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A FRACTIONAL-ORDER DYNAMICAL APPROACH TO VECTOR EQUILIBRIUM PROBLEMS WITH PARTIAL ORDER INDUCED BY A POLYHEDRAL CONE

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Abstract. In this paper, we propose a specific dynamical model for solving a class of vector equilibrium problems with partial order induced by a polyhedral cone which is generated by some matrix. Unlike the traditional dynamical models, it particularly possesses the feature of fractional-order system. The so-called Mittag-Leffler stability of the dynamical system is studied, which verifies the convergence to the solution of the corresponding vector equilibrium problems. This result is established by applying the techniques involving Caputo fractional derivatives, Lipschitz-type continuity, and strong pseudo-monotonicity assumptions with partial ordering based on a polyhedral cone. Numerical implementations are demonstrated to illustrate the proposed approach. In addition, a real-world application to the general framework of vector network equilibrium models based on polyhedral cone ordering is presented.

Keywords. Fractional-order dynamical system; Mittag-Leffler stability; Polyhedral cone; Vector equilibrium problem.

1. Introduction

The traditional scalar equilibrium problem is in form of

$$\mathbf{EP}(C,h): \text{find } z^* \in C \text{ such that } h(z^*,u) \geq 0, \text{ for all } u \in C,$$

where C is a given set and $h\colon C \times C \to \mathbb{R}$ is a bifunction satisfying $h(z,z) = 0$ for all $z \in C$. The form of inequality in $\mathbf{EP}(C,h)$ was first used by Nikaido and Isoda [25] for a class of non-cooperative convex games. Problem $\mathbf{EP}(C,h)$ is also known under the term "equilibrium problem" in the papers of Blum and Oettli [3], and Muu and Oettli [24]. Equilibrium problems are generalizations of important problems, such as variational inequalities, Nash equilibrium problems, complementarity problems, and so on. Numerous remarkable applications of equilibrium problems were studied in mechanics, network analysis, economics, transportation, and operations research. For details, we refer the readers to [2, 16, 20] and the references therein.

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In recent years, the scalar equilibrium problems were extended to vector equilibrium problems, whose cost function is a vector-valued function based on partial order induced by various kinds of cones ,such as nonnegative orthant, lexicographic cone and p-order cone ($p \geq 2$); see, e.g., [16,18,19,34]. In particular, Gutiérrez et al. [9,10] considered a class of multiobjective optimization problems with partial ordering by a polyhedral cone generated by some matrix. They claimed that their obtained results based on this conic ordering are attractive from a computational point of view. Using the partial order introduced in [10], Hai et al. [12] further explored a vector equilibrium problem with a polyhedral cone ordering and employed variants of the Ekeland variational principle to study approximate proper solutions for this class of problems. Very recently, Hung et al. [14] investigated the vector equilibrium problem with polyhedral cone ordering and established error bounds under assumptions involving polyhedral cone ordering. They also proposed an interesting application to a model of traffic network equilibrium problems of the vector type in light of polyhedral cone ordering, which is a general framework of network equilibrium models in many references; see, e.g., [5,36,37] and the references therein. In addition, Tam [33] built up the Holder continuity of solutions for vector network equilibrium model with a polyhedral cone ordering studied in [14] under parametric perturbations. This article is devoted to finding new and novel approaches to solve the vector equilibrium problems with polyhedral cone ordering and their applications to traffic network equilibrium models.

On the other hand, there exist a lot of results of dynamical systems by using ordinary differential equations for solving variational inequalities, variational inclusions, and fixed point problems; see [4, 6, 11, 13, 31, 32]. The key character of dynamical systems is that their trajectories can reach a stable state, and then the equilibrium point of the dynamical system is the solution of corresponding problems. Very recently, Vuong and Strodiot [35] introduced and investigated a dynamical system for solving problem $\mathbf{EP}(C,h)$ under the strong pseudo-monotonicity and Lipschitz-type continuity of cost function h. Ju et al. [15] developed a finite-time converging proximal dynamical model to deal with problem $\mathbf{EP}(C,h)$ under some mild conditions.

To the contrast, a fractional-order system is a dynamical system that is modeled by a fractional differential equation containing derivatives of non-integer order. In particular, fractional differential equations based on fractional-order calculus [26] were explored to various application fields in electronics, automatic control, and some interdisciplinary sciences [17, 23, 38, 39, 41]. This specific fractional-order system is also adopted for optimization problems. For example, using the fractional-order calculus, Liang et al. [22] studied a class of dynamical systems by employing a fractional differential equations based algorithm for solving convex optimizations. Moreover, Liang et al. [22] claimed that the convergence speed of the algorithm by using fractional-order derivatives (in the sense of Mittag–Leffler stabilization) is faster than those with the integer orders for some optimization problems.

Inspired by the above-given equilibrium problems and dynamical systems methods, we decide to revisit the vector equilibrium problem based on polyhedral cone ordering, considered in [14] and explore the method of fractional-order dynamical systems in [22] for solving such problem. Nonetheless, the approach and techniques applied to the vector equilibrium problem with polyhedral cone ordering are new. We highlight the main contributions of this paper as below.

(1) This is the first article considering a dynamical system involving fractional-order derivative operators for solving the vector equilibrium problem with polyhedral ordering cone.
(2) The second novelty is that we provide the stability of the dynamical system in the Mittag-Leffler sense by employing some computational technologies involving Caputo fractional-order derivatives, strong pseudo-monotonicity, strong monotonicity, and Lipschitz-type continuity assumptions with respect to partial order constructed by a polyhedral cone.
(3) A real-world problem, the traffic network equilibrium problems of the vector type based on partial order given by a polyhedral cone, is demonstrated to verify the theoretical results.

The structure of this paper is as follows. In Section 2, we recall some basic definitions, concepts and properties on convex analysis, fractional-order derivative operators and revisit a framework of the vector equilibrium problem with polyhedral ordering cones. Our main results in this paper are presented in Section 3 which include imposing the hypotheses on the data, establishing the fractional-order dynamical model for solving the vector equilibrium problem with polyhedral ordering cone and proposing the Mittag-Leffler stability of the dynamical system. Finally, in Section 4, we provide an application to a vector network equilibrium problem to illustrate our theoretical results considered in Section 3.

2. Preliminaries

In this section, we review some notations and concepts for subsequent needs. Throughout the paper, let $\mathbb{R}^l$ be the l-dimensional Euclidean space and

$$\mathbb{R}^l_+ = \{(z_1,\dots,z_l) \in \mathbb{R}^l : z_i \geq 0, \forall i = 1,\dots,l\}.$$

For any two vectors $z = (z_1,\dots,z_l)^\top$ and $u = (u_1,\dots,u_l)^\top$ in $\mathbb{R}^l$, we define the relationships between tese two vectors:

(i) $z \leq u$ if and only if $z_i \leq u_i$ for all $i \in \{1,\dots,l\}$;
(ii) $z < u$ if and only if $z_i < u_i$ for all $i \in \{1,\dots,l\}$.

A nonempty set $\mathscr{P} \subset \mathbb{R}^l$ is a cone if $\lambda z \in \mathscr{P}$ for all $z \in \mathscr{P}$ and $\lambda \geq 0$. A cone $\mathscr{P}$ is said to be pointed if $\mathscr{P} \cap -\mathscr{P} = \{\mathbf{0}\}$, where $\mathbf{0} = (0,\dots,0)^\top \in \mathbb{R}^l$. A set $\mathscr{P} \subset \mathbb{R}^m$ is a polyhedral cone if $\mathscr{P}$ has a representation of the form $\mathscr{P} = \{z \in \mathbb{R}^m : \langle a_i, z\rangle \geq 0, \forall i = 1,\dots,l\}$ for some positive integer l and some $a_i \in \mathbb{R}^m$, $i = 1,\dots,l$; see [28].

Definition 2.1. Let $A \in \mathbb{R}^{l\times m}$. Then, the set

$$\mathscr{P}_A = \{z \in \mathbb{R}^m : Az \geq \mathbf{0}\}, \tag{2.1}$$

is called a cone generated by A.

It is known that $\mathscr{P}_A$ is polyhedral, so it is also convex and closed; see [7]. Given a matrix $A \in \mathbb{R}^{l\times m}$, the mapping defined by the matrix A (also denoted by A), i.e., $A : \mathbb{R}^m \to \mathbb{R}^l$ with $z \mapsto Az$ (or $A(z)$), is a bounded linear mapping.

Proposition 2.1. [29, Proposition 4 and Proposition 5] *Let $A \in \mathbb{R}^{l\times m}$, where $l \geq m$. Then the following assertions hold.*

(i) *The cone $\mathscr{P}_A$ defined by (2.1) is pointed if and only if* $\operatorname{rank}(A) = m$.

(ii) *If the matrix A has no zero rows, then* $\text{int}(\mathscr{P}_A) = \{z \in \mathbb{R}^m : Az > \mathbf{0}\}$.

Lemma 2.1. [40, Lemma 1] *Let $A \in \mathbb{R}^{l\times m}$. If $\mathscr{P}_A = \{\mathbf{0}\}$, then* $\text{rank}(A) = m$ *and $l > m$.*

Proposition 2.2. [30, Proposition 4.1] *Let A be a mapping defined by a matrix $A \in \mathbb{R}^{l\times m}$. Assume that the set $\{z \in \mathbb{R}^m : Az \geq \mathbf{0}\}$ is a pointed cone, or, equivalently, that* $\text{rank}(A) = m$ *and $l \geq m$. Then, the following statements are satisfied.*

(i) *The mapping A is injective.*
(ii) *The image of the set $\{z \in \mathbb{R}^m : Az \geq \mathbf{0}\}$ under the mapping A is a convex cone included in $\mathbb{R}^l_+$.*
(iii) *If $l = m$, then the image of the space $\mathbb{R}^m$ under the mapping A is $\mathbb{R}^l$ and the image of the cone $\{z \in \mathbb{R}^m : Az \geq \mathbf{0}\}$ is $\mathbb{R}^l_+$.*
(iv) *If $l > m$, then the image of the space $\mathbb{R}^m$ under the mapping A is a proper subset of $\mathbb{R}^l$ and the image of the cone $\{z \in \mathbb{R}^m : Az \geq \mathbf{0}\}$ is a proper subset of $\mathbb{R}^l_+$.*

In the following, let C be a nonempty subset of an Euclidean space, $A = (a_{ij})$ be a real matrix with l rows and m columns where l,m are positive integers such that $l \geq m$ and $\text{rank}(A) = m$, and $\mathscr{P}_A$ be a polyhedral cone generated by A with $\text{int}(\mathscr{P}_A) \neq \emptyset$. To proceed, we recall the notion of $\mathscr{P}_A$-convexity of a vector-valued function.

Definition 2.2. [14] For each $k \in \{1,\dots,m\}$, let $\psi_k : C \to \mathbb{R}$ be a function. A function $\psi := (\psi_1,\dots,\psi_m)$ defined by $\psi(x) = (\psi_1(x),\dots,\psi_m(x))$ is said to be $\mathscr{P}_A$-convex if, for all $z,u \in C$ and $\lambda \in [0,1]$, there holds $\lambda\psi(z) + (1-\lambda)\psi(u) - \psi(\lambda z + (1-\lambda)u) \in \mathscr{P}_A$.

