x} = 0, \tag{1.5}$$

where the summation convention is used. In (1.5), $\mathbf{x} \in \mathbb{R}^{d+1}$ represent the space-time independent variables and $u : \mathbb{R}^{d+1} \to \mathbb{R}^n$ are the dependent variables. The coefficients $A^{ij}_{\mu\nu}$: $\mathbb{R}^{d+1} \times \mathbb{R}^n \to \mathbb{R}$ are smooth functions and satisfy $A^{ij}_{\mu\nu} = A^{ij}_{\nu\mu} = A^{ji}_{\mu\nu}$. We consider the simple case $n = 1$ and $d = 1$ and take $(A^{ij})_{2\times 2}$ as the special form

$$(A^{ij}(x,u))_{2\times 2} = \begin{pmatrix} \alpha^2 & \beta \\ \beta & -\gamma^2 \end{pmatrix}(x,u),$$

to obtain

$$\delta \int \left\{ \alpha^2(x,u)u_t^2 + 2\beta(x,u))u_t u_x + \gamma^2(x,u)u_x^2 \right\} \mathrm{d}x\mathrm{d}t = 0. \tag{1.6}$$

Eq. (1.1) is the Euler-Lagrange equation of (1.6). A special case of (1.1) is the well-known variational wave equation

$$u_{tt} - c(u)[c(u)u_x]_x = 0, \tag{1.7}$$

with a given positive smooth function $c(\cdot)$, which is used to model the propagation of the orientation waves in the director field of a nematic liquid crystal; see [16, 23, 24] for the details.

Many efforts have been made to study the Cauchy problem for variational wave Eq. (1.7) and its related models. We refer to, for example, the works on the singularity formation [13, 15, 16, 20, 27], on the existence of dissipative weak solutions [6, 28, 29, 30], on the existence and uniqueness of conservative weak solutions [2, 5, 8, 17], on the stability and generic regularity of conservative weak solutions [3, 4], and on the systems of variational wave equations [9, 14, 18, 31, 32]. Moreover, Eq. (1.7) with $c(u) = u$ corresponds to the second sound equation in one space dimension and its local existence and singularity of solutions were examined in [25, 26]. We refer to [22] for its existence of the degenerate Cauchy problem. In [19], Hu applied the method of energy-dependent coordinates introduced by Bressan, Zhang and Zheng [7, 8] to establish the global existence of conservative weak solutions to the Cauchy problem of (1.1) for initial data of finite energy; see also [21] for a somewhat more general equation. Recently, the uniqueness of energy conservative weak solutions of (1.1) was investigated in [10] and a Finsler type Lipschitz optimal transport metric was constructed in [11]. Moreover, Zeng and Hu [33] established the global existence of weak solutions to the initial-boundary value and initial problems for two classes of two extreme cases of (1.1).

In this paper, we study the initial-boundary value problem for one-dimensional nonlinear variational wave equation (1.1). To the best of our knowledge, the study of the initial boundary value problem for variational wave equations (1.7) and (1.1) are missing for many years. In a recent paper [12], Chen, Hu, and Zhang explored the initial-boundary value problem for the Poiseuille flow of one-dimensional hyperbolic-parabolic Ericksen-Leslie model of nematic liquid crystals. It was discussed the global existence of Hölder continuous solution for the initial-boundary value problems with different types of boundary conditions. The hyperbolic-parabolic coupled model considered in [12] reads that

$$\begin{cases} u_t = (u_x + \theta_t)_x, \\ \theta_{tt} + 2\theta_t = c(\theta)(c(\theta)\theta_x)_x - u_x, \end{cases}$$

and then the existence result on the initial-boundary value problem there can be applied to Eq. (1.7). In the current paper, we investigate the existence of weak solutions to the variational wave equations (1.1) with the homogeneous first boundary conditions (1.3). These homogeneous boundary conditions result in the disappearance of boundary energies on $x = a, b$, which leads to the conservation of energy within the finite interval $[a,b]$.

The main result of this paper can be stated below.

Theorem 1.1. *Let condition (1.4) be satisfied. Then initial-boundary value problem (1.1)-(1.3) admits a global weak solution $u = u(t,x)$ defined for all $(t,x) \in \mathbb{R}^+ \times [a,b]$, as follows:*
(i) *The function $u(t,x)$ is locally Hölder continuous with exponent $1/2$ in $\mathbb{R}^+ \times [a,b]$. The function $t \mapsto u(t,\cdot)$ is continuously differentiable as a map with values in L^θ for all $1 \le \theta < 2$. Moreover, it is Lipschitz continuous with respect to the L^2 distance, that is, there exists a constant L such that $\|u(t,\cdot) - u(s,\cdot)\|_{L^2([a,b])} \le L|t-s|$ for all $t, s \in \mathbb{R}^+$.*
(ii) *The function $u(t,x)$ takes on the initial condition in (1.2) pointwise, while its temporal derivative holds in L^θ for $\theta \in [1,2)$.*
(iii) *The boundary conditions in (1.3) are satisfied pointwise.*
(iv) *The equation (1.1) is satisfied in the distributional sense, that is,*

$$\int_0^\infty \int_a^b \left[\phi_t(\alpha^2 u_t + \beta u_x) + \phi_x(\beta u_t - \gamma^2 u_x) + \phi(\alpha\alpha_u u_t^2 + \beta_u u_t u_x - \gamma\gamma_u u_x^2)\right] \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\infty \phi(\beta u_t - \gamma^2 u_x)\Big|_{x=a}^{x=b} \mathrm{d}t = 0, \tag{1.8}$$

for all test functions $\phi \in \mathscr{F}$, where

$$\mathscr{F} := \left\{ f \in C^\infty(\mathbb{R}^+ \times (a,b)) : \ \partial_t^i \partial_x^j f\Big|_{t=0,\infty} = 0, \ \forall\, i,j = 0,1,2\cdots \right\}.$$

Moreover, there holds

$$\mathscr{E}(t) \le \mathscr{E}(0) =: \mathscr{E}_0, \tag{1.9}$$

where

$$\mathscr{E}(t) := \frac{1}{2}\int_a^b \left[\alpha^2(x,u(t,x))u_t^2(t,x) + \gamma^2(x,u(t,x))u_x^2(t,x)\right] \mathrm{d}x,$$

is the total energy. In addition, the solution $u(t,x)$ constructed above is conservative in the sense that the total energy represented by a Radon measure is conserved in time.

The paper is organized as follows. In Section 2, we introduce the energy-dependent coordinates into the finite interval $[a,b]$ and then formulate the problem in the energy-dependent coordinates. In Section 3, we establish the global existence of solutions for the problem in the energy energy-dependent coordinates. In Section 4, the last section, we return the solution to the original variables and then complete the proof of Theorem 1.1.

2. New Formulation in Energy-dependent Coordinates

In this section, we introduce the energy-dependent coordinates into the finite interval $[a,b]$ to transform the original problem into a new one.

2.1. **Energy-dependent coordinates.** Denote

$$R := \alpha u_t + c_2 u_x, \quad S := \alpha u_t + c_1 u_x, \tag{2.1}$$

so that

$$u_t = \frac{c_2 S - c_1 R}{\alpha(c_2 - c_1)}, \quad u_x = \frac{R-S}{c_2 - c_1}, \tag{2.2}$$

where $c_1 := \alpha\lambda_- < 0$ and $c_2 := \alpha\lambda_+ > 0$. By direct calculations, for smooth solutions, Eq. (1.1) can be transformed to

$$\begin{cases} \alpha(x,u)R_t + c_1(x,u)R_x = a_1 R^2 - (a_1+a_2)RS + a_2 S^2 + c_2 bS - d_1 R, \\ \alpha(x,u)S_t + c_2(x,u)S_x = -a_1 R^2 + (a_1+a_2)RS - a_2 S^2 + c_1 bR - d_2 S, \\ \alpha(x,u)u_t + c_1(x,u)u_x = S, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} a_i &= \frac{c_i \partial_u \alpha - \alpha \partial_u c_i}{2\alpha(c_2 - c_1)}, \quad b = \frac{\alpha \partial_x (c_1 - c_2) + (c_1 - c_2)\partial_x \alpha}{2\alpha(c_2 - c_1)}, \\ d_i &= \frac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{2(c_2 - c_1)} + \frac{\alpha \partial_x c_i - c_i \partial_x \alpha}{2\alpha}, \qquad (i = 1,2) \end{aligned} \tag{2.4}$$

and ∂_x and ∂_u represent partial derivatives with respect to x and u, respectively.

For any point $(t,x) \in [0,\infty) \times [a,b]$, let $x = x_\pm(s;t,x)(s \le t)$ be the forward and backward characteristics passing through the point (t,x) defined as follows

$$\begin{cases} \dfrac{\mathrm{d}x_\pm(s;t,x)}{\mathrm{d}s} = \lambda_\pm(u(x_\pm(s;t,x),s)), \\ x_\pm(t;t,x) = x. \end{cases} \tag{2.5}$$

Let us start defining the coordinate transformation $(t,x) \to (X,Y)$ on $[0,\infty) \times [a,b]$. We first specify this transformation to transform the lines $x = a$ and $x = b$ with $t \ge 0$ into the lines $Y = X$ with $X \ge 0$ and $Y = X - \widetilde{X}$ with $X \ge \widehat{X}$, respectively, where

$$\widetilde{X} = \int_a^b [1 + R_0^2(z)]\,\mathrm{d}z - \int_b^a [1 + S_0^2(z)]\,\mathrm{d}z, \quad \widehat{X} = \int_a^b [1 + R_0^2(z)]\,\mathrm{d}z, \tag{2.6}$$

where

$$\begin{aligned} R_0(x) &= \alpha(x, u_0(x))u_1(x) + c_2(x, u_0(x))u_0'(x), \\ S_0(x) &= \alpha(x, u_0(x))u_1(x) + c_1(x, u_0(x))u_0'(x), \quad \forall\, x \in [a,b]. \end{aligned} \tag{2.7}$$

Due to the assumption (1.2), we know that $\widetilde{X}$ and $\widehat{X}$ are well-defined. For the segment $t = 0$ with $x \in [a,b]$, we specify that it is transformed to a piece of curve $\Gamma_0 : Y = \varphi(X)(X \in [0,\widehat{X}])$ defined through a parametric $x \in [a,b]$

$$X = \int_a^x (1 + R_0^2(z))\,\mathrm{d}z, \quad Y = \int_x^a (1 + S_0^2(z))\,\mathrm{d}z. \tag{2.8}$$

Obviously, the two functions $X = X(x)$ and $Y = Y(x)$ with $x \in [a,b]$ are absolutely continuous satisfying that $X(x)$ is strictly increasing while $Y(x)$ is strictly decreasing. Thus we acquire that the function $Y = \varphi(X)$ is continuous and strictly decreasing.

Let (t,x) be any point in $[0,\infty)\times[a,b]$. We now draw the backward characteristic $x_-(s;t,x)$ up to a point P_1 on $x=b$, and then draw the forward characteristic $x_+(s;P_1)$ up to a point P_2 on $x=a$. Doing the same process, we can reach the segment $t=0 (x\in[a,b])$ through finite steps by (1.4). Assume that P_l is the last point on $x=a$ or $x=b$ such that the backward characteristic $x_-(s;P_l)$ or the forward characteristic $x_+(s;P_l)$ intersects the segment $t=0(x\in[a,b])$ at a point $P^*(0,x_{P^*})$. Clearly, the points P_i are on $x=b$ for odd numbers $i\le l$, and on $x=a$ for even numbers $i\le l$. See Fig. 1 (a) for the illustration. We denote the coordinates of P_i by (t_i,b) for odd numbers i, and by (t_i,a) for even numbers i. The numbers $t_i(i=1,\cdots,l)$ and x_{P^*} can be determined sequentially as follows

$$\begin{cases} x_-(t_1;x,t)=b, \\ x_+(t_2;P_1)=a, \\ x_-(t_3;P_2)=b, \\ \cdots \\ x_+(t_l;P_{l-1})=a, \quad x_{P^*}=x_-(0;P_l), \quad \text{if } l \text{ is an even number}, \\ x_-(t_l;P_{l-1})=b, \quad x_{P^*}=x_+(0;P_l), \quad \text{if } l \text{ is an odd number}. \end{cases} \tag{2.9}$$

Then the value $X(t,x)$ can be defined in the following form

$$\begin{aligned} X(t,x) =& X(P_1)=Y(P_1)+\widetilde{X}=Y(P_2)+\widetilde{X}=X(P_2)+\widetilde{X} \\ =& X(P_3)+\widetilde{X}=(Y(P_3)+\widetilde{X})+\widetilde{X}=Y(P_4)+2\widetilde{X} \\ =& \cdots = \begin{cases} X(P_l)+k\widetilde{X}, & l=2k \\ Y(P_l)+(k+1)\widetilde{X}, & l=2k+1 \end{cases} \\ =& \begin{cases} \displaystyle\int_a^{x_{P^*}} (1+R_0^2(z))\,\mathrm{d}z+k\widetilde{X}, & l=2k, \\ \displaystyle\int_{x_{P^*}}^{a} (1+S_0^2(z))\,\mathrm{d}z+(k+1)\widetilde{X}, & l=2k+1. \end{cases} \end{aligned} \tag{2.10}$$

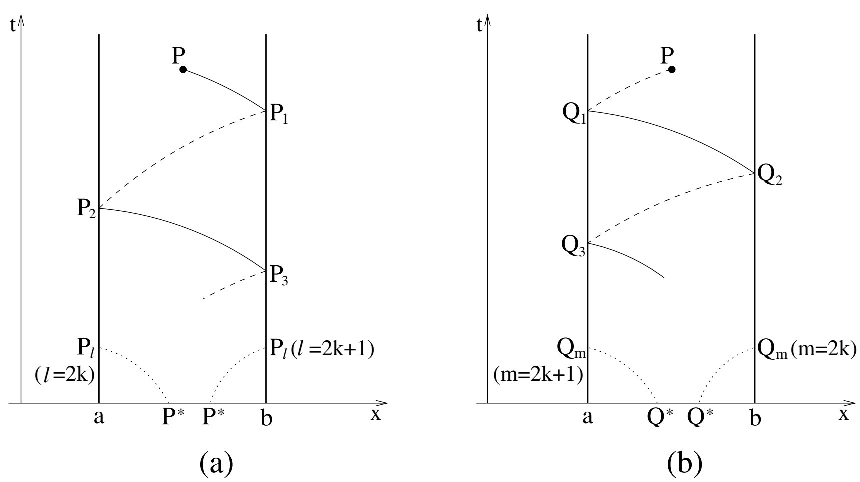

FIGURE 1. Characteristic curves.

Similarly, in order to define the value $Y(t,x)$, one draws the forward characteristic $x_+(s;t,x)$ up to a point Q_1 on $x=a$, and then draws the backward characteristic $x_-(s;Q_1)$ up to a point Q_2 on $x=b$. Repeating the above process, the characteristics can be reached the segment $t=0(x\in[a,b])$ through finite steps by the assumption (1.4). Let Q_m be the last point on $x=a$ or $x=b$ such that the backward characteristic $x_-(s;Q_m)$ or the forward characteristic $x_+(s;Q_m)$ intersects the segment $t=0(x\in[a,b])$ at a point $Q^*(0,x_{Q^*})$. We see that, for $i\le m$, Q_i is on $x=a$ if i is odd, and is on $x=b$ if i is even. See Fig. 1 (b) for the illustration. The coordinates of Q_i are denoted by $(\tilde{t}_i,a)$ for odd numbers i, and by $(\tilde{t}_i,b)$ for even numbers i. One can also determine the numbers $\tilde{t}_i(i=1,\cdots,m)$ and x_{Q^*} sequentially

$$\begin{cases} x_+(\tilde{t}_1;x,t)=a,\\ x_-(\tilde{t}_2;Q_1)=b,\\ x_+(\tilde{t}_3;Q_2)=a,\\ \cdots\\ x_-(\tilde{t}_m;Q_{m-1})=b,\;\; x_{Q^*}=x_+(0;Q_m), & \text{if } m \text{ is an even number},\\ x_+(\tilde{t}_m;Q_{m-1})=a,\;\; x_{Q^*}=x_-(0;Q_m), & \text{if } m \text{ is an odd number}.\end{cases} \tag{2.11}$$

Thus we can define the value $Y(t,x)$ by

$$\begin{aligned} Y(t,x)=&Y(Q_1)=X(Q_1)=X(Q_2)=Y(Q_2)+\widetilde{X}=Y(Q_3)+\widetilde{X}\\ =&X(Q_3)+\widetilde{X}=X(Q_4)+\widetilde{X}=(Y(Q_4)+\widetilde{X})+\widetilde{X}=Y(Q_4)+2\widetilde{X}\\ =&\cdots=\begin{cases} Y(Q_m)+k\widetilde{X}, & m=2k\\ X(Q_m)+k\widetilde{X}, & m=2k+1\end{cases}\\ =&\begin{cases} \displaystyle\int_{x_{Q^*}}^{a}(1+S_0^2(z))\,\mathrm{d}z+k\widetilde{X}, & m=2k,\\ \displaystyle\int_{a}^{x_{Q^*}}(1+R_0^2(z))\,\mathrm{d}z+k\widetilde{X}, & m=2k+1.\end{cases}\end{aligned} \tag{2.12}$$