If $a_{ik} \geq 0$ and ψ_k is a convex function for all $i \in \{1,\dots,l\}$ and $k \in \{1,\dots,m\}$, then ψ is $\mathscr{P}_A$-convex. However, the reverse implication does not hold; see [14, Remark 3.1(ii)]. Besides, the concept of strong convexity and some properties of a minimization problem are needed for subsequent analysis.

Definition 2.3. (see [2]) A function $\phi : C \to \mathbb{R}$ is said to be strongly convex with modulus τ if there exists a constant $\tau > 0$ such that for all $z,u \in C$ and for each $\lambda \in [0,1]$, we have

$$\phi(\lambda z + (1-\lambda)u) \leq \lambda\phi(z) + (1-\lambda)\phi(u) - \lambda(1-\lambda)\frac{\tau}{2}\|z-u\|^2.$$

If $\tau = 0$ in the above inequality, we obtain the usual definition of convexity of the function ϕ.

Lemma 2.2. [2, Theorem 2.1.9] *Let C be a nonempty closed and convex subset. If $\phi : C \to \mathbb{R}$ is strongly convex with modulus τ, then the minimization problem* $\min\{\phi(z) : z \in C\}$ *has a unique solution z^* and any $z \in C$ satisfies $\phi(z) \geq \phi(z^*) + \frac{\tau}{2}\|z-z^*\|^2$.*

Next, we recall a class of vector equilibrium problems with partial order induced by a polyhedral cone which is generated by some matrix A. It is indeed studied by Hung et al. [14] and in form of $\mathbf{VEP}(C,H,\mathscr{P}_A)$: Find $z^* \in C$ such that

$$H(z^*,u) \notin -\text{int}(\mathscr{P}_A), \quad \forall u \in C, \tag{2.2}$$

where $H : C \times C \to \mathbb{R}^m$ is a vector-valued function such that H is continuous and $H(z,z) = 0$ for all $z \in C$. For convenience, we denote the solution set of problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$ by $\text{Sol}(C,H,\mathscr{P}_A)$. Since the existence of vector equilibrium problems has been well investigated in previous literature, we always assume that $\text{Sol}(C,H,\mathscr{P}_A) \neq \emptyset$.

From the computational point of view, applying Proposition 2.1, Proposition 2.2, and the linearity of mapping A, Hung et al. [14] already showed that problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$ can be converted to a usual vector equilibrium problem. In other words, problem (2.2) is equivalent to finding $z^* \in C$ such that

$$(A \circ H)(z^*,u) \notin -\mathrm{int}(\mathbb{R}^l_+), \quad \forall u \in C, \tag{2.3}$$

where the function H is given by $H(z,u) = (H_1(z,u),\ldots,H_m(z,u))^\top \in \mathbb{R}^m$ for all $z,u \in C$, and the composition of the mapping $A \circ H$ is defined by

$$A \circ H = ((A \circ H)_1,\ldots,(A \circ H)_l)^\top \text{ and } (A \circ H)(z,u) = A[H(z,u)],\ \forall u,z \in C.$$

We point out that problem (2.3) is a kind of equilibrium problems of the weak vector type in finite-dimensional spaces depending on the data of matrix A and component functions of H. This makes an interesting aspect from the computational point of view (see Example 3.1 in the next section). In particular, by the form of problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$ in (2.3), if $m = l = 1, A \equiv 1$, and $H_1 = h$, then problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$ reduces to scalar equilibrium problem $\mathbf{EP}(C,h)$ mentioned in the first part of the Introduction.

To end this section, we recall some definitions and properties of fractional-order differential operators which are used in the sequel. Grunwald–Letnikov derivative, Riemann–Liouville derivative, and Caputo derivative are three well-known definitions of differential operators using fractional-order calculus [17,26]. Unlike the other two derivatives, the Caputo fractional-order derivative was commonly used in physical applications because of its advantage that the Caputo derivative of a constant is equal to zero and it satisfies linear relationship and takes the same form as for classical differential equations in the initial conditions (see Remark 2.1). Therefore, in this paper, we always adopt the fractional derivative in the Caputo sense for fractional-order dynamic systems.

Definition 2.4. [17,26] The Riemann-Liouville fractional integral of order $q > 0$ for a suitable function z is defined by

$$\mathbf{I}^q_{t_0} z(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\zeta)^{q-1} z(\zeta)\, d\zeta, \qquad \text{for a.e. } t > t_0,$$

where t_0 is the initial time and $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(q) = \int_0^{+\infty} t^{q-1} e^{-t}\, dt.$$

It is well known that $\Gamma(1) = 1$ and $\Gamma(q+1) = q\Gamma(q)$ for all $q > 0$.

Definition 2.5. [17,26] For a suitable function z given on the interval $(t_0,+\infty)$, the Caputo fractional-order derivative of z of order $q > 0$ is defined by

$${}^{\mathbb{C}}_{t_0}\mathbf{D}^q_t z(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{z^{(n)}(\zeta)}{(t-\zeta)^{q+1-n}} d\zeta,$$

where t_0 is the initial time, $n = \min\{k \in \mathbb{N} : k > q\}$, and $z^{(n)}(t)$ is the nth-order derivative of $z(t)$. More specifically, it is noticed that

(i) when q is equal to a positive integer n, we define ${}^{\mathbb{C}}_{t_0}\mathbf{D}^q_t z(t) = z^{(n)}(t)$;

(ii) when $0 < q < 1$, we have

$$ {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{z'(\zeta)}{(t-\zeta)^q} d\zeta. $$

Remark 2.1. There are a few facts worthy of being mentioned.

(a): For any constant S, it is clear that ${}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q S = 0$.

(b): The linearity of Caputo fractional-order derivative holds, that is,

$$ {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q(\alpha z(t) + \beta u(t)) = \alpha {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t) + \beta {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q u(t), $$

where α, β are constants.

(c): Note that ${}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t) = \mathbf{I}_{t_0}^{n-q} z^{(n)}(t)$ and the Laplace transform of the Caputo fractional-order derivative is given by

$$ \mathbf{L}\left\{ {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t) \right\} = s^q Z(s) - \sum_{k=0}^{n-1} s^{q-k-1} z^{(k)}(t_0), \quad n-1 < q \le n, $$

where s is the variable in Laplace domain and $Z(s)$ denotes the Laplace transform of $z(t)$.

Lemma 2.3. [1] *Let $z(t) \in \mathbb{R}^n$ be a differentiable vector-valued function. Then, for any time instant $t > t_0$, there holds*

$$ {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q \left[\frac{1}{2} \|z(t)\|^2 \right] \le \left\langle z(t), {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t) \right\rangle. \tag{2.4} $$

Lemma 2.4. [21] *Let $\mathscr{L}$ be a continuous function on $[t_0, +\infty)$ which satisfies ${}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q \mathscr{L}(t) \le \eta \mathscr{L}(t)$, where $0 < q < 1$, $\eta \in \mathbb{R}$ and t_0 is the initial time. Then, $\mathscr{L}(t) \le \mathscr{L}(t_0) E_q(\eta (t-t_0)^q)$, where $E_q(\cdot)$ is a Mittag-Leffler function, given by*

$$ E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}. \tag{2.5} $$

Note that from Lemma 2.4 $E_q(0) = 1$ and $E_1(z) = e^z$.

3. Fractional-order Dynamical Model and Stability Analysis

In this section, we propose a novel fractional-order dynamical model, which is capable of solving problem $\mathbf{VEP}(C, H, \mathscr{P}_A)$. Then, the Mittag-Leffler stability of the system is established under suitable assumptions. This result derives the convergence of the trajectory of fractional-order dynamical system to the solution of problem $\mathbf{VEP}(C, H, \mathscr{P}_A)$.

Before proceeding, we impose the following hypotheses on the data of $\mathbf{VEP}(C, H, \mathscr{P}_A)$.

$(\mathbf{I}_A)$: The matrix $A \in \mathbb{R}^{l \times m}$ with $A = (a_{ij})$ and $l \ge m$ satisfies that $\operatorname{rank}(A) = m$ and

$$ \min_{1 \le i \le l} \left\{ \sum_{k=1}^{m} a_{ik} \right\} > 0. $$

$(\mathbf{I}_C)$: C is a nonempty closed and convex subset.

$(\mathbf{I}_H)_0$: H is $\mathscr{P}_A$-convex in the second component;

$(\mathbf{I}_H)_1$: H is $(\mathscr{P}_A,\mu)$-strongly pseudo-monotone, i.e., there exists $\mu>0$ such that

$$H(z,u)\notin -\text{int}\mathscr{P}_A \Longrightarrow H(u,z)+\mu\|z-u\|^2\mathbf{e}\in -\mathscr{P}_A, \tag{3.1}$$

for all $z,u\in C$ and $\mathbf{e}=(1,1,\ldots,1)^\top\in\mathbb{R}^m$;

$(\mathbf{I}_H)_2$: H is $(\mathscr{P}_A,L)$-Lipschitz-type continuous in the sense that there exists $L>0$ such that

$$H(z,u)+H(u,v)-H(z,v)+L\|z-u\|\|u-v\|\mathbf{e}\in\mathscr{P}_A, \tag{3.2}$$

for all $z,u,v\in C$.

Remark 3.1. Here are some remarks regarding the hypotheses above.

(a): The $(\mathscr{P}_A,\mu)$-strong pseudo-monotonicity in $(\mathbf{I}_H)_1$ and the $(\mathscr{P}_A,L)$-Lipschitz-type continuity in $(\mathbf{I}_H)_2$ are defined on partial order provided by a polyhedral cone which depends on the data of matrix A.

(b): In special cases of $m=l=1$, $A\equiv 1$ and $H_1=h$, conditions (3.1) and (3.2) are equivalent to

$$h(z,u)\geq 0\Longrightarrow h(u,z)\leq -\mu\|z-u\|^2,$$

and

$$h(z,u)+h(u,v)\geq h(z,v)-L\|z-u\|\|u-v\|, \tag{3.3}$$

respectively for all $z,u,v\in C$. Thus, $(\mathbf{I}_H)_1$ reduces to the usual definition of strongly pseudo-monotone bifunctions and $(\mathbf{I}_H)_2$ reduces to the Lipschitz-type continuity studied by Quoc and Muu [27] for scalar equilibrium problem $\mathbf{EP}(C,h)$.

(c): The condition (3.2) is indeed equivalent to

$$A\left[H(z,u)+H(u,v)-H(z,v)+L\|z-u\|\|u-v\|\mathbf{e}\right]\in\mathbb{R}^l_+$$

that is, for any $i\in\{1,\ldots,l\}$,

$$\sum_{k=1}^m a_{ik}\left[H_k(z,u)+H_k(u,v)-H_k(z,v)+L\|z-u\|\|u-v\|\right]\geq 0. \tag{3.4}$$

In fact, we observe that, if $a_{ik}\geq 0$ and H_k is a Lipschitz-type continuous function with constant $L_k>0$ in the form (3.3) for all $i\in\{1,\ldots,l\}$ and $k\in\{1,\ldots,m\}$, then H is $(\mathscr{P}_A,L)$-Lipschitz-type continuous with $L>0$. Indeed, since H_k is Lipschitz-type continuous with constant L_k for all $k\in\{1,\ldots,m\}$, it follows from (3.3) that

$$H_k(z,u)+H_k(u,v)-H_k(z,v)\geq -L_k\|z-u\|\|u-v\|$$

for all $z,u,v\in C$. Then for each $i\in\{1,\ldots,l\}$, we have

$$\sum_{k=1}^m a_{ik}\left[H_k(z,u)+H_k(u,v)-H_k(z,v)+\max_{1\leq k\leq m}\{L_k\}\|z-u\|\|u-v\|\right]\geq 0,$$

for all $z,u,v\in C$. Thus, the inequality (3.4) holds with $L=\max_{1\leq k\leq m}\{L_k\}>0$.