It is obvious that if (t,x) is a point on $t=0(x\in[a,b])$, the transformation defined in (2.10) and (2.12) are reduced to (2.8). Moreover, we know by the construction process of the transformation $(t,x)\to(X,Y)$ that X and Y are constants along backward and forward characteristics, respectively; that is,

$$\alpha(x,u)X_t+c_1(x,u)X_x=0,\quad \alpha(x,u)Y_t+c_2(x,u)Y_x=0, \tag{2.13}$$

from which we obtain for any smooth function f

$$\begin{aligned} \alpha(x,u)f_t+c_1(x,u)f_x=(\alpha Y_t+c_1Y_x)f_Y=(c_1-c_2)Y_xf_Y,\\ \alpha(x,u)f_t+c_2(x,u)f_x=(\alpha X_t+c_2X_x)f_X=(c_2-c_1)X_xf_X.\end{aligned} \tag{2.14}$$

2.2. **A semilinear system.** For convenience to deal with possibly unbounded values of (R,S), we introduce a new set of dependent variables

$$\ell:=\frac{R}{1+R^2},\quad h:=\frac{1}{1+R^2},\quad m:=\frac{S}{1+S^2},\quad g:=\frac{1}{1+S^2}, \tag{2.15}$$

so that

$$\ell^2+h^2=h,\quad m^2+g^2=g. \tag{2.16}$$

Furthermore, we introduce

$$p := \frac{1+R^2}{X_x}, \quad q := \frac{1+S^2}{-Y_x}, \tag{2.17}$$

and then by (2.15)

$$\frac{1}{X_x} = \frac{p}{1+R^2} = ph, \quad \frac{1}{-Y_x} = \frac{q}{1+S^2} = qg. \tag{2.18}$$

By performing direct calculations, we can achieve a semilinear hyperbolic system with smooth coefficients for the variables g,h,ℓ,m,p,q,u,x in (X,Y) coordinates as follows:

$$\begin{cases} \ell_Y = \frac{q(2h-1)}{c_2-c_1}\Big\{a_1g+a_2h-(a_1+a_2)(gh+m\ell)+c_2bhm-d_1g\ell\Big\}, \\ m_X = \frac{p(2g-1)}{c_2-c_1}\Big\{-a_1g-a_2h+(a_1+a_2)(gh+m\ell)+c_1bg\ell-d_2hm\Big\}, \\ u_X = \frac{1}{c_2-c_1}p\ell\left(\text{or } u_Y = \frac{1}{c_2-c_1}qm\right), \quad x_X = \frac{c_2}{c_2-c_1}ph\left(\text{or } x_Y = \frac{c_1}{c_2-c_1}qg\right), \\ h_Y = -\frac{2q\ell}{c_2-c_1}\Big\{a_1g+a_2h-(a_1+a_2)(gh+m\ell)+c_2bhm-d_1g\ell\Big\}, \\ g_X = -\frac{2pm}{c_2-c_1}\Big\{-a_1g-a_2h+(a_1+a_2)(gh+m\ell)+c_1bg\ell-d_2hm\Big\}, \\ p_Y = \frac{2pq}{c_2-c_1}\Big\{a_2(\ell-m)+(a_1+a_2)(hm-g\ell)+c_2bm\ell+d_1gh+\frac{c_1\partial_xc_2-c_2\partial_xc_1}{2(c_2-c_1)}g\Big\}, \\ q_X = \frac{2pq}{c_2-c_1}\Big\{a_1(\ell-m)+(a_1+a_2)(hm-g\ell)+c_1bm\ell+d_2gh+\frac{c_1\partial_xc_2-c_2\partial_xc_1}{2(c_2-c_1)}h\Big\}. \end{cases} \tag{2.19}$$

In addition, there also hold

$$\begin{aligned} t_X = \frac{\alpha ph}{c_2-c_1}, \quad t_Y = \frac{\alpha qg}{c_2-c_1}, \quad \left(\frac{\alpha qg}{c_2-c_1}\right)_X = \left(\frac{\alpha ph}{c_2-c_1}\right)_Y, \\ \left(\frac{qm}{c_2-c_1}\right)_X = \left(\frac{p\ell}{c_2-c_1}\right)_Y, \quad \left(\frac{c_1qg}{c_2-c_1}\right)_X = \left(\frac{c_2ph}{c_2-c_1}\right)_Y, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \partial_Y(h^2+\ell^2-h) = 0, \quad \partial_X(g^2+m^2-g) = 0, \\ \left(\frac{c_2q(1-g)}{c_2-c_1}\right)_X - \left(\frac{c_1p(1-h)}{c_2-c_1}\right)_Y = 0. \end{aligned} \tag{2.21}$$

The detailed derivations of (2.19)-(2.21) can be found in Hu [19]. It follows by (2.19) and (2.20) that

$$\mathrm{d}x\mathrm{d}t = \frac{\alpha pqgh}{c_2-c_1}\mathrm{d}X\mathrm{d}Y. \tag{2.22}$$

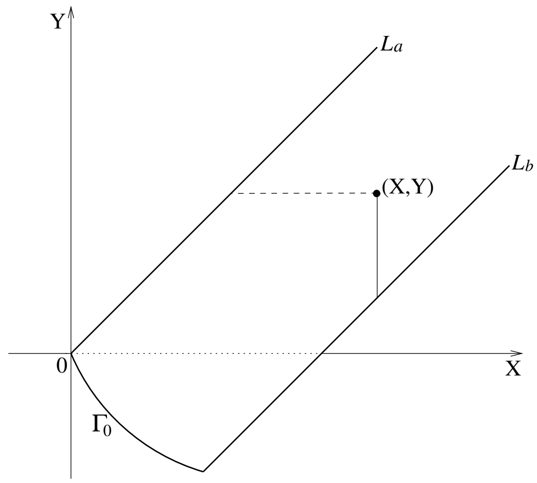

FIGURE 2. The region in the (X,Y) plane.

2.3. **The boundary conditions in the (X,Y) coordinates.** We now consider the boundary conditions of system (2.19) in the energy-dependent coordinates (X,Y), corresponding to (1.2)-(1.3) in the original coordinates (t,x).

It is easy to see by the construction of the coordinate transformation $(t,x)\to(X,Y)$ that the segment $t=0(x\in[a,b])$ is transformed into a piece of continuous and strictly decreasing curve $\Gamma_0: Y=\varphi(X)(X\in[0,\widehat{X}])$ defined in (2.8) by a parametric $x\in[a,b]$. Due to the construction, we also know that the lines $x=a(t\geq 0)$ and $x=b(t\geq 0)$ are transformed into the lines $L_a: Y=X(X\geq 0)$ and $L_b: Y=X-\widetilde{X}(X\geq\widehat{X})$, respectively. See Fig. 2. Since Γ_0 is parameterized by the parameter x, we can thus assign the boundary data $(\bar{h},\bar{g},\bar{\ell},\bar{m},\bar{p},\bar{q},\bar{u},\bar{x})\in L^\infty$ defined by

$$\begin{cases}\bar{h}=\dfrac{1}{1+R^2(0,x)},\\ \bar{g}=\dfrac{1}{1+S^2(0,x)},\end{cases}\quad \begin{cases}\bar{\ell}=R(0,x)\bar{h},\\ \bar{m}=S(0,x)\bar{g},\end{cases}\quad \begin{cases}\bar{p}=1,\\ \bar{q}=1,\end{cases}\quad \bar{u}=u_0(x),\quad \bar{x}=x, \tag{2.23}$$

where $R_0(x)$ and $S_0(x)$ are given in (2.7) and the data of (p,q) come from (2.17). Furthermore, according to (1.3), one obtains $u_t=0$ on $x=a,b$, which means by (2.2) and (2.15) that $c_2S-c_1R=0$ and then

$$c_2mh-c_1g\ell=0,\quad \text{on } L_a,L_b. \tag{2.24}$$

Combining (2.16) and (2.24) gives

$$c_2^2h(1-g)=c_1^2g(1-h),\quad \text{on } L_a,L_b, \tag{2.25}$$

from which one acquires

$$g=\frac{c_2^2h}{c_1^2(1-h)+c_2^2h},\quad \text{on } L_a;\quad h=\frac{c_1^2g}{c_2^2(1-g)+c_1^2g},\quad \text{on } L_b. \tag{2.26}$$