Remark 3.2. If conditions $(\mathbf{I}_A)$ and $(\mathbf{I}_H)_1$ hold, then $\mathbf{VEP}(C,H,\mathscr{P}_A)$ has a unique solution. To see this, let $z_1,z_2\in\text{Sol}(C,H,\mathscr{P}_A)$, namely, $z_1,z_2\in C$ such that

$$H(z_j,u)\notin -\text{int}(\mathscr{P}_A) \tag{3.5}$$

for all $u \in C$ and for $j = 1,2$. Since H is $(\mathscr{P}_A, \mu)$-strong pseudo-monotone, there exists $\mu > 0$ such that

$$H(u,z_j) + \mu \|z_j - u\|^2 \mathbf{e} \in -\mathscr{P}_A \tag{3.6}$$

for all $u \in C$ and for $j = 1,2$. Taking $u = z_2$ if $j = 1$ in (3.5) and $u = z_1$ if $j = 2$ in (3.6), we have

$$\begin{cases} A\,[H(z_1,z_2)] \notin -\mathrm{int}(\mathbb{R}^l_+), \\ A\left[H(z_1,z_2) + \mu\|z_1 - z_2\|^2 \mathbf{e}\right] \in -\mathbb{R}^l_+. \end{cases}$$

Then, there exists $i_0 \in \{1,\dots,l\}$ such that

$$\begin{cases} \sum_{k=1}^m a_{i_0k} H_k(z_1,z_2) \geq 0, \\ \sum_{k=1}^m a_{i_0k} H_k(z_1,z_2) + \sum_{k=1}^m a_{i_0k}\mu\|z_1 - z_2\|^2 \leq 0. \end{cases}$$

Hence, there holds

$$\sum_{k=1}^m a_{i_0k}\mu\|z_1 - z_2\|^2 \leq 0. \tag{3.7}$$

In addition, due to

$$\sum_{k=1}^m a_{i_0k}\mu \geq \min_{1\leq i\leq l} \left\{ \sum_{k=1}^m a_{ik} \right\} \mu > 0,$$

inequality (3.7) implies that $z_1 = z_2$. Thus, $\mathbf{VEP}(C,H,\mathscr{P}_A)$ has a unique solution.

3.1. **Fractional-order Dynamical Model.** To construct the fractional-order dynamical system for solving problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$, we need the following technical results which connects some properties of the solutions of problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$. First, given $\rho > 0$ we consider the function $\varphi_\rho : C \times C \to \mathbb{R}$ defined by

$$\varphi_\rho(z,u) = \rho \max_{1\leq i\leq l} \left\{ \sum_{k=1}^m a_{ik} H_k(z,u) \right\} + \frac{1}{2}\|u - z\|^2 \quad \forall z,u \in C. \tag{3.8}$$

Lemma 3.1. *Suppose that assumptions* $(\mathbf{I}_A)$, $(\mathbf{I}_C)$, *and* $(\mathbf{I}_H)_0$ *hold. Then, for any* $\rho > 0$ *and* $z \in C$, *the function* $u \mapsto \varphi_\rho(z,u)$ *has a unique minimizer over* C.

Proof. Applying the $\mathscr{P}_A$-convexity in the second component of H, for all $i \in \{1,\dots,l\}$, $u_1, u_2 \in C$ and $\lambda \in [0,1]$, thanks to [14, Remark 3.1(i)], we obtain

$$\sum_{k=1}^m a_{ik} H_k(z, \lambda u_1 + (1-\lambda)u_2) \leq \lambda \sum_{k=1}^m a_{ik} H_k(z,u_1) + (1-\lambda) \sum_{k=1}^m a_{ik} H_k(z,u_2)$$

for all $z \in C$. This implies that, for all $i \in \{1,\dots,l\}$, the function

$$u \mapsto \sum_{k=1}^m a_{ik} H_k(z,u)$$

is convex. Hence

$$u \mapsto \max_{1\leq i\leq l} \left\{ \sum_{k=1}^m a_{ik} H_k(z,u) \right\}$$

is also a convex function. Then, for any $\rho > 0$ and $z \in C$, the function

$$u \mapsto \varphi_\rho(z,u) = \rho \max_{1\le i\le l} \left\{ \sum_{k=1}^{m} a_{ik} H_k(z,u) \right\} + \frac{1}{2}\|u-z\|^2$$

is strongly convex with modulus 1 on C. Since C is a nonempty, closed, and convex set, it follows from Lemma 2.2 that the function $u \mapsto \varphi_\rho(z,u)$ has a unique minimizer over C. □

For any $\rho > 0$, we now consider the function Δ_ρ given by

$$\Delta_\rho(z) = -\frac{1}{\rho} \inf_{u\in C} \varphi_\rho(z,u), \quad \forall z \in C. \tag{3.9}$$

The following result provides some properties of Δ_ρ.

Lemma 3.2. *Suppose that assumptions* $(\mathbf{I}_A)$, $(\mathbf{I}_C)$, *and* $(\mathbf{I}_H)_0$ *hold. Then, for any* $\rho > 0$, *the function* Δ_ρ *defined by* (3.9) *satisfies the following conditions:*

(i) $\Delta_\rho(z) \ge 0$ *for all* $z \in C$;
(ii) $\Delta_\rho(z) = 0$ *if and only if* $z \in \mathrm{Sol}(C,H,\mathscr{P}_A)$.

Proof. By the definition of function Δ_ρ in (3.9), we have

$$\Delta_\rho(z) = \sup_{u\in C} \left(-\max_{1\le i\le l} \left\{ \sum_{k=1}^{m} a_{ik} H_k(z,u) \right\} - \frac{1}{2\rho}\|u-z\|^2 \right), \quad \forall z \in C.$$

From [14, Theorem 3.1], we know that Δ_ρ is a gap function of $\mathbf{VEP}(C,H,\mathscr{P}_A)$, so it follows from [14, Definition 3.1] that $\Delta_\rho(z) \ge 0$ for all $z \in C$ and $\Delta_\rho(z) = 0$ if and only if $z \in \mathrm{Sol}(C,H,\mathscr{P}_A)$. □

Using Lemma 3.1 and Lemma 3.2, we establish a characteristic of the minimizer of φ_ρ and the solution set $\mathrm{Sol}(C,H,\mathscr{P}_A)$.

Proposition 3.1. *Suppose that assumptions* $(\mathbf{I}_A)$, $(\mathbf{I}_C)$, *and* $(\mathbf{I}_H)_0$ *hold. Then, for any* $\rho > 0$, $z^* \in \mathrm{Sol}(C,H,\mathscr{P}_A)$ *if and only if* $z^* = \mathbf{prox}_C^{\rho H}(z^*)$, *where*

$$\mathbf{prox}_C^{\rho H}(z) = \arg\min_{u\in C} \varphi_\rho(z,u), \quad \forall z \in C. \tag{3.10}$$

Proof. $(\Longrightarrow)$ Suppose that $z^* \in \mathrm{Sol}(C,H,\mathscr{P}_A)$. Without loss of generality, we may assume that there exists $\bar{i} \in \{1,\dots,l\}$ such that

$$\max_{1\le i\le l} \left\{ \sum_{k=1}^{m} a_{ik} H_k(z,u) \right\} = \sum_{k=1}^{m} a_{\bar{i}k} H_k(z,u)$$

for all $z,u \in C$, so (3.8) can be rewritten as

$$\varphi_\rho(z,u) = \rho \sum_{k=1}^{m} a_{\bar{i}k} H_k(z,u) + \frac{1}{2}\|u-z\|^2 \quad \forall z,u \in C. \tag{3.11}$$

It follows from the proof of Lemma 3.1 that φ_ρ is a strongly convex function with modulus 1 on C. Moreover, from (3.8), (3.10) and Lemma 3.1, we also obtain that, for each $z \in C$, $\mathbf{prox}_C^{\rho H}(z)$ is a unique solution to the minimization problem

$$\min\{\varphi_\rho(z,u) : u \in C\}.$$

Then, applying Lemma 2.2 yields

$$\frac{1}{2}\left\|u-\mathbf{prox}_C^{\rho H}(z)\right\|^2 \leq \varphi_\rho(z,u)-\varphi_\rho\left(z,\mathbf{prox}_C^{\rho H}(z)\right), \quad \forall z,u\in C. \tag{3.12}$$

Putting $u=z^*$ and $z=z^*$ into (3.12) leads to

$$\frac{1}{2}\left\|z^*-\mathbf{prox}_C^{\rho H}(z^*)\right\|^2 \leq \varphi_\rho(z^*,z^*)-\varphi_\rho\left(z^*,\mathbf{prox}_C^{\rho H}(z^*)\right). \tag{3.13}$$

Since $z^*\in \mathrm{Sol}(C,H,\mathscr{P}_A)$, it follows from Lemma 3.2 that $\Delta_\rho(z^*)$ =0. By (3.9), we have $\inf_{u\in C}\varphi_\rho(z^*,u)=0$. Hence, from Lemma 3.1 and (3.10), one has $\varphi_\rho\left(z^*,\mathbf{prox}_C^{\rho H}(z^*)\right)=0$. Furthermore, we note that $\varphi_\rho(z^*,z^*)=0$, so inequality (3.13) follows

$$\left\|z^*-\mathbf{prox}_C^{\rho H}(z^*)\right\|^2 \leq 0.$$

Thus, it verifies $z^*=\mathbf{prox}_C^{\rho H}(z^*)$.

($\Longleftarrow$) Let $z^*=\mathbf{prox}_C^{\rho H}(z^*)$. It follows from Lemma 3.1 and (3.10) that

$$\Delta_\rho(z^*)=-\frac{1}{\rho}\inf_{u\in C}\varphi_\rho(z^*,u)=-\frac{1}{\rho}\varphi_\rho\left(z^*,\mathbf{prox}_C^{\rho H}(z^*)\right).$$

Since $z^*=\mathbf{prox}_C^{\rho H}(z^*)$, $\varphi_\rho\left(z^*,\mathbf{prox}_C^{\rho H}(z^*)\right)=\varphi_\rho(z^*,z^*)=0$, then $\Delta_\rho(z^*)=0$. Applying Lemma 3.2 immediately gives $z^*\in\mathrm{Sol}(C,H,\mathscr{P}_A)$. □

In light of Proposition 3.1, for solving problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$, we propose the fractional-order dynamical model:

$$\begin{cases} {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t)=-\kappa\left[z(t)-\mathbf{prox}_C^{\rho H}(z(t))\right] & \forall t>t_0,\\ z(t_0)=z_0, \end{cases} \tag{3.14}$$

where $q\in(0,1]$, $\kappa>0$, and $z_0\in C$ is fixed.

For further understanding this model, some special cases of (3.14) are elaborated below.

(a): If $q=1$, then (3.14) reduces to the following dynamical system:

$$\begin{cases} \dot{z}(t)=-\kappa\left[z(t)-\mathbf{prox}_C^{\rho H}(z(t))\right] & \forall t>t_0,\\ z(t_0)=z_0\in C, \end{cases} \tag{3.15}$$

where $\kappa>0$.