Moreover, by means of (2.19), we have

$$\mathrm{d}x=x_X\mathrm{d}X+x_Y\mathrm{d}Y=\frac{c_2ph}{c_2-c_1}\mathrm{d}X+\frac{c_1qg}{c_2-c_1}\mathrm{d}Y,$$

which implies by (2.21) and the definitions of L_a, L_b that

$$c_2ph + c_1qg = 0, \quad \text{on } L_a, L_b,$$

from which and (2.26) we obtain the boundary values of (p,q)

$$q = \frac{c_1^2(1-h)+c_2^2h}{-c_1c_2}p, \ \text{ on } L_a; \quad p = \frac{c_2^2(1-g)+c_1^2g}{-c_1c_2}q, \ \text{ on } L_b. \tag{2.27}$$

Summing up (2.23), (2.26), and (2.27), the new boundary value problem in the (X,Y) coordinate plane is the semilinear system (2.19) supplemented with

$$\begin{aligned}
&(h,g,\ell,m,p,q,u,x) = (\bar{h},\bar{g},\bar{\ell},\bar{m},\bar{p},\bar{q},\bar{u},\bar{x}), && \text{on } \Gamma_0,\\
&g = \frac{c_2^2h}{c_1^2(1-h)+c_2^2h}, \ \ q = \frac{c_1^2(1-h)+c_2^2h}{-c_1c_2}p, \ \ x = a, && \text{on } L_a,\\
&h = \frac{c_1^2g}{c_2^2(1-g)+c_1^2g}, \ \ p = \frac{c_2^2(1-g)+c_1^2g}{-c_1c_2}q, \ \ x = b, && \text{on } L_b,\\
&c_2mh - c_1g\ell = 0, \ \ \ell^2+h^2 = h, \ \ m^2+g^2 = g, \ \ u = 0, && \text{on } L_a, L_b.
\end{aligned} \tag{2.28}$$

For convenience, we use Ω to denote the region bounded by $t=0, x=a$, and $x=b$ in the (t,x) plane, and use $\widetilde{\Omega}$ to represent its image in the (X,Y) plane which is bounded by Γ_0, L_a and L_b.

3. Solutions in (X,Y) Coordinates

In this section, we show that boundary value problem (2.19)-(2.28) admits a unique global solution in the energy coordinates (X,Y).

3.1. **The local existence.** We first use the level lines of X and Y to divide the region $\widetilde{\Omega}$ into a series of subregions $\widetilde{\Omega} = \bigcup_{n=0}^{\infty}\Omega^n$, where $\Omega^0 = \Omega_1^0 \cup \Omega_2^0 \cup \Omega_3^0$ with

$$\begin{aligned}
\Omega_1^0 &= \{(X,Y):\ 0 < X \le \widehat{X},\ 0 < Y \le X\},\\
\Omega_2^0 &= \{(X,Y):\ 0 \le X \le \widehat{X},\ \varphi(X) \le Y \le 0\},\\
\Omega_3^0 &= \{(X,Y):\ \widehat{X} < X \le \widetilde{X},\ X-\widetilde{X} \le Y \le 0\},
\end{aligned}$$

and $\Omega^n = \Omega_1^n \cup \Omega_2^n \cup \Omega_3^n$ with

$$\begin{aligned}
\Omega_1^n &= \{(X,Y):\ \widehat{X}+k\widetilde{X} < X \le (k+1)\widetilde{X},\ \widehat{X}+k\widetilde{X} < Y \le X\},\\
\Omega_2^n &= \{(X,Y):\ \widehat{X}+k\widetilde{X} < X \le (k+1)\widetilde{X},\ k\widetilde{X} < Y \le \widehat{X}+k\widetilde{X}\},\\
\Omega_3^n &= \{(X,Y):\ (k+1)\widetilde{X} < X \le \widehat{X}+(k+1)\widetilde{X},\ X-\widetilde{X} \le Y \le \widehat{X}+k\widetilde{X}\},
\end{aligned}$$

for $n = 2k+1 (k = 0,1,2,\cdots)$, and with

$$\begin{aligned}
\Omega_1^n &= \{(X,Y):\ k\widetilde{X} < X \le \widehat{X}+k\widetilde{X},\ k\widetilde{X} < Y \le X\},\\
\Omega_2^n &= \{(X,Y):\ k\widetilde{X} < X \le \widehat{X}+k\widetilde{X},\ \widehat{X}+(k-1)\widetilde{X} < Y \le k\widetilde{X}\},\\
\Omega_3^n &= \{(X,Y):\ \widehat{X}+k\widetilde{X} < X \le (k+1)\widetilde{X},\ X-\widetilde{X} \le Y \le k\widetilde{X}\},
\end{aligned}$$

for $n = 2k (k = 1,2,3,\cdots)$. See Fig. 3 for the illustration.

It is easily seen that the existence result in the region Ω_2^0 can be obtained directly by the result of the Cauchy problem in Hu [19]. To establish the local existence of boundary value problem (2.19)-(2.28), it suffices to discuss the local existence of solutions in the region Ω_1^0 near point $(0,0)$ and in the region Ω_3^0 near point $(\widehat{X}, \varphi(\widehat{X}))$. We here only consider the case near

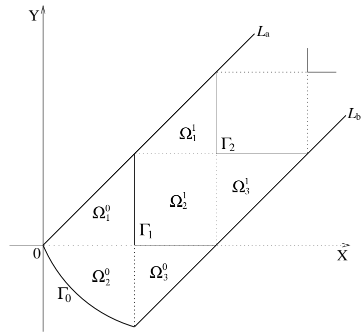

FIGURE 3. The region in the (X,Y) plane.

point $(0,0)$ and the other case can be handled analogously. For any point (X,Y) in Ω_1^0, we use (2.19)-(2.28) and the solution on $Y=0 (X\in[0,\widehat{X}])$ to construct a map $\mathscr{T}$ as follows

$$(\hat{h},\hat{g},\hat{\ell},\hat{m},\hat{p},\hat{q},\hat{u},\hat{x}) = \mathscr{T}(h,g,\ell,m,p,q,u,x),$$

where

$$\begin{cases} \hat{h}(X,Y) = h(X,0) - \displaystyle\int_0^Y \frac{2q\ell}{c_2-c_1}\Big\{a_1g+a_2h \\ -(a_1+a_2)(gh+m\ell)+c_2bhm-d_1g\ell\Big\}(X,Y')\,\mathrm{d}Y', \\ \hat{\ell}(X,Y) = \ell(X,0) + \displaystyle\int_0^Y \frac{q(2h-1)}{c_2-c_1}\Big\{a_1g+a_2h \\ -(a_1+a_2)(gh+m\ell)+c_2bhm-d_1g\ell\Big\}(X,Y')\,\mathrm{d}Y', \\ \hat{p}(X,Y) = p(X,0) + \displaystyle\int_0^Y \frac{2pq}{c_2-c_1}\Big\{a_2(\ell-m)+(a_1+a_2)(hm-g\ell) \\ +c_2bm\ell+d_1gh+\dfrac{c_1\partial_xc_2-c_2\partial_xc_1}{2(c_2-c_1)}g\Big\}(X,Y')\,\mathrm{d}Y', \end{cases} \tag{3.1}$$

$$\begin{cases} \hat{g}(X,Y) = \dfrac{c_{2a}^2h(Y,Y)}{c_{1a}^2(1-h(Y,Y))+c_{2a}^2h(Y,Y)} - \displaystyle\int_Y^X \frac{2pm}{c_2-c_1}\Big\{-a_1g-a_2h \\ +(a_1+a_2)(gh+m\ell)+c_1bg\ell-d_2hm\Big\}(X',Y)\,\mathrm{d}X', \\ \hat{m}(X,Y) = \dfrac{c_{1a}c_{2a}\ell(Y,Y)}{c_{1a}^2(1-h(Y,Y))+c_{2a}^2h(Y,Y)} + \displaystyle\int_Y^X \frac{p(2g-1)}{c_2-c_1}\Big\{-a_1g-a_2h \\ +(a_1+a_2)(gh+m\ell)+c_1bg\ell-d_2hm\Big\}(X',Y)\,\mathrm{d}X', \\ \hat{q}(X,Y) = \dfrac{c_{1a}^2(1-h(Y,Y))+c_{2a}^2h(Y,Y)}{-c_{1a}c_{2a}}p(Y,Y) + \displaystyle\int_Y^X \frac{2pq}{c_2-c_1}\Big\{a_1(\ell-m) \\ +(a_1+a_2)(hm-g\ell)+c_1bm\ell+d_2gh+\dfrac{c_1\partial_xc_2-c_2\partial_xc_1}{2(c_2-c_1)}h\Big\}(X',Y)\,\mathrm{d}X', \end{cases} \tag{3.2}$$

and

$$\begin{cases} \hat{u}(X,Y) = \displaystyle\int_Y^X \frac{1}{c_2-c_1} p\ell(X',Y)\,\mathrm{d}X', \\ \hat{x}(X,Y) = a + \displaystyle\int_Y^X \frac{c_2}{c_2-c_1} ph(X',Y)\,\mathrm{d}X'. \end{cases} \tag{3.3}$$