(b): If $m=l=1$, $A\equiv 1$ and $H_1=h$, then (3.14) reduces to the following dynamical system:

$$\begin{cases} {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t)=-\kappa\left[z(t)-\widetilde{\mathbf{prox}}_C^{\rho h}(z(t))\right] & \forall t>t_0,\\ z(t_0)=z_0\in C, \end{cases} \tag{3.16}$$

where $q\in(0,1]$, $\kappa>0$ and

$$\widetilde{\mathbf{prox}}_C^{\rho h}(z)=\arg\min_{u\in C}\left\{\rho h(z,u)+\frac{1}{2}\|u-z\|^2\right\} \quad \forall z\in C.$$

(c): If $q = 1$, $m = l = 1$, $A \equiv 1$ and $H_1 = h$, then (3.14) reduces to the following dynamical system for solving problem $\mathbf{EP}(C,h)$, which was recently been studied by Vuong and Strodiot [35]:

$$\begin{cases} \dot{z}(t) = -\kappa \left[z(t) - \widetilde{\mathbf{prox}}_C^{\rho h}(z(t)) \right] & \forall t > t_0, \\ z(t_0) = z_0 \in C, \end{cases} \tag{3.17}$$

where $\kappa > 0$.

Remark 3.3. The dynamical system (3.15) in the case of (a) invoking to ordinary differential equations is used to solve the problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$. Besides, we obtain that the fractional-order dynamical system (3.16) in the case of (b) solves the solution of scalar equilibrium problem $\mathbf{EP}(C,h)$. From the case of (c), it is clear to see that (3.16) includes the dynamical system (3.17). Therefore, dynamical systems (3.15) and (3.16) are considered as useful tools for solving problems $\mathbf{VEP}(C,H,\mathscr{P}_A)$ and $\mathbf{EP}(C,h)$, respectively. To the best of our knowledge, they have not been explored.

3.2. **Mittag-Leffler stability.** Now, in order to establish the stability of the proposed fractional-order dynamical system, we describe the definitions of an equilibrium point and Mittag-Leffler stability of dynamical system (3.14).

Definition 3.1. A point $z^* \in C$ is said to be an equilibrium point of dynamical system (3.14) if z^* satisfies the following

$$\mathbf{0} = -\kappa \left[z^* - \mathbf{prox}_C^{\rho H}(z^*) \right]. \tag{3.18}$$

Since $\kappa > 0$, condition (3.18) is equivalent to $z^* = \mathbf{prox}_C^{\rho H}(z^*)$. Thus, from Remark 3.2 and Proposition 3.1, we can conclude that dynamical system (3.14) has a unique equilibrium point, which is the unique solution to $\mathbf{VEP}(C,H,\mathscr{P}_A)$.

Definition 3.2. The equilibrium point z^* of dynamical system (3.14) is said to be Mittag-Leffler stable if there exist some positive constants M, α, and ω such that, for any solution $z(t)$ of dynamical system (3.14) with any initial value $z(t_0) = z_0$,

$$\|z(t) - z^*\| \le \left\{ M\|z_0 - z^*\|^{\alpha} E_q\left(-\omega(t-t_0)^q\right) \right\}^{\frac{1}{\alpha}}. \tag{3.19}$$

Moreover, dynamical system (3.14) is said to be Mittag-Leffler stable, if its equilibrium point is Mittag-Leffler stable.

Obviously, Mittag–Leffler stability implies asymptotical stability, i.e., $\lim_{t\to+\infty} z(t) = z^*$. When $q = 1$, inequality (3.19) is equivalent to the form

$$\|z(t) - z^*\| \le M'\|z_0 - z^*\| e^{-\omega'(t-t_0)},$$

where M' and ω' are positive constants. This shows that the Mittag–Leffler stability of dynamical system (3.14) reduces to the exponential stability of dynamical system (3.15).

We now present the key result on establishing an estimation to analysis the Mittag-Leffler stability of dynamical system (3.14).

Proposition 3.2. *Assume that hypotheses* $(\mathbf{I}_A)$*,* $(\mathbf{I}_C)$*,* $(\mathbf{I}_H)_0$*–*$(\mathbf{I}_H)_2$ *hold. Let* z^* *be the unique solution to* $\mathbf{VEP}(C,H,\mathscr{P}_A)$ *and* $\rho > 0$*. Then, for any* $z \in C$*,*

$$\left[1+\rho\left(2\mathbf{a}_{\min}\mu-\rho\left(\mathbf{a}_{\max}L\right)^2\right)\right]\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2 \leq \|z-z^*\|^2, \tag{3.20}$$

where

$$\mathbf{a}_{\min} = \min_{1\leq i\leq l}\left\{\sum_{k=1}^m a_{ik}\right\} \quad \text{and} \quad \mathbf{a}_{\max} = \max_{1\leq i\leq l}\left\{\sum_{k=1}^m a_{ik}\right\}. \tag{3.21}$$

Proof. Let z^* be the unique solution to $\mathbf{VEP}(C,H,\mathscr{P}_A)$ and $\rho > 0$. We recall the inequality (3.12) in the proof of Proposition 3.1:

$$\frac{1}{2}\left\|u-\mathbf{prox}_C^{\rho H}(z)\right\|^2 \leq \varphi_\rho(z,u)-\varphi_\rho\left(z,\mathbf{prox}_C^{\rho H}(z)\right), \quad \forall z,u\in C.$$

Inserting $u = z^*$ into the above inequality and using the definition of φ_ρ in (3.11), we obtain

$$\begin{aligned}\left\|z^*-\mathbf{prox}_C^{\rho H}(z)\right\|^2 &\leq \|z-z^*\|^2-\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2\\ &\quad +2\rho\sum_{k=1}^m a_{\bar{i}k}\left[H_k(z,z^*)-H_k\left(z,\mathbf{prox}_C^{\rho H}(z)\right)\right],\end{aligned} \tag{3.22}$$

for some $\bar{i}\in\{1,\ldots,l\}$. Using the condition $(\mathbf{I}_H)_2$ with $u = \mathbf{prox}_C^{\rho H}(z)$ and $v = z^*$, it follows from (3.2) that

$$\begin{aligned}A\Big[H\left(z,\mathbf{prox}_C^{\rho H}(z)\right)+H\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)-H\left(z,z^*\right)\\ +L\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|\mathbf{e}\Big]\in\mathbb{R}^l_+.\end{aligned}$$

This further implies that

$$\begin{aligned}&\sum_{k=1}^m a_{\bar{i}k}\left[H_k(z,z^*)-H_k\left(z,\mathbf{prox}_C^{\rho H}(z)\right)\right]\\ &\leq \sum_{k=1}^m a_{\bar{i}k}H_k\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)+\sum_{k=1}^m a_{\bar{i}k}L\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|.\end{aligned} \tag{3.23}$$

For $\rho > 0$, using Young's inequality, we obtain

$$\begin{aligned}&\sum_{k=1}^m a_{\bar{i}k}L\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|\\ &\qquad\leq \frac{1}{2\rho}\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2+\frac{\rho\left(\sum_{k=1}^m a_{\bar{i}k}L\right)^2}{2}\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2.\end{aligned}$$

Hence, it follows from inequality (3.23) that

$$\begin{aligned}&\sum_{k=1}^{m} a_{\bar{i}k}\left[H_k(z,z^*)-H_k\left(z,\mathbf{prox}_C^{\rho H}(z)\right)\right]\\&\leq \sum_{k=1}^{m} a_{\bar{i}k}H_k\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)+\frac{1}{2\rho}\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2+\frac{\rho\left(\sum_{k=1}^{m} a_{\bar{i}k}L\right)^2}{2}\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2\\&\leq \sum_{k=1}^{m} a_{\bar{i}k}H_k\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)+\frac{1}{2\rho}\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2+\frac{\rho\left(\mathbf{a}_{\max}L\right)^2}{2}\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2,\end{aligned} \tag{3.24}$$

where $\mathbf{a}_{\max}=\max_{1\leq i\leq l}\left\{\sum_{k=1}^{m} a_{ik}\right\}$. Since z^* is a solution to $\mathbf{VEP}(C,H,\mathscr{P}_A)$, $H\left(z^*,\mathbf{prox}_C^{\rho H}(z)\right)\notin -\text{int}\mathscr{P}_A$. By the $(\mathscr{P}_A,\mu)$-strong pseudo-monotonicity of H, one has

$$H\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)+\mu\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2\mathbf{e}\in-\mathscr{P}_A,$$

that is,

$$A\left[H\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)+\mu\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2\mathbf{e}\right]\in-\mathbb{R}_+^l$$

so we obtain

$$\begin{aligned}\sum_{k=1}^{m} a_{\bar{i}k}H_k\left(\mathbf{prox}_C^{\rho H}(z),z^*\right)&\leq-\sum_{k=1}^{m} a_{\bar{i}k}\mu\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2\\&\leq-\mathbf{a}_{\min}\mu\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2,\end{aligned} \tag{3.25}$$

where $\mathbf{a}_{\min}=\min_{1\leq i\leq l}\left\{\sum_{k=1}^{m} a_{ik}\right\}$. Combining (3.24) and (3.25), we have

$$\begin{aligned}&\sum_{k=1}^{m} a_{\bar{i}k}\left[H_k(z,z^*)-H_k\left(z,\mathbf{prox}_C^{\rho H}(z)\right)\right]\\&\leq\frac{1}{2\rho}\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2+\left(\frac{\rho\left(\mathbf{a}_{\max}L\right)^2}{2}-\mathbf{a}_{\min}\mu\right)\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2.\end{aligned} \tag{3.26}$$

Putting (3.26) into (3.22), we obtain

$$\begin{aligned}\left\|z^*-\mathbf{prox}_C^{\rho H}(z)\right\|^2&\leq\|z-z^*\|^2-\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2\\&\quad+\left\|z-\mathbf{prox}_C^{\rho H}(z)\right\|^2+\rho\left(\rho\left(\mathbf{a}_{\max}L\right)^2-2\mathbf{a}_{\min}\mu\right)\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2\\&=\|z-z^*\|^2+\rho\left(\rho\left(\mathbf{a}_{\max}L\right)^2-2\mathbf{a}_{\min}\mu\right)\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2.\end{aligned}$$

This implies that

$$\left[1+\rho\left(2\mathbf{a}_{\min}\mu-\rho\left(\mathbf{a}_{\max}L\right)^2\right)\right]\left\|\mathbf{prox}_C^{\rho H}(z)-z^*\right\|^2\leq\|z-z^*\|^2.$$

Thus, inequality (3.20) holds. □

In light of Proposition 3.2, the stability of fractional-order dynamical system (3.14) is established as in the following theorem.

Theorem 3.1. *Suppose that hypotheses* $(\mathbf{I}_A)$, $(\mathbf{I}_C)$, $(\mathbf{I}_H)_0$–$(\mathbf{I}_H)_2$ *are satisfied, and let*

$$0 < \rho < \frac{2\mathbf{a}_{\min}\mu}{(\mathbf{a}_{\max}L)^2}, \tag{3.27}$$

where $\mathbf{a}_{\min}$ *and* $\mathbf{a}_{\max}$ *are given by* (3.21). *Then,* (3.14) *is Mittag-Leffler stable.*

Proof. Let z^* be the unique equilibrium point of system (3.14). We now consider the Lyapunov function given by

$$\mathscr{L}(t) := \frac{1}{2}\|z(t) - z^*\|^2, \quad \forall z(t) \in C.$$

From (3.14), using inequality (2.4) in calculating fractional derivative of order q leads to

$$\begin{aligned} {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q \mathscr{L}(t) &\le \left\langle z(t) - z^*, {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q z(t) \right\rangle \\ &= \kappa \left\langle z^* - z(t), z(t) - \mathbf{prox}_C^{\rho H}(z(t)) \right\rangle \\ &= \kappa \left\langle z^* - z(t), z^* - \mathbf{prox}_C^{\rho H}(z(t)) \right\rangle - \kappa \|z(t) - z^*\|^2. \end{aligned} \tag{3.28}$$

Applying Cauchy–Schwarz inequality, it follows from (3.20) and (3.27) that

$$\begin{aligned} \left\langle z^* - z(t), z^* - \mathbf{prox}_C^{\rho H}(z(t)) \right\rangle &\le \|z^* - z(t)\| \left\| z^* - \mathbf{prox}_C^{\rho H}(z(t)) \right\| \\ &\le \frac{1}{\sqrt{1 + \rho\left(2\mathbf{a}_{\min}\mu - \rho\left(\mathbf{a}_{\max}L\right)^2\right)}} \|z(t) - z^*\|^2. \end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29), we have

$${}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q \mathscr{L}(t) \le \kappa \left[\frac{1}{\sqrt{1 + \rho\left(2\mathbf{a}_{\min}\mu - \rho\left(\mathbf{a}_{\max}L\right)^2\right)}} - 1 \right] \|z(t) - z^*\|^2 = -\omega \mathscr{L}(t),$$

where

$$\omega = 2\kappa \left[1 - \frac{1}{\sqrt{1 + \rho\left(2\mathbf{a}_{\min}\mu - \rho\left(\mathbf{a}_{\max}L\right)^2\right)}} \right] > 0.$$

Thus, thanks to Lemma 2.4, we obtain $\mathscr{L}(t) \le \mathscr{L}(t_0) E_q\left(-\omega(t - t_0)^q\right)$, that is,

$$\|z(t) - x^*\|^2 \le \|z_0 - x^*\|^2 E_q\left(-\omega(t - t_0)^q\right).$$

Hence,

$$\|z(t) - z^*\| \le \left\{ M\|z_0 - z^*\|^\alpha E_q\left(-\omega(t - t_0)^q\right) \right\}^{\frac{1}{\alpha}}, \tag{3.30}$$

where $M = 1, \alpha = 2$. Thus (3.14) is Mittag-Leffler stable. $\square$

For $0 < q < 1$, the Mittag-Leffler function $E_q\left(-\omega(t - t_0)^q\right)$ has the asymptotic behavior as $\mathscr{O}(t^{-q})$ as $t \to +\infty$; see [26, page 34]. Then, it follows from (3.30) that $z(t) \to z^*$ as $t \to +\infty$ with at least the Mittag-Leffler convergence rate.

To illustrate Theorem 3.1, we present the below numerical example.

Example 3.1. For problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$, let $l=m=3$ and the matrix $A=(a_{ij})\in\mathbb{R}^{l\times m}$ be defined by

$$A=\begin{pmatrix} 1 & 0.5 & 0.4 \\ 0.2 & 1 & 0 \\ 1 & 0 & 0.5 \end{pmatrix},$$

the constraint set C be defined by

$$C=\left\{z\in\mathbb{R}^5 : z_1+z_2+z_3+z_4+z_5\geq 0, \quad -4.5\leq z_i\leq 4.5, \quad \forall i=1,2,3,4,5\right\}.$$

and $H=(H_1,H_2,H_3)^\top : C\times C\to\mathbb{R}^3$ be defined as follows:

$$\begin{aligned} H(z,u) &= (H_1(z,u),H_2(z,u),H_3(z,u))^\top, \\ H_1(z,u) &= \langle u-z, 0.3u+0.02z-0.44\mathbf{e}\rangle, \\ H_2(z,u) &= \langle u-z, 0.15u+0.03z-0.4\mathbf{e}\rangle, \\ H_3(z,u) &= \langle u-z, 0.2u+0.02z-0.9\mathbf{e}\rangle, \end{aligned}$$

where $z=(z_1,z_2,z_3,z_4,z_5)^\top$, $u=(u_1,u_2,u_3,u_4,u_5)^\top\in C$, and $\mathbf{e}=(1,1,1,1,1)^\top\in\mathbb{R}^5$. Then, $\operatorname{rank}(A)=3$ and we have

$$\begin{aligned} \mathscr{P}_A &= \{z=(z_1,z_2,z_3)^\top\in\mathbb{R}^3 : Az\geq\mathbf{0}\} \\ &= \{z=(z_1,z_2,z_3)^\top\in\mathbb{R}^3 : z_1+0.5z_2+0.4z_3\geq 0,\ 0.2z_1+z_2\geq 0,\ z_1+0.5z_3\geq 0\}, \end{aligned}$$

$$\mathbf{a}_{\min}=\min_{1\leq i\leq 3}\left\{\sum_{k=1}^{3}a_{ik}\right\}=1.2 \quad \text{and} \quad \mathbf{a}_{\max}=\max_{1\leq i\leq 3}\left\{\sum_{k=1}^{3}a_{ik}\right\}=1.9.$$

and

$$\begin{aligned} (A\circ H)(z,u) &= \left(\sum_{k=1}^{3}a_{1k}H_k(z,u), \sum_{k=1}^{3}a_{2k}H_k(z,u), \sum_{k=1}^{3}a_{3k}H_k(z,u)\right)^\top \\ &= (f_1(z,u),f_2(z,u),f_2(z,u))^\top, \end{aligned}$$

where

$$\begin{cases} f_1(z,u)=\langle u-z, 0.455u+0.045z-\mathbf{e}\rangle = 0.455\|z-u\|^2+0.5\langle u-z, z-2\mathbf{e}\rangle, \\ f_2(z,u)=\langle u-z, 0.21u+0.034z-0.488\mathbf{e}\rangle = 0.21\|z-u\|^2+0.244\langle u-z, z-2\mathbf{e}\rangle, \\ f_3(z,u)=\langle u-z, 0.4u+0.045z-0.89\mathbf{e}\rangle = 0.4\|z-u\|^2+0.445\langle u-z, z-2\mathbf{e}\rangle. \end{cases} \tag{3.31}$$

Problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$ is equivalent to finding $z\in C$ such that

$$(f_1(z,u),f_2(z,u),f_3(z,u))^\top\notin -\operatorname{int}(\mathbb{R}^3_+),\ \forall u\in C.$$

By the direct calculation, it follows that $z^*=(2,2,2,2,2)^\top$ is a unique solution to $\mathbf{VEP}(C,H,\mathscr{P}_A)$. It is easy to see that conditions $(\mathbf{I}_A)$ and $(\mathbf{I}_C)$ hold. By the convexity of $\mathbb{R}^5\ni u\mapsto H_k(z,u)$ for all $z\in\mathbb{R}^5$ and $k=1,2,3$, we can show that $\mathbb{R}^5\ni u\mapsto H(z,u)$ is $\mathscr{P}_A$-convex, so condition $(\mathbf{I}_H)_0$ is satisfied.

Next, we check that assumptions $(\mathbf{I}_H)_1$ and $(\mathbf{I}_H)_2$ hold.

$(\mathbf{I}_H)_1$: For each $(z,u) \in C \times C$, we have

$$H(z,u)+H(u,z) = \begin{pmatrix} H_1(z,u)+H_1(u,z) \\ H_2(z,u)+H_2(u,z) \\ H_3(z,u)+H_3(u,z) \end{pmatrix}$$
$$= \begin{pmatrix} \langle u-z, 0.3u+0.02z-0.44\mathbf{e}\rangle + \langle z-u, 0.3z+0.02u-0.44\mathbf{e}\rangle \\ \langle u-z, 0.15u+0.03z-0.4\mathbf{e}\rangle + \langle z-u, 0.15z+0.03u-0.4\mathbf{e}\rangle \\ \langle u-z, 0.2u+0.02z-0.9\mathbf{e}\rangle + \langle z-u, 0.2z+0.02u-0.9\mathbf{e}\rangle \end{pmatrix}$$
$$= \begin{pmatrix} -0.28\|z-u\|^2 \\ -0.12\|z-u\|^2 \\ -0.18\|z-u\|^2 \end{pmatrix}.$$

Then, for all $i \in \{1,2,3\}$, we obtain

$$\sum_{k=1}^{3} a_{ik}[H_k(z,u)+H_k(u,z)] = -(0.28a_{i1}+0.12a_{i2}+0.18a_{i3})\|z-u\|^2$$
$$\leq -\sum_{k=1}^{3} a_{ik}0.12\|z-u\|^2, \quad \forall z,u \in C. \tag{3.32}$$

For any $z,u \in C$, assume that $H(z,u) \notin -\mathrm{int}\mathscr{P}_A$, i.e., $A[H(z,u)] \notin -\mathrm{int}\mathbb{R}^3$. Then there exists $i^* \in \{1,2,3\}$ such that $f_{i^*}(z,u) = \sum_{k=1}^{3} a_{i^*k}[H_k(z,u)] \geq 0$ for all $z,u \in C$. It follows from (3.31) that for all $i \in \{1,2,3\}$, we have

$$f_i(z,u) = \sum_{k=1}^{3} a_{ik}[H_k(z,u)] \geq 0, \quad \forall z,u \in C. \tag{3.33}$$

Combining (3.32) and (3.33), we get that for all $i \in \{1,2,3\}$,

$$\sum_{k=1}^{3} a_{ik}\left[H_k(z,u)+0.12\|z-u\|^2\right] \leq 0, \quad \forall z,u \in C.$$

This implies $H(u,z)+0.12\|z-u\|^2\mathbf{e} \in -\mathscr{P}_A$ for all $z,u \in C$. Thus, H is $(\mathscr{P}_A,\mu)$-strongly pseudo-monotone with $\mu = 0.12$.