In (3.2), $c_{1a} = c_1(a,0)$, $c_{2a} = c_2(a,0)$, and

$$\begin{cases} h(Y,Y) = h(Y,0) - \displaystyle\int_0^Y \frac{2q\ell}{c_2-c_1}\Big\{a_1 g + a_2 h \\ -(a_1+a_2)(gh+m\ell) + c_2 bhm - d_1 g\ell\Big\}(Y,Y')\,\mathrm{d}Y', \\ \ell(Y,Y) = \ell(Y,0) + \displaystyle\int_0^Y \frac{q(2h-1)}{c_2-c_1}\Big\{a_1 g + a_2 h \\ -(a_1+a_2)(gh+m\ell) + c_2 bhm - d_1 g\ell\Big\}(Y,Y')\,\mathrm{d}Y', \\ p(Y,Y) = p(Y,0) + \displaystyle\int_0^Y \frac{2pq}{c_2-c_1}\Big\{a_2(\ell-m) + (a_1+a_2)(hm-g\ell) \\ +c_2 bm\ell + d_1 gh + \dfrac{c_1\partial_x c_2 - c_2 \partial_x c_1}{2(c_2-c_1)} g\Big\}(Y,Y')\,\mathrm{d}Y'. \end{cases} \tag{3.4}$$

Let

$$V = (h,g,\ell,m,p,q,u,x), \quad \hat{V} = (\hat{h},\hat{g},\hat{\ell},\hat{m},\hat{p},\hat{q},\hat{u},\hat{x}),$$

and

$$V_0(X) = (h,g,\ell,m,p,q,u,x)(X,0), \ \ \forall\, X \in [0,\widehat{X}].$$

Due to the existence result in the region Ω_2^0 in Hu [19], we can obtain the vector function $V_0(X)$, which is Lipschitz continuous. It is easy to check by (3.1)-(3.4) that

$$\hat{V}(X,0) = V_0(X). \tag{3.5}$$

We use K to denote a sufficiently large positive constant depending only on $\|V_0(X)\|_{\mathrm{Lip}}$ and $\delta < \widehat{X}$ is a small positive constant. Denote

$$\mathscr{K} = \left\{ V \;\middle|\; \begin{array}{l} \|V(X,Y)\|_{\mathrm{Lip}(\Omega^0_{1\delta})} \le K, \ \ V(X,0) = V_0(X), \\ g(X,X) = \dfrac{c_{2a}^2 h(X,X)}{c_{1a}^2(1-h(X,X)) + c_{2a}^2 h(X,X)}, \\ q(X,X) = \dfrac{c_{1a}^2(1-h(X,X)) + c_{2a}^2 h(X,X)}{-c_{1a}c_{2a}} p(X,X), \\ (c_2 mh - c_1 g\ell)(X,X) = 0, \ \ (\ell^2+h^2-h)(X,X) = 0, \\ (m^2+g^2-g)(X,X) = 0, \ \ u(X,X) = 0, \ \ x(X,X) = a \end{array} \right\}, \tag{3.6}$$

where

$$\Omega^0_{1\delta} = \big\{(X,Y) \in \Omega^0_1 : \ \mathrm{dist}((X,Y),(0,0)) \le \delta\big\}.$$

Obviously, $V_0(X) \in \mathscr{K}$ by (3.5). By selecting a constant K large enough and then a constant δ small enough, one can show that $\mathscr{T}$ maps $\mathscr{K}$ to itself and is a strict contraction under C^0 norm. Then the local existence of solutions is based on the fixed point theorem. We omit the proof since it is similar to that in Bressan and Zheng [8] or Chen, Huang and Liu [13].

3.2. **The global existence.** The local solution constructed in Subsection 3.1 can be extended to the whole region $\widetilde{\Omega}$ by a priori global estimates on p and q.

Lemma 3.1. *Let $(h,g,\ell,m,p,q,u,x)(X,Y)$ be a solution of boundary value problem (2.19)-(2.28) on $\widetilde{\Omega}$. Then, for any point $(X^*,Y^*)\in\widetilde{\Omega}$, there exist two positive constants M and N depending only on the boundary data on Γ_0 and (X^*,Y^*) such that*

$$0<M\le \max_{(X,Y)\in\widetilde{\Omega}^*}\{p(X,Y),q(X,Y)\}\le N,$$

where $\widetilde{\Omega}^*=\widetilde{\Omega}\cap\{X\le X^*,Y\le Y^*\}$.

Proof. Denote the equations of p and q in (2.19) as

$$p_Y=Fpq,\quad q_X=Gpq. \tag{3.7}$$

From (3.7) and the boundary conditions of p,q on Γ_0,L_a,L_b, we see that p and q are positive on Ω^0. Moreover, thanks to (2.21), one has $0\le h,g\le 1$ and $|\ell|,|m|\le\frac{1}{2}$, which indicate by (1.4) that F and G satisfy $|F|,|G|\le\widehat{K}$ for some positive constant $\widehat{K}$.

Let us first discuss the boundedness of p and q on the region Ω^0. For any point (X,Y) in Ω_1^0, let Σ_1^0 be the region enclosed by a horizontal segment between (Y,Y) and (X,Y), a vertical segment between (X,Y) and $(X,\varphi(X))$, Γ_0 and L_a. In view of Green's theorem, we achieve by (2.21)

$$\begin{aligned}&\int_{\partial\Sigma_1^0}\frac{-c_1p(1-h)}{c_2-c_1}\,\mathrm{d}X'-\frac{c_2q(1-g)}{c_2-c_1}\,\mathrm{d}Y'\\ =&-\iint_{\Sigma_1^0}\left(\frac{c_2q(1-g)}{c_2-c_1}\right)_X+\left(\frac{-c_1p(1-h)}{c_2-c_1}\right)_Y\,\mathrm{d}X'\mathrm{d}Y'=0,\end{aligned} \tag{3.8}$$

where $\partial\Sigma_1^0$ denotes the boundary of Σ_1^0. It follows by (3.8) that

$$\begin{aligned}&\int_Y^X\frac{-c_1p(1-h)}{c_2-c_1}(X',Y)\,\mathrm{d}X'+\int_{\varphi(X)}^Y\frac{c_2q(1-g)}{c_2-c_1}(X,Y')\,\mathrm{d}Y'\\ =&\int_{\Gamma_0}\frac{-c_1p(1-h)}{c_2-c_1}\,\mathrm{d}X'-\frac{c_2q(1-g)}{c_2-c_1}\,\mathrm{d}Y'\\ &+\int_{(0,0)}^{(Y,Y)}\frac{c_1p(1-h)}{c_2-c_1}\,\mathrm{d}X'+\frac{c_2q(1-g)}{c_2-c_1}\,\mathrm{d}Y'.\end{aligned} \tag{3.9}$$

Applying (2.28) yield

$$\begin{aligned}\int_{\Gamma_0}\frac{-c_1p(1-h)}{c_2-c_1}\,\mathrm{d}X'-\frac{c_2q(1-g)}{c_2-c_1}\,\mathrm{d}Y'=&\int_0^X\left(\frac{-c_1(1-\bar{h})}{c_2-c_1}-\frac{c_2(1-\bar{g})}{c_2-c_1}\varphi'(X)\right)\mathrm{d}X'\\ \le&X-\varphi(X),\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\int_{(0,0)}^{(Y,Y)}\frac{c_1p(1-h)}{c_2-c_1}\,\mathrm{d}X'+\frac{c_2q(1-g)}{c_2-c_1}\,\mathrm{d}Y'\\ =&\int_0^Y\frac{1}{c_2-c_1}[c_1p(1-h)-c_2q(1-g)](X',X')\,\mathrm{d}X'=0.\end{aligned} \tag{3.11}$$