$(\mathbf{I}_H)_2$: For each $(z,u) \in C \times C$, we have

$$\begin{aligned} &H_1(z,u)+H_1(u,v)-H_1(z,v) \\ &= \langle u-z, 0.3u+0.02z-0.44\mathbf{e}\rangle + \langle v-u, 0.3v+0.02u-0.44\mathbf{e}\rangle \\ &\quad - \langle v-z, 0.3v+0.02z-0.44\mathbf{e}\rangle \\ &= 0.28\langle u-z, u-v\rangle \geq -0.28\|z-u\|\|u-v\|. \end{aligned}$$

Similarly, we also obtain

$$H_2(z,u)+H_2(u,v)-H_2(z,v) \geq -0.12\|z-u\|\|u-v\|$$

and

$$H_3(z,u)+H_3(u,v)-H_3(z,v) \geq -0.18\|z-u\|\|u-v\|.$$

Hence, for all $i \in \{1,2,3\}$,

$$\sum_{k=1}^{3} a_{ik}\left[H_k(z,u)+H_k(u,v)-H_k(z,v)\right] \geq -\sum_{k=1}^{3} a_{ik}\left[0.28\|z-u\|\|u-v\|\right],$$

that is,

$$\sum_{k=1}^{3} a_{ik}\left[H_k(z,u)+H_k(u,v)-H_k(z,v)+0.28\|z-u\|\|u-v\|\right] \geq 0.$$

This implies that

$$H(z,u)+H(u,v)-H(z,v)+0.28\|z-u\|\|u-v\|\mathbf{e} \in \mathscr{P}_A.$$

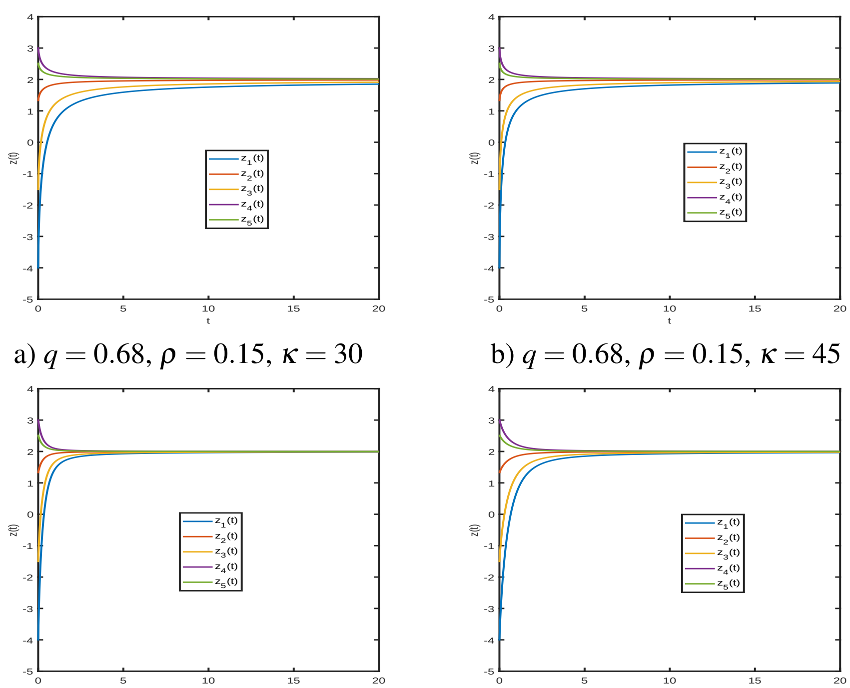
Hence, H is $(\mathscr{P}_A, L)$-Lipschitz-type continuous with $L=0.28$.

Thus, we conclude that all the assumptions of Theorem 3.1 are fulfilled with

$$0<\rho<\frac{2\mathbf{a}_{\min}\mu}{(\mathbf{a}_{\max}L)^2} \approx 1.018,$$

so Theorem 3.1 is valid.

In what follows, we provide Figure 1 and Figure 2 which depict the trajectories generated by the dynamical system (3.14) with the initial values $z_0^1 = (-4.021, 1.308, -1.524, 3.025, 2.534)^\top$ and $z_0^2 = (2.221, 0.508, 1.324, -1.025, 3.534)^\top$ and some values of parameters ρ, κ and order $q \in (0,1)$. It is clear that $z(t)$ converges to the unique equilibrium point $z^* = (2,2,2,2,2)^\top$.



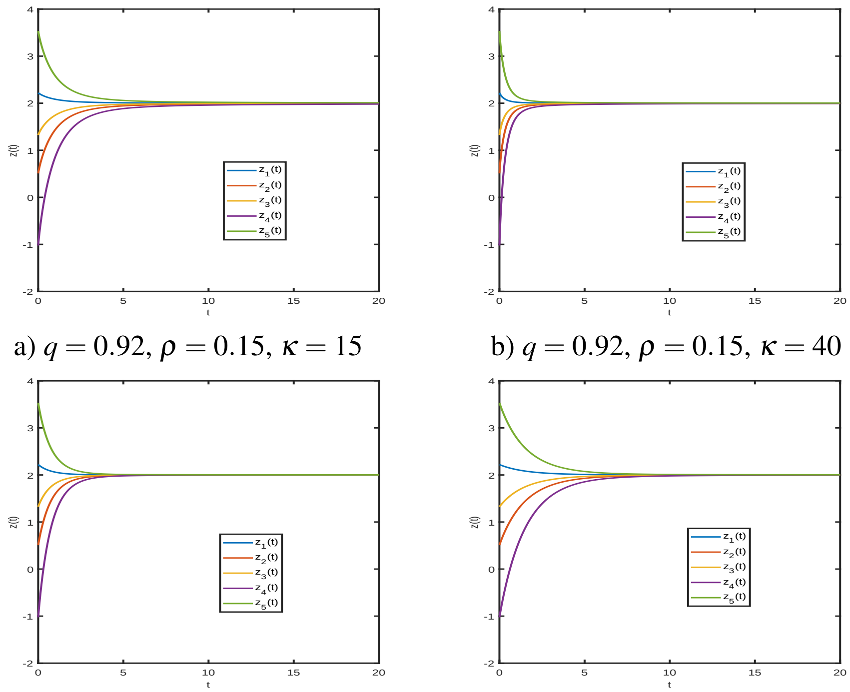
a) $q=0.68$, $\rho=0.15$, $\kappa=30$

b) $q=0.68$, $\rho=0.15$, $\kappa=45$

c) $q=0.88$, $\rho=0.15$, $\kappa=45$

d) $q=0.88$, $\rho=0.15$, $\kappa=25$

FIGURE 1. Transient behavior of the trajectory of dynamical system (3.14) with the initial value z_0^1.



a) $q = 0.92$, $\rho = 0.15$, $\kappa = 15$ b) $q = 0.92$, $\rho = 0.15$, $\kappa = 40$

c) $q = 0.98$, $\rho = 0.15$, $\kappa = 20$ d) $q = 0.98$, $\rho = 0.15$, $\kappa = 15$

FIGURE 2. Transient behavior of the trajectory of dynamical system (3.14) with the initial value z_0^2.

The Caputo fractional differential equations in system (3.14) were solved in Matlab with the code `fde12.m` modified by the basic Predictor-Corrector method developed by Garrappa [8] with $t_0 = 0, t_{\text{final}} = 20$ and order $q \in (0,1)$. Step size for `fde12.m` was set to $h = 2^{-6}$.

We now introduce two assumptions for problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$ studied by Hung et al. [14] and establish a result with using those assumptions.

$(\mathbf{I}_H)_3$: $\bigcap_{i=1}^{l} \mathscr{S}_i := \left\{z \in C : \sum_{k=1}^{m} a_{ik} H_k(z,u) \geq 0, \forall u \in C\right\} \neq \emptyset$;
$(\mathbf{I}_H)_4$: H is $(\mathscr{P}_A,\mu)$-strongly monotone, i.e., there exists $\mu > 0$ if, for each $(z,u) \in C \times C$,

$$H(z,u) + H(u,z) + \mu \|z-u\|^2 \mathbf{e} \in -\mathscr{P}_A.$$

Lemma 3.3. *Let z^* be the unique solution to* $\mathbf{VEP}(C,H,\mathscr{P}_A)$ *and $z \in C$. If conditions $(\mathbf{I}_H)_3$ and $(\mathbf{I}_H)_4$ are satisfied, then inequality (3.25) holds.*

Proof. Let z^* be the unique solution to $\mathbf{VEP}(C,H,\mathscr{P}_A)$ and $z \in C$. Using the condition $(\mathbf{I}_H)_4$, we have

$$A\left[H\left(z^*, \mathbf{prox}_C^{\rho H}(z)\right) + H\left(\mathbf{prox}_C^{\rho H}(z), z^*\right) + \mu \left\|\mathbf{prox}_C^{\rho H}(z) - z^*\right\|^2 \mathbf{e}\right] \in -\mathbb{R}_+^l,$$

which implies

$$\sum_{k=1}^{m} a_{\bar{i}k} \left[H_k\left(z^*, \mathbf{prox}_C^{\rho H}(z)\right) + H_k\left(\mathbf{prox}_C^{\rho H}(z), z^*\right) + \mu \left\|\mathbf{prox}_C^{\rho H}(z) - z^*\right\|^2\right] \leq 0,$$

for some $\bar{i} \in \{1,\ldots,l\}$, so

$$\begin{aligned}
&\sum_{k=1}^{m} a_{\bar{i}k} H_k\left(\mathbf{prox}_C^{\rho H}(z), z^*\right) \\
&\leq -\sum_{k=1}^{m} a_{\bar{i}k} H_k\left(z^*, \mathbf{prox}_C^{\rho H}(z)\right) - \sum_{k=1}^{m} a_{\bar{i}k}\mu \left\|\mathbf{prox}_C^{\rho H}(z) - z^*\right\|^2 \\
&\leq -\sum_{k=1}^{m} a_{\bar{i}k} H_k\left(z^*, \mathbf{prox}_C^{\rho H}(z)\right) - \mathbf{a}_{\min}\mu \left\|\mathbf{prox}_C^{\rho H}(z) - z^*\right\|^2. \qquad (3.34)
\end{aligned}$$

Furthermore, by the condition $(\mathbf{I}_H)_3$, without loss of generality, we may assume that $x^* \in \mathscr{S}_{\bar{i}}$, i.e.,

$$\sum_{k=1}^{m} a_{\bar{i}k} H_k\left(z^*, \mathbf{prox}_C^{\rho H}(z)\right) \geq 0.$$

Hence, (3.34) implies that

$$\sum_{k=1}^{m} a_{\bar{i}k} H_k\left(\mathbf{prox}_C^{\rho H}(z), z^*\right) \leq -\mathbf{a}_{\min}\mu \left\|\mathbf{prox}_C^{\rho H}(z) - z^*\right\|^2.$$

which implies that inequality (3.25) holds. □

From Lemma 3.3, if we replace condition $(\mathbf{I}_H)_1$ by conditions $(\mathbf{I}_H)_3$ and $(\mathbf{I}_H)_4$, then the result of Proposition 3.2 is still valid. Therefore, the following corollary can be directly obtained from Theorem 3.1.

Corollary 3.1. *Suppose that hypotheses $(\mathbf{I}_A)$, $(\mathbf{I}_C)$, $(\mathbf{I}_H)_0$ and $(\mathbf{I}_H)_2$–$(\mathbf{I}_H)_4$ hold, and let*

$$0 < \rho < \frac{2\mathbf{a}_{\min}\mu}{(\mathbf{a}_{\max}L)^2},$$

where $\mathbf{a}_{\min}$ and $\mathbf{a}_{\max}$ are given by (3.21). Then, (3.14) is Mittag-Leffler stable.

Remark 3.4. Our main results in this paper, Theorem 3.1 and Corollary 3.1, provide the Mittag-Leffler stability of the dynamical system (3.14) which derives the convergence to the solution of problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$. As mentioned in Introduction, the dynamical system (3.14) involving Caputo fractional derivative operators for solving $\mathbf{VEP}(C,H,\mathscr{P}_A)$ has not been considered in any previous literature. Thus, Theorem 3.1 and Corollary 3.1 are novel contributions in this work.

Remark 3.5. For $q = 1$ and $t_0 = 0$, we have $E_q\left(-\omega(t-t_0)^q\right) = e^{-\omega t}$, so (3.30) implies that

$$\|z(t) - z^*\| \leq \|z_0 - z^*\| e^{-\hat{\omega} t},$$

where

$$\hat{\omega} = \kappa \left[1 - \frac{1}{\sqrt{1 + \rho\left(2\mathbf{a}_{\min}\mu - \rho\left(\mathbf{a}_{\max}L\right)^2\right)}}\right].$$

Also, (3.14) reduces to (3.15). By Theorem 3.1 and Corollary 3.1, we conclude that the trajectories of (3.15) converge to z^* at least with the exponential convergence rate, i.e., the unique equilibrium point z^* of dynamical system (3.15) is exponentially stable.