Here we used the fact $c_1p(1-h)=c_2q(1-g)$ on L_a by the following equations

$$\begin{aligned} c_2q(1-g)&=c_2\frac{c_1^2(1-h)+c_2^2h}{-c_1c_2}p\cdot\left(1-\frac{c_2^2h}{c_1^2(1-h)+c_2^2h}\right)=c_1p(1-h), \quad \text{on } L_a,\\ c_1p(1-h)&=c_1\frac{c_2^2(1-g)+c_1^2g}{-c_1c_2}q\cdot\left(1-\frac{c_1^2g}{c_2^2(1-g)+c_1^2g}\right)=c_2q(1-g), \quad \text{on } L_b. \end{aligned} \tag{3.12}$$

Putting (3.10) and (3.11) into (3.9) gives

$$\int_Y^X \frac{-c_1p(1-h)}{c_2-c_1}(X',Y)\,\mathrm{d}X'+\int_{\varphi(X)}^Y \frac{c_2q(1-g)}{c_2-c_1}(X,Y')\,\mathrm{d}Y'\le X-\varphi(X),$$

which means by (1.4) that

$$\int_Y^X p(1-h)(X',Y)\,\mathrm{d}X'+\int_{\varphi(X)}^Y q(1-g)(X,Y')\,\mathrm{d}Y'\lesssim X-\varphi(X). \tag{3.13}$$

Here and below, the notation $A\lesssim B$ expresses $A\le CB$ for some uniform constant C. Moreover, we utilize Green's theorem for (2.20) on the region Σ_1^0 to acquire similarly

$$\int_Y^X \frac{c_2ph}{c_2-c_1}(X',Y)\,\mathrm{d}X'+\int_{\varphi(X)}^Y \frac{-c_1qg}{c_2-c_1}(X,Y')\,\mathrm{d}Y'\le X-\varphi(X),$$

and then

$$\int_Y^X ph(X',Y)\,\mathrm{d}X'+\int_{\varphi(X)}^Y qg(X,Y')\,\mathrm{d}Y'\lesssim X-\varphi(X). \tag{3.14}$$

Combining (3.13) and (3.14) leads to

$$\int_Y^X p(X',Y)\,\mathrm{d}X'+\int_{\varphi(X)}^Y q(X,Y')\,\mathrm{d}Y'\lesssim X-\varphi(X). \tag{3.15}$$

Based on (3.7) and (3.15), there exist two positive constants M_{01} and N_{01} depending only on the boundary data on Γ_0 such that

$$0<M_{01}\le \max_{(X,Y)\in\Omega_1^0}\{p(X,Y),q(X,Y)\}\le N_{01}. \tag{3.16}$$

From symmetry, we can gain that

$$0<M_{03}\le \max_{(X,Y)\in\Omega_3^0}\{p(X,Y),q(X,Y)\}\le N_{03}, \tag{3.17}$$

for some uniform constants M_{03} and N_{03}. Moreover, the boundedness of p and q on the region Ω_2^0 can be obtained by the result in Hu [19], that is, there exist two uniform positive constants M_{02} and N_{02} such that

$$0<M_{02}\le \max_{(X,Y)\in\Omega_2^0}\{p(X,Y),q(X,Y)\}\le N_{02}. \tag{3.18}$$

Combining (3.16)-(3.18) yields

$$0<M_{0}\le \max_{(X,Y)\in\Omega^0}\{p(X,Y),q(X,Y)\}\le N_{0}, \tag{3.19}$$

where $M_0=\min\{M_{01},M_{02},M_{03}\}$ and $N_0=\max\{N_{01},N_{02},N_{03}\}$. According to (3.19), we have

$$0<M_{0}\le \max_{(X,Y)\in\Gamma_1}\{p(X,Y),q(X,Y)\}\le N_{0},$$

where $\Gamma_1 = \Gamma_{11} \cup \Gamma_{12}$,

$$\Gamma_{11} = \{(X,Y):\ X = \widehat{X},\ 0 \le Y \le \widehat{X}\}, \quad \Gamma_{12} = \{(X,Y):\ \widehat{X} \le X \le \widetilde{X},\ Y = 0\},$$

which are two boundaries of the region Ω^1. For any point (X,Y) in Ω^1, for example, $(X,Y) \in \Omega_3^1$, we consider the region Σ_3^1 enclosed by a horizontal segment between $(\widehat{X},Y)$ and (X,Y), a vertical segment between (X,Y) and $(X, X-\widetilde{X})$, L_b, Γ_{12} and Γ_{11}. By performing similarly calculations on region Σ_3^1 as on Σ_1^0, we can obtain

$$0 < M_{13} \le \max_{(X,Y)\in\Omega_3^1} \{p(X,Y), q(X,Y)\} \le N_{13},$$

for some uniform constants M_{13} and N_{13} depending only on the boundary data on Γ_1. Thus there exist two positive constants M_1 and N_1 depending only on the boundary data on Γ_1 such that $0 < M_1 \le \max_{(X,Y)\in\Omega^1}\{p(X,Y), q(X,Y)\} \le N_1$. For any point $(X^*,Y^*) \in \widetilde{\Omega}$, we assume that $(X^*,Y^*) \in \Omega^l$ for some integer $l \ge 1$. Then from the above analysis, we know that there exist positive constants M_i and N_i such that

$$0 < M_i \le \max_{(X,Y)\in\Omega^i} \{p(X,Y), q(X,Y)\} \le N_i, \ \ i = 0, 1, \cdots, l-1,$$

and then $0 < M_{l-1} \le \max_{(X,Y)\in\Gamma_l}\{p(X,Y), q(X,Y)\} \le N_{l-1}$, where $\Gamma_l = \Gamma_{l1} \cup \Gamma_{l2}$,

$$\Gamma_{l1} = \{(X,Y):\ X = X_l,\ Y_l \le Y \le X_l\}, \quad \Gamma_{l2} = \{(X,Y):\ X_l \le X \le Y_l + \widetilde{X},\ Y = Y_l\}.$$

Here X_l and Y_l are given as follows

$$(X_l, Y_l) = \begin{cases} (\widehat{X} + k\widetilde{X}, k\widetilde{X}), & l = 2k+1, \\ (k\widetilde{X}, \widehat{X} + (k-1)\widetilde{X}), & l = 2k. \end{cases}$$

Clearly, Γ_{l1} and Γ_{l2} are two boundaries of the region Ω^l. For any point $(X,Y) \in \Omega^l$, we repeat the above process to obtain $0 < M_l \le \max_{(X,Y)\in\Omega^l}\{p(X,Y), q(X,Y)\} \le N_l$ for some uniform constants M_l and N_l depending only on the boundary data on Γ_l and the number l. The proof of the lemma is completed by taking $M = \min\{M_0, M_1, \cdots, M_l\}$ and $N = \max\{N_0, N_1, \cdots, N_l\}$. □

Therefore we have the following theorem.

Theorem 3.1. *Suppose that* (1.2)-(1.4) *hold. Then boundary value problem* (2.19)-(2.28) *admits a unique global solution defined for all* $(X,Y) \in \widetilde{\Omega}$.

4. Solutions in Original Variables

In this section, we express the function $u(X,Y)$ constructed in Section 3 into the (t,x) plane and then verify that it is a weak solution of (1.1) on the region Ω.

4.1. **Inverse transformation.** We recall (2.20) to see that $t_{XY} = t_{YX}$, which implies that one can integrate one of the equations t_X and t_Y to obtain the function $t = t(X,Y)$. Hence we have the function $(t,x) = (t(X,Y), x(X,Y))$. It is pointed out that the map $(X,Y) \mapsto (t,x)$ may not be one-to-one mapping. However, we can show an assertion that if $x(X_1,Y_1) = x(X_2,Y_2)$ and $t(X_1,Y_1) = t(X_2,Y_2)$ for two points (X_1,Y_1) and (X_2,Y_2) in $\widetilde{\Omega}$, there holds

$$u(t(X_1,Y_1), x(X_1,Y_1)) = u(t(X_2,Y_2), x(X_2,Y_2)). \tag{4.1}$$

The above fact indicates that the values of u do not depend on the choice of (X,Y) and then we can choose an arbitrary point (X,Y) satisfying $t(X,Y)=t$, $x(X,Y)=x$ and define $u(t,x):=u(X,Y)$. The proof of (4.1) is divided into two cases: Case 1. $X_1\le X_2, Y_1\le Y_2$ and Case 2. $X_1\le X_2, Y_1\ge Y_2$. For Case 1, if $x(X_1,Y_1)=x(X_2,Y_2)=x^*\in(a,b)$, as in [19], we consider the set $D_{X^*}:=\{(X,Y):\ x(X,Y)\le x^*\}$, and use ∂D_{X^*} to denote its boundary. Since x is increasing with X and decreasing with Y, this boundary is a Lipschitz continuous curve in $\widetilde{\Omega}$. Hence we can construct a Lipschitz continuous curve γ_1 connecting points (X_1,Y_1) and (X_2,Y_2), which consists a horizontal segment $Y\equiv Y_1$, ∂D_{X^*} and a vertical segment $X\equiv X_2$. On γ_1, there hold $x(X,Y)\equiv x(X_1,Y_1)$ and $t(X,Y)\equiv t(X_1,Y_1)$ by (2.19) and (2.20). Hence one has $ph\mathrm{d}X=qg\mathrm{d}Y=0$ and then $p\ell\mathrm{d}X=qm\mathrm{d}Y=0$ along the curve Γ_1. Thus

$$u(t(X_2,Y_2),x(X_2,Y_2))-u(t(X_1,Y_1),x(X_1,Y_1))=\int_{\gamma_1}\frac{p\ell}{c_2-c_1}\,\mathrm{d}X+\frac{qm}{c_2-c_1}\,\mathrm{d}Y=0.$$

If $x(X_1,Y_1)=x(X_2,Y_2)=b$, we construct a Lipschitz continuous curve γ_2 connecting points (X_1,Y_1) and (X_2,Y_2) by a horizontal segment $Y\equiv Y_1$, L_b and a vertical segment $X\equiv X_2$. According to $t(X_1,Y_1)=t(X_2,Y_2)$, it concludes by (2.20) and (2.28) that

$$\begin{aligned}0=&t(X_2,Y_2)-t(X_1,Y_1)=t(X_2,X_2-\widetilde{X})-t(Y_1+\widetilde{X},Y_1)\\
=&\int_{(Y_1+\widetilde{X},Y_1)}^{(X_2,X_2-\widetilde{X})}\frac{\alpha ph}{c_2-c_1}\,\mathrm{d}X+\frac{\alpha qg}{c_2-c_1}\,\mathrm{d}Y\\
=&\int_{Y_1+\widetilde{X}}^{X_2}\frac{\alpha}{c_2-c_1}\left(\frac{c_2^2(1-g)+c_1^2g}{-c_1c_2}q\cdot\frac{c_1^2g}{c_2^2(1-g)+c_1^2g}+qg\right)(X,X-\widetilde{X})\,\mathrm{d}X\\
=&\int_{Y_1+\widetilde{X}}^{X_2}\frac{\alpha}{c_2}qg(X,X-\widetilde{X})\,\mathrm{d}X,\end{aligned}$$

which indicates that $ph(X,X-\widetilde{X})=qg(X,X-\widetilde{X})=0$ and then $p\ell(X,X-\widetilde{X})=qm(X,X-\widetilde{X})=0$ for $X\in[Y_1+\widetilde{X},X_2]$. Thus we get along γ_2

$$\begin{aligned}u(t(X_2,Y_2),x(X_2,Y_2))-u(t(X_1,Y_1),x(X_1,Y_1))=&u(t(X_2,X_2-\widetilde{X}),b)-u(t(Y_1+\widetilde{X},Y_1),b)\\
=&\int_{(Y_1+\widetilde{X},Y_1)}^{(X_2,X_2-\widetilde{X})}\frac{p\ell}{c_2-c_1}\,\mathrm{d}X+\frac{qm}{c_2-c_1}\,\mathrm{d}Y=0.\end{aligned}$$

The proof of Case 1 is finished and the Case 2 can be checked similarly.

4.2. **The energy estimate.** This subsection is devoted to deriving the energy estimate (1.9), that is,

$$\mathscr{E}(t)=\frac{1}{2}\int_a^b\left(\alpha^2(x,u(t,x))u_t^2(t,x)+\gamma^2(x,u(t,x))u_x^2(t,x)\right)\mathrm{d}x\le\mathscr{E}(0).$$

We use $\Gamma_t\subset\widetilde{\Omega}$ to represent the transformation of the horizontal segment of t with $x\in[a,b]$ in the (t,x) plane. Let (X_a,X_a) and $(X_b,X_b-\widetilde{X})$ be, respectively, the coordinates of the intersection points of Γ_t and the lines L_a, L_b. Denote $\Omega_t=[0,t]\times[a,b]$ and $\widetilde{\Omega}_t$ the corresponding region of

Ω_t in the (X,Y) plane. Then one applies Green's theorem for (2.21) on the region $\widetilde{\Omega}_t$ to gain

$$\begin{aligned}
&\int_a^b \Big(\alpha^2(x,u(t,x))u_t^2(t,x)+\gamma^2(x,u(t,x))u_x^2(t,x)\Big)\,\mathrm{d}x\\
=&\int_{\Gamma_t\cap\{h\neq 0\}} \frac{-c_1(1-h)p}{c_2-c_1}\,\mathrm{d}X-\int_{\Gamma_t\cap\{g\neq 0\}}\frac{c_2(1-g)q}{c_2-c_1}\,\mathrm{d}Y\\
\leq&\int_{\Gamma_t}\frac{-c_1(1-h)p}{c_2-c_1}\,\mathrm{d}X-\frac{c_2(1-g)q}{c_2-c_1}\,\mathrm{d}Y\\
=&\left\{\int_{\Gamma_0}+\int_{(\widehat{X},\phi(\widehat{X}))}^{(X_b,X_b-\widetilde{X})}-\int_{(0,0)}^{(X_a,X_a)}\right\}\frac{-c_1(1-h)p}{c_2-c_1}\,\mathrm{d}X-\frac{c_2(1-g)q}{c_2-c_1}\,\mathrm{d}Y\\
&-\iint_{\widetilde{\Omega}_t}\left(\frac{c_2q(1-g)}{c_2-c_1}\right)_X+\left(\frac{-c_1p(1-h)}{c_2-c_1}\right)_Y\,\mathrm{d}X\mathrm{d}Y\\
=&\int_{\Gamma_0}\frac{-c_1(1-h)p}{c_2-c_1}\,\mathrm{d}X-\frac{c_2(1-g)q}{c_2-c_1}\,\mathrm{d}Y\\
=&\int_a^b\Big(\alpha^2(x,u(0,x))u_t^2(0,x)+\gamma^2(x,u(0,x))u_x^2(0,x)\Big)\,\mathrm{d}x.
\end{aligned}\tag{4.2}$$

Here we used (3.12) to obtain the following facts

$$\left\{\int_{(\widehat{X},\phi(\widehat{X}))}^{(X_b,X_b-\widetilde{X})}-\int_{(0,0)}^{(X_a,X_a)}\right\}\frac{-c_1(1-h)p}{c_2-c_1}\,\mathrm{d}X-\frac{c_2(1-g)q}{c_2-c_1}\,\mathrm{d}Y=0.$$

The energy estimate (1.9) follows from (4.2).

In addition, one can show as in Hu [19] that $u=u(t,x)$ is Hölder continuous with exponent 1/2 and all characteristic curves are C^1 with Hölder continuous derivative. Moreover, from the energy estimate, we know that the functions R and S in (2.1) are square integrable. we also have

$$\begin{aligned}
\alpha(x,u)u_t+c_2(x,u)u_x&=(c_2-c_1)u_XX_x=\frac{\ell}{h}=R,\\
\alpha(x,u)u_t+c_1(x,u)u_x&=(c_1-c_2)u_YY_x=\frac{m}{g}=S.
\end{aligned}\tag{4.3}$$

Thus functions R and S at (2.1) are indeed the same as recovered from (2.15).

4.3. **Proof of (1.8).** In this subsection, we check that the function $u(t,x)$ constructed in Subsection 4.1 satisfies (1.1) in the distributional sense, that is, (1.8) holds.