Furthermore, with $0 < \rho < \dfrac{2\mathbf{a}_{\min}\mu}{(\mathbf{a}_{\max}L)^2}$, we can control the parameter ρ to obtain the maximum convergence rate of the trajectory $z(t)$ given by

$$\hat{\omega} = \hat{\omega}^* = \kappa \left[1 - \frac{\mathbf{a}_{\max}L}{\sqrt{(\mathbf{a}_{\min}\mu)^2 + (\mathbf{a}_{\max}L)^2}} \right],$$

when $\rho = \rho^* = \dfrac{\mathbf{a}_{\min}\mu}{(\mathbf{a}_{\max}L)^2}$.

To sum up, Theorem 3.1 and Corollary 3.1 are a generalization of [35, Theorem 3.2 and Corollary 3.2] and [15, Theorem 1 and Corollary 1] according to the following aspects; see also Figure 3:

- Dynamical system (3.14) is based on Caputo fractional derivative operators.
- $\mathbf{VEP}(C,H,\mathscr{P}_A)$ is an equilibrium problem of the vector type with partial order given by a polyhedral cone.
- Using computational technologies involving Caputo fractional-order derivatives, strong pseudo-monotonicity, strong monotonicity, and Lipschitz-type continuity assumptions with respect to partial order are constructed by a polyhedral cone.

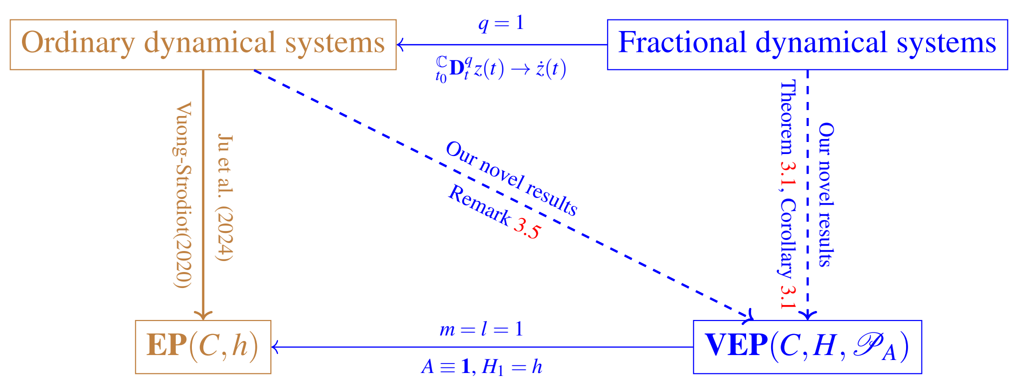


FIGURE 3. Illustration of the development of dynamical systems for solving different kinds of $\mathbf{EP}(C,h)$ and $\mathbf{VEP}(C,H,\mathscr{P}_A)$.

4. APPLICATION TO A TRAFFIC NETWORK EQUILIBRIUM PROBLEM

The aim of this section is to provide the applicability of the theoretical results studied in Section 3. In particular, we look into the traffic network equilibrium problems of the vector type based on polyhedral cone ordering introduced in [14]. Indeed, it is a general framework of network equilibrium models that received much attentions recently; see, e.g., [5, 36, 37].

Let us given a transportation network $\mathbf{M} = (\mathscr{N}, \mathscr{A})$, where $\mathscr{N}$ denotes the set of nodes and $\mathscr{A}$ denotes the set of directed arcs. We denote by Ω the set of origin-destination (O-D) pairs and $\mathscr{G}_\omega$, $\omega \in \Omega$ denotes the set of available paths joining O-D pair ω. Let $\mathbf{d} = (\mathbf{d}_\omega)_{\omega\in\Omega}$ denote the demand vector, where $\mathbf{d}_\omega$ denotes the demand of traffic flow on O-D pair ω. For a given path $p \in \mathscr{G}_\omega$, we denote the traffic flow on p by f_p and $\mathbf{f} = (f_1, \ldots, f_N)^\top \in \mathbb{R}^N$, where $N = \sum_{\omega\in\Omega} |\mathscr{G}_\omega|$

being $|\cdot|$ the cardinality of $\mathscr{G}_\omega$. The path flow vector $\mathbf{f}$ induces a flow v_e on each arc $e \in \mathscr{A}$ given by

$$v_e = \sum_{\omega\in\Omega}\sum_{p\in\mathscr{G}_\omega} \pi_{ep} f_p,$$

where $[\pi_{ep}] \in \mathbb{R}^{\tau\times N}$ ($\tau = |\mathscr{A}|$) is the arc path incidence matrix with

$$\pi_{ep} = \begin{cases} 1 & \text{if arc } e \text{ belongs to path } p; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{v} = (v_1, \ldots, v_\tau)^\top \in \mathbb{R}^\tau$ be the vector of arc flow. A path flow $\mathbf{f}$ satisfies demands if $\sum_{p\in\mathscr{G}_\omega} f_p = \mathbf{d}_\omega$ for all $\omega \in \Omega$. A path flow $\mathbf{f}$ is said to be feasible if $\mathbf{f} \geq \mathbf{0}$ satisfying the demand. Let us consider the set

$$\mathbf{F} = \left\{ \mathbf{f} \in \mathbb{R}^N : \mathbf{f} \geq \mathbf{0}, \sum_{p\in\mathscr{G}_\omega} f_p = \mathbf{d}_\omega, \forall\omega \in \Omega \right\}.$$

Assume that $\mathbf{F} \neq \emptyset$. It is easy to check that $\mathbf{F}$ is compact and convex. Let $\mathbf{c}_e \colon \mathbb{R}^\tau \to \mathbb{R}^m$ be a vector-valued cost function for arc e. In general, $\mathbf{c}_e$ is a function of all the arc flows. We assume that $\mathbf{c}_e(\mathbf{v}) = (\mathbf{c}_e^1(\mathbf{v}), \ldots, \mathbf{c}_e^m(\mathbf{v}))^\top \in \mathbb{R}^m$. Let $\mathscr{K}_p \colon \mathbb{R}^N \to \mathbb{R}^m$ be a vector-valued cost function along the path p. For each $\omega \in \Omega$ and $p \in \mathscr{G}_\omega$, the vector cost $\mathscr{K}_p$ is assumed to be the sum of all the arc cost of the flow f_p through arcs, which belong to the path p, i.e.,

$$\mathscr{K}_p(\mathbf{f}) = \sum_{e\in\mathscr{A}} \pi_{ep}\mathbf{c}_e(\mathbf{v}) = \begin{pmatrix} \sum_{e\in\mathscr{A}} \pi_{ep}\mathbf{c}_e^1(\mathbf{v}) \\ \vdots \\ \sum_{e\in\mathscr{A}} \pi_{ep}\mathbf{c}_e^m(\mathbf{v}) \end{pmatrix}.$$

For each $\omega \in \Omega$, $p \in \mathscr{G}_\omega$, $j \in \{1, \ldots, m\}$, $\mathbf{v} \in \mathbb{R}^\tau$ and $\mathbf{f} \in \mathbf{F}$, let

$$\mathscr{K}_p^j(\mathbf{f}) = \sum_{e\in\mathscr{A}} \pi_{ep}\mathbf{c}_e^j(\mathbf{v}) \text{ and } \mathscr{K}^j(\mathbf{f}) = (\mathscr{K}_1^j(\mathbf{f}), \ldots, \mathscr{K}_N^j(\mathbf{f}))^\top \in \mathbb{R}^N.$$

We always assume that $\mathscr{K}_p^j(\cdot)$ is continuous for all $\omega \in \Omega, p \in \mathscr{G}_\omega$ and $j \in \{1, \ldots, m\}$. Then, for each $\mathbf{f} \in \mathbf{F}$, let

$$\mathscr{K}(\mathbf{f}) = (\mathscr{K}^1(\mathbf{f}), \ldots, \mathscr{K}^m(\mathbf{f}))^\top = (\mathscr{K}_1(\mathbf{f}), \ldots, \mathscr{K}_N(\mathbf{f})) \in \mathbb{R}^{m\times N},$$

that is,

$$\mathscr{K}(\mathbf{f}) = \begin{pmatrix} \mathscr{K}_1^1(\mathbf{f}) & \mathscr{K}_2^1(\mathbf{f}) & \cdots & \mathscr{K}_N^1(\mathbf{f}) \\ \mathscr{K}_1^2(\mathbf{f}) & \mathscr{K}_2^2(\mathbf{f}) & \cdots & \mathscr{K}_N^2(\mathbf{f}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{K}_1^m(\mathbf{f}) & \mathscr{K}_2^m(\mathbf{f}) & \cdots & \mathscr{K}_N^m(\mathbf{f}) \end{pmatrix}.$$

Definition 4.1. [14] A flow $\mathbf{f} \in \mathbf{F}$ is said to be in $\mathscr{P}_A$-equilibrium if for all $\omega \in \Omega, p \in \mathscr{G}_\omega$ and $k \in \mathscr{G}_\omega$,

$$\mathscr{K}_p(\mathbf{f}) - \mathscr{K}_k(\mathbf{f}) \in \operatorname{int}(\mathscr{P}_A) \quad \Longrightarrow \quad f_p = 0, \tag{4.1}$$

where $\mathscr{P}_A$ is the polyhedral cone generated by matrix A.

If $l = m$ and A is the identity matrix of size m, then $\mathscr{P}_A = \{z \in \mathbb{R}^m : Az \geq \mathbf{0}\} = \mathbb{R}_+^m$. Accordingly, (4.1) becomes

$$\mathscr{K}_p(\mathbf{f}) - \mathscr{K}_k(\mathbf{f}) \in \operatorname{int}(\mathbb{R}_+^m) \quad \Longrightarrow \quad f_p = 0.$$

Then, the flow $\mathbf{f}$ is in weak vector equilibrium; see [5, Definition 3.2].

Proposition 4.1. [14] *The path flow* $\mathbf{f}^* \in \mathbf{F}$ *is in* $\mathscr{P}_A$*-equilibrium if* $\mathbf{f}^*$ *solves the vector variational inequality* (*for short,* $\mathbf{VVI}(\mathbf{F},\mathscr{K},\mathscr{P}_A)$) :

$$\sum_{\omega\in\Omega}\sum_{p\in\mathscr{G}_\omega}(h_p - f_p^*)\mathscr{K}_p(\mathbf{f}^*) \notin -\mathrm{int}(\mathscr{P}_A), \quad \forall \mathbf{h}\in\mathbf{F}.$$

Denote by $\mathbf{S}(\mathbf{F},\mathscr{K},\mathscr{P}_A)$ the solution set of problem $\mathbf{VVI}(\mathbf{F},\mathscr{K},\mathscr{P}_A)$. We always assume that $\mathbf{S}(\mathbf{F},\mathscr{K},\mathscr{P}_A)$ is nonempty. Now, we assume that the condition $(\mathbf{I}_A)$ is satisfied and impose the following assumptions for problem $\mathbf{VVI}(\mathbf{F},\mathscr{K},\mathscr{P}_A)$:

$(\mathbf{I}_{\mathscr{K}})_1$: $\mathscr{K}$ is $(\mathscr{P}_A,\hat{\mu})$-strongly pseudo-monotone, i.e., there exists $\hat{\mu}>0$ such that

$$\sum_{\omega\in\Omega}\sum_{p\in\mathscr{G}_\omega}(h_p - f_p)\mathscr{K}_p(\mathbf{f}) \notin -\mathrm{int}\mathscr{P}_A$$
$$\Longrightarrow \sum_{\omega\in\Omega}\sum_{p\in\mathscr{G}_\omega}(f_p - h_p)\mathscr{K}_p(\mathbf{h}) + \hat{\mu}\|\mathbf{f}-\mathbf{h}\|^2\mathbf{e} \in -\mathscr{P}_A,$$

for all $\mathbf{f},\mathbf{h}\in\mathbf{F}$;
$(\mathbf{I}_{\mathscr{K}})_2$: For each $i\in\{1,\ldots,l\}$, there exists $L_i>0$ such that

$$\sum_{j=1}^m a_{ij}\left[\left\langle \mathscr{K}^j(\mathbf{f}) - \mathscr{K}^j(\mathbf{h}), \mathbf{g}-\mathbf{h}\right\rangle + L_i\|\mathbf{f}-\mathbf{h}\|\|\mathbf{g}-\mathbf{h}\|\right] \geq 0,$$

for all $\mathbf{f},\mathbf{h},\mathbf{g}\in\mathbf{F}$.

Now, let us consider the following function

$$\widehat{\mathbf{prox}}_{\mathbf{F}}^{\rho\mathscr{K}}(\mathbf{f}) = \arg\min_{\mathbf{h}\in\mathbf{F}}\left(\rho\max_{1\le i\le l}\left\{\sum_{j=1}^m a_{ij}\sum_{\omega\in\Omega}\sum_{p\in\mathscr{G}_\omega}(h_p - f_p)\mathscr{K}_p^j(\mathbf{f})\right\} + \frac{1}{2}\|\mathbf{f}-\mathbf{h}\|^2\right),$$

for all $\mathbf{f}\in\mathbf{F}$. In order to apply the results presented in Section 3, for each $j\in\{1,\ldots,m\}$, let $C=\mathbf{F}$ and functions $H_j\colon \mathbf{F}\times\mathbf{F}\to\mathbb{R}$ and $H\colon \mathbf{F}\times\mathbf{F}\to\mathbb{R}^m$ be defined by

$$H_j(\mathbf{f},\mathbf{h}) = \sum_{\omega\in\Omega}\sum_{p\in\mathscr{G}_\omega}(h_p - f_p)\mathscr{K}_p^j(\mathbf{f}),$$
$$H(\mathbf{f},\mathbf{h}) = (H_1(\mathbf{f},\mathbf{h}),\ldots,H_m(\mathbf{f},\mathbf{h}))^\top \in\mathbb{R}^m \quad \forall \mathbf{f},\mathbf{h}\in\mathbf{F}.$$

Then, problem $\mathbf{VVI}(\mathbf{F},\mathscr{K},\mathscr{P}_A)$ is equivalent to problem $\mathbf{VEP}(C,H,\mathscr{P}_A)$, i.e., $\widehat{\mathbf{prox}}_{\mathbf{F}}^{\rho\mathscr{K}}$ is equivalent to the function $\mathbf{prox}_C^{\rho H}$.

Since $\mathbf{F}\ni\mathbf{h}\mapsto H_j(\mathbf{f},\mathbf{h})$ is an affine function for all $j\in\{1,\ldots,m\}$, it is easy to show that H is $\mathscr{P}_A$-convex in the second component. By assumptions, $\mathbf{F}$ is a nonempty, compact, and convex set. Moreover, for all $i \in \{1,\dots,l\}$, one has

$$\begin{aligned}
&\sum_{j=1}^{m} a_{ij}\left[H_j(\mathbf{f},\mathbf{h})+H_j(\mathbf{h},\mathbf{g})-H_j(\mathbf{f},\mathbf{g})\right]\\
&=\sum_{j=1}^{m} a_{ij}\sum_{\omega\in\Omega}\sum_{p\in\mathcal{G}_\omega}\left[(h_p-f_p)\mathcal{K}_p^j(\mathbf{f})+(g_p-h_p)\mathcal{K}_p^j(\mathbf{h})-(g_p-f_p)\mathcal{K}_p^j(\mathbf{f})\right]\\
&=\sum_{j=1}^{m} a_{ij}\sum_{\omega\in\Omega}\sum_{p\in\mathcal{G}_\omega}(g_p-h_p)\left(\mathcal{K}_p^j(\mathbf{h})-\mathcal{K}_p^j(\mathbf{f})\right)\\
&=\sum_{j=1}^{m} a_{ij}\left\langle \mathcal{K}^j(\mathbf{f})-\mathcal{K}^j(\mathbf{h}),\mathbf{g}-\mathbf{h}\right\rangle,
\end{aligned}$$

for all $\mathbf{f},\mathbf{h},\mathbf{g}\in\mathbf{F}$. Using conditions $(\mathbf{I}_A)$ and $(\mathbf{I}_{\mathcal{K}})_2$, we obtain

$$\begin{aligned}
\sum_{j=1}^{m} a_{ij}\left\langle \mathcal{K}^j(\mathbf{f})-\mathcal{K}^j(\mathbf{h}),\mathbf{g}-\mathbf{h}\right\rangle &\geq -\sum_{j=1}^{m} a_{ij}L_i\|\mathbf{f}-\mathbf{h}\|\|\mathbf{g}-\mathbf{h}\|\\
&\geq -\sum_{j=1}^{m} a_{ij}\max_{1\leq i\leq l}\{L_i\}\|\mathbf{f}-\mathbf{h}\|\|\mathbf{g}-\mathbf{h}\|
\end{aligned}$$

Hence, for all $i\in\{1,\dots,l\}$,

$$\sum_{j=1}^{m} a_{ij}\left[H_j(\mathbf{f},\mathbf{h})+H_j(\mathbf{h},\mathbf{g})-H_j(\mathbf{f},\mathbf{g})+\max_{1\leq i\leq l}\{L_i\}\|\mathbf{h}-\mathbf{f}\|\|\mathbf{g}-\mathbf{h}\|\right]\geq 0,$$

that is,

$$A\left[H(\mathbf{f},\mathbf{h})+H(\mathbf{h},\mathbf{g})-H(\mathbf{f},\mathbf{g})+\max_{1\leq i\leq l}\{L_i\}\|\mathbf{h}-\mathbf{f}\|\|\mathbf{g}-\mathbf{h}\|\mathbf{e}\right]\in\mathbb{R}_+^l.$$

for all $\mathbf{f},\mathbf{h},\mathbf{g}\in\mathbf{F}$. Thus

$$H(\mathbf{f},\mathbf{h})+H(\mathbf{h},\mathbf{g})-H(\mathbf{f},\mathbf{g})+\max_{1\leq i\leq l}\{L_i\}\|\mathbf{h}-\mathbf{f}\|\|\mathbf{g}-\mathbf{h}\|\mathbf{e}\in\mathcal{P}_A,$$

for all $\mathbf{f},\mathbf{h},\mathbf{g}\in\mathbf{F}$, so H is $(\mathcal{P}_A,L)$-Lipschitz-type continuous with $L=\max_{1\leq i\leq l}\{L_i\}>0$. It follows from assumption $(\mathbf{I}_{\mathcal{K}})_1$ that H is $(\mathcal{P}_A,\hat{\mu})$-strongly pseudo-monotone. Therefore, all hypotheses $(\mathbf{I}_C)$ and $(\mathbf{I}_H)_0$–$(\mathbf{I}_H)_2$ are satisfied with $C=\mathbf{F}$, $\mu=\hat{\mu}$ and $L=\max_{1\leq i\leq l}\{L_i\}$.

From the above discussions and Propositions 3.1-3.2 and Theorem 3.1, an immediate consequence follows.

Theorem 4.1. *For problem* $\mathbf{VVI}(\mathbf{F},\mathcal{K},\mathcal{P}_A)$*, suppose that assumptions* $(\mathbf{I}_A)$*,* $(\mathbf{I}_{\mathcal{K}})_1$*, and* $(\mathbf{I}_{\mathcal{K}})_2$ *hold, and let*

$$0<\rho<\frac{2\mathbf{a}_{\min}\hat{\mu}}{\left(\mathbf{a}_{\max}\max_{1\leq i\leq l}\{L_i\}\right)^2}.$$

Then,

$$\|\mathbf{f}(t)-\mathbf{f}^*\|\leq\|\mathbf{f}_0-\mathbf{f}^*\|\sqrt{E_q(-\theta(t-t_0)^q)}, \tag{4.2}$$

where

- $\mathbf{f}^*$ *is the unique solution of* $\mathbf{VVI}(\mathbf{F},\mathcal{K},\mathcal{P}_A)$*;*

- $\mathbf{f}(t)$ *is the trajectory of the following fractional-order dynamic system*

$$\begin{cases} {}_{t_0}^{\mathbb{C}}\mathbf{D}_t^q \mathbf{f}(t) = -\kappa \left[\mathbf{f}(t) - \widehat{\mathbf{prox}}_{\mathbf{F}}^{\rho\mathscr{K}}(\mathbf{f}(t))\right] \ \forall t > t_0, \\ \mathbf{f}(t_0) = \mathbf{f}_0 \in \mathbf{F},\ \kappa > 0; \end{cases} \tag{4.3}$$

- *the parameter* θ *is given by*

$$\theta = 2\kappa \left[1 - \frac{1}{\sqrt{1 + \rho\left[2\mathbf{a}_{\min}\hat{\mu} - \rho\left(\mathbf{a}_{\max} \max_{1\le i\le l}\{L_i\}\right)^2\right]}}\right] > 0.$$

- $\mathbf{a}_{\min}$ *and* $\mathbf{a}_{\max}$ *are given by* (3.21).
- $E_q(\cdot)$ *is the Mittag-Leffler function, as defined by* (2.5).

It follows from (4.2) that the trajectory $\mathbf{f}(t)$ of (4.3) converges to $\mathbf{f}^*$ as $t \to +\infty$ with at least the Mittag-Leffler convergence rate. Also, (4.3) is Mittag-Leffler stable with the unique equilibrium point $\mathbf{f}^*$. In addition, applying Lemma 3.3 and Corollary 3.1, if condition $(\mathbf{I}_{\mathscr{K}})_2$ is replaced by the following conditions:

$(\mathbf{I}_{\mathscr{K}})_3$: $\bigcap_{i=1}^{l} \widehat{\mathscr{S}_i} := \left\{\mathbf{f} \in \mathbf{F} : \sum_{j=1}^{m} a_{ij} \sum_{\omega\in\Omega} \sum_{p\in\mathscr{G}_\omega} (h_p - f_p)\mathscr{K}_p^j(\mathbf{f}) \ge 0, \forall \mathbf{h} \in \mathbf{F}\right\} \neq \emptyset$;

$(\mathbf{I}_{\mathscr{K}})_4$: There exists $\widehat{\mu} > 0$ if, for each $(\mathbf{f},\mathbf{h}) \in \mathbf{F}\times\mathbf{F}$,

$$\sum_{\omega\in\Omega} \sum_{p\in\mathscr{G}_\omega} [(h_p - f_p)\mathscr{K}_p(\mathbf{f}) + (f_p - h_p)\mathscr{K}_p(\mathbf{h})] + \widehat{\mu}\|\mathbf{f}-\mathbf{h}\|^2\mathbf{e} \in -\mathscr{P}_A,$$

then the result of Theorem 4.1 is still valid for (4.3).

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