In order to prove (1.8), for any test functions $\phi\in\mathscr{F}$, we only need to check

$$\begin{aligned}
0=&\int_0^\infty\int_a^b\Big\{(\alpha\phi_t+c_1\phi_x+\alpha\phi_t+c_2\phi_x)\alpha u_t+(\alpha\phi_t+c_1\phi_x)c_2u_x+(\alpha\phi_t+c_2\phi_x)c_1u_x\\
&+2\phi(\alpha\partial_u\alpha u_t^2+\partial_u\beta u_tu_x-\gamma\partial_u\gamma u_x^2)\Big\}\,\mathrm{d}x\mathrm{d}t-\int_0^\infty 2\phi(\beta u_t-\gamma^2u_x)\Big|_{x=a}^{x=b}\,\mathrm{d}t\\
=&\int_0^\infty\int_a^b\Big\{\Big[(\alpha\phi_t+c_1\phi_x)(\alpha u_t+c_2u_x)+(\alpha\phi_t+c_2\phi_x)(\alpha u_t+c_1u_x)\Big]\\
&+2\phi\Big(\alpha\partial_u\alpha u_t^2+\partial_u\beta u_tu_x-\gamma\partial_u\gamma u_x^2\Big)\Big\}\,\mathrm{d}x\mathrm{d}t-\int_0^\infty 2\phi(\beta u_t-\gamma^2u_x)\Big|_{x=a}^{x=b}\,\mathrm{d}t.
\end{aligned}\tag{4.4}$$

Expressing the first part of double integral (4.4) in terms of the variables X, Y by (2.22) leads to

$$\begin{aligned}
&\int_0^\infty \int_a^b \left\{ (\alpha\phi_t + c_1\phi_x)(\alpha u_t + c_2 u_x) + (\alpha\phi_t + c_2\phi_x)(\alpha u_t + c_1 u_x) \right\} \mathrm{d}x\mathrm{d}t \\
=& \iint_{\widetilde{\Omega}} \left\{ (c_2 - c_1)(-Y_x)\phi_Y \frac{\ell}{h} + (c_2 - c_1)(X_x)\phi_X \frac{m}{g} \right\} \cdot \frac{\alpha pqgh}{c_2 - c_1} \mathrm{d}X\mathrm{d}Y \\
=& \iint_{\widetilde{\Omega}} \left\{ \alpha qm\phi_X + \alpha p\ell\phi_Y \right\} \mathrm{d}X\mathrm{d}Y \\
=& \iint_{\widetilde{\Omega}} \left\{ (\alpha qm\phi)_X + (\alpha p\ell\phi)_Y \right\} \mathrm{d}X\mathrm{d}Y - \iint_{\widetilde{\Omega}} \phi \left\{ (\alpha qm)_X + (\alpha p\ell)_Y \right\} \mathrm{d}X\mathrm{d}Y.
\end{aligned} \tag{4.5}$$

By direct calculations, one obtains

$$(\alpha qm)_X + (\alpha p\ell)_Y = \frac{2\alpha pqgh}{c_2 - c_1} \left\{ a_2 \frac{1-g}{g} - a_1 \frac{1-h}{h} + \left(a_1 - a_2 + \frac{\partial_u \alpha}{\alpha} \right) \frac{\ell m}{hg} \right\},$$

from which and (4.3) we see that

$$\begin{aligned}
&- \iint_{\widetilde{\Omega}} \phi \left\{ (\alpha qm)_X + (\alpha p\ell)_Y \right\} \mathrm{d}X\mathrm{d}Y \\
=& - \int_0^\infty \int_a^b 2\phi \left\{ a_2 S^2 - a_1 R^2 + \left(a_1 - a_2 + \frac{\partial_u \alpha}{\alpha} \right) RS \right\} \mathrm{d}x\mathrm{d}t \\
=& - \int_0^\infty \int_a^b 2\phi \left\{ \alpha\partial_u \alpha u_t^2 + \partial_u \beta u_t u_x - \gamma\partial_u \gamma u_x^2 \right\} \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.6}$$

Furthermore, it follows by Green's theorem that

$$\begin{aligned}
&\iint_{\widetilde{\Omega}} \left\{ (\alpha qm\phi)_X + (\alpha p\ell\phi)_Y \right\} \mathrm{d}X\mathrm{d}Y \\
=& \int_{(\widehat{X}, \varphi(\widehat{X}))}^{(\infty,\infty)} -\alpha p\ell\phi \ \mathrm{d}X + \alpha qm\phi \ \mathrm{d}Y - \int_{(0,0)}^{(\infty,\infty)} -\alpha p\ell\phi \ \mathrm{d}X + \alpha qm\phi \ \mathrm{d}Y \\
=& \int_{\widehat{X}}^\infty \alpha\phi(qm - p\ell)(X, X - \widetilde{X}) \ \mathrm{d}X - \int_0^\infty \alpha\phi(qm - p\ell)(X, X) \ \mathrm{d}X.
\end{aligned} \tag{4.7}$$

In addition, one acquires by (2.20)

$$\mathrm{d}t = t_X \mathrm{d}X + t_Y \mathrm{d}Y = \frac{\alpha ph}{c_2 - c_1}\mathrm{d}X + \frac{\alpha qg}{c_2 - c_1}\mathrm{d}Y = \frac{\alpha(ph + qg)}{c_2 - c_1}\mathrm{d}X, \quad \text{on } x = a, b,$$

from which we achieve

$$\begin{aligned}
&\int_0^\infty 2\phi(\beta u_t-\gamma^2 u_x)\Big|_{x=a}^{x=b}\,\mathrm{d}t\\
=&\int_{\widehat{X}}^\infty 2\phi\Big(\beta(u_XX_t+u_YY_t)-\gamma^2(u_XX_x+u_YY_x)\Big)\frac{\alpha(ph+qg)}{c_2-c_1}(X,X-\widetilde{X})\,\mathrm{d}X\\
&-\int_0^\infty 2\phi\Big(\beta(u_XX_t+u_YY_t)-\gamma^2(u_XX_x+u_YY_x)\Big)\frac{\alpha(ph+qg)}{c_2-c_1}(X,X)\,\mathrm{d}X\\
=&\int_{\widehat{X}}^\infty \frac{2\phi\alpha(ph+qg)}{(c_2-c_1)^2gh}\Big(\frac{\beta}{\alpha}(c_2hm-c_1g\ell)-\gamma^2(g\ell-hm)\Big)(X,X-\widetilde{X})\,\mathrm{d}X\\
&-\int_0^\infty \frac{2\phi\alpha(ph+qg)}{(c_2-c_1)^2gh}\Big(\frac{\beta}{\alpha}(c_2hm-c_1g\ell)-\gamma^2(g\ell-hm)\Big)(X,X)\,\mathrm{d}X. \qquad (4.8)
\end{aligned}$$

Note that

$$\frac{\beta}{\alpha}c_2+\gamma^2=\frac{c_2(c_2-c_1)}{2},\quad \frac{\beta}{\alpha}c_1+\gamma^2=\frac{-c_1(c_2-c_1)}{2}.$$

Inserting the above into (4.8) and using the boundary conditions $c_2ph+c_1qg=0$ on $L_{a,b}$ concludes

$$\begin{aligned}
&\int_0^\infty 2\phi(\beta u_t-\gamma^2 u_x)\Big|_{x=a}^{x=b}\,\mathrm{d}t\\
=&\int_{\widehat{X}}^\infty \frac{\phi\alpha(ph+qg)}{(c_2-c_1)gh}(c_2hm+c_1g\ell)(X,X-\widetilde{X})\,\mathrm{d}X\\
&-\int_0^\infty \frac{\phi\alpha(ph+qg)}{(c_2-c_1)gh}(c_2hm+c_1g\ell)(X,X)\,\mathrm{d}X\\
=&\int_{\widehat{X}}^\infty \alpha\phi(qm-p\ell)(X,X-\widetilde{X})\,\mathrm{d}X-\int_0^\infty \alpha\phi(qm-p\ell)(X,X)\,\mathrm{d}X. \qquad (4.9)
\end{aligned}$$

Summing up (4.4)-(4.7) and (4.9) completes the proof of (1.8).

Finally, we can verify that $t\mapsto u(t,\cdot)$ is Lipschitz continuous in the L^2 distance and is continuously differentiable as a map with values in L^θ, for all $1\le\theta<2$. Moreover, for any fixed time t, we define

$$\mu_t([a,b])=\mu_t^-([a,b])+\mu_t^+([a,b]),$$

where

$$\mu_t^-([a,b])=\int_{\Gamma_t}\frac{-c_1(1-h)p}{2(c_2-c_1)}\,\mathrm{d}X,\quad \mu_t^+([a,b])=-\int_{\Gamma_t}\frac{c_2(1-g)q}{2(c_2-c_1)}\,\mathrm{d}Y.$$

It suggests by (4.2) that $\mu_t([a,b])=\mathscr{E}_0$. One can show that, for each t, the absolutely continuous part of μ_t has density $\frac{1}{2}(\alpha^2(x,u)u_t^2+\gamma^2(x,u)u_x^2)$ with respect to the Lebesgue measure, while for almost every t, the singular part of μ_t is concentrated on the set where $\partial_u\lambda_-=0$ or $\partial_u\lambda_+=0$. Thus the total energy characterized by the measure μ is conserved in time. This energy may concentrate on a set of times of zero measure or at points where $\partial_u\lambda_-$ or $\partial_u\lambda_+$ vanishes, which means that the set $\{t;\ \mathscr{E}(t)<\mathscr{E}_0\}$ has measure zero provided $\partial_u\lambda_\pm\neq 0$. The proofs of the above properties of the solution are identical to that in Sections 6 and 7 in Hu [19], so we omit them here.

